Revised Edition

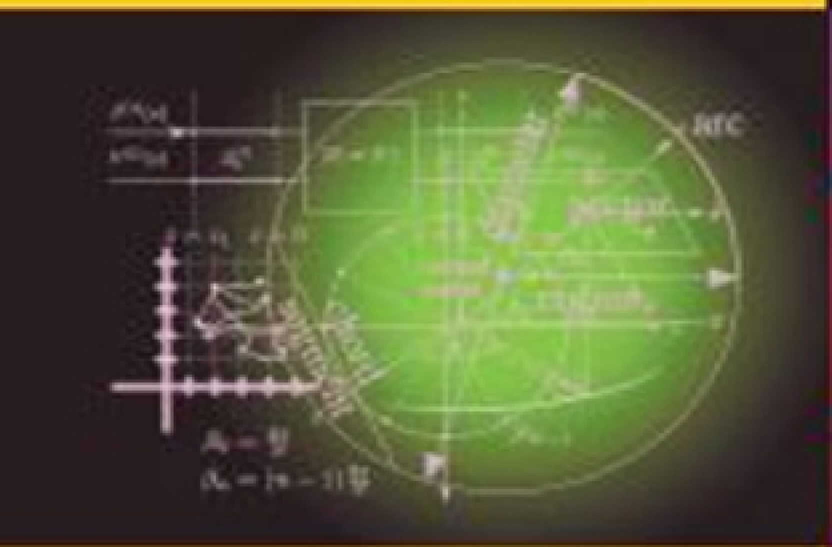
A TEXTBOOK OF B.Sc. MATHEMATICS

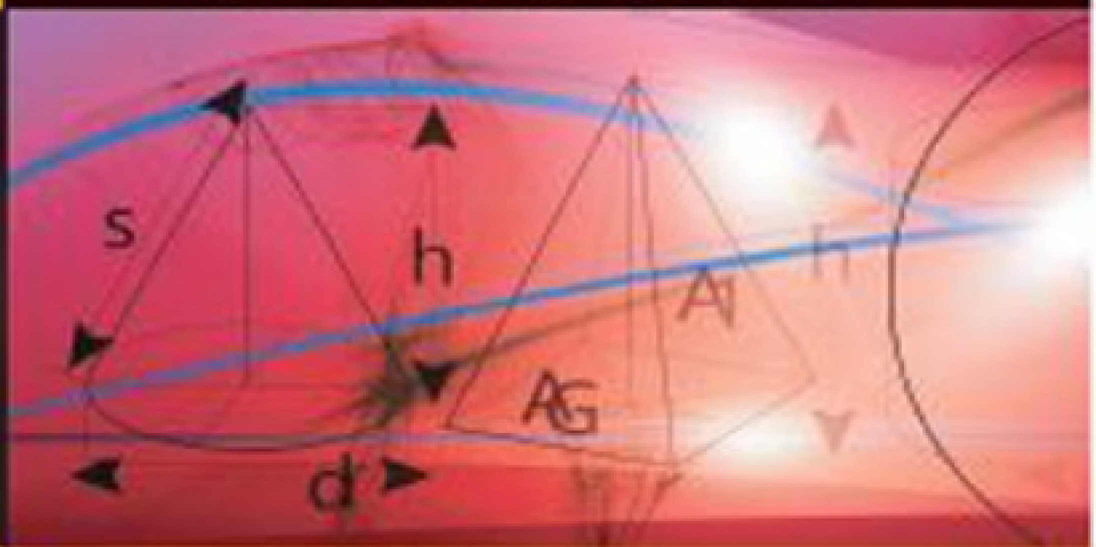
VOLUME – I

(THEORY AND PRACTICAL)

For B.Sc. First Year Second Semester Students of the Universities in Andhra Pradesh

Key Book also Included





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S ANJANEYA SASTRY
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S. CHAND

A TEXTBOOK OF

B.Sc.
MATHEMATICS

First Year, A.P.

SEMESTER II

(*1st Year 2nd Semester*)

(THEORY & PRACTICAL)

&

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(As per New Common Core Syllabus 2015-16 (revised in 2016) based on CBCS for B. Sc. First Year Second Semester Students of the Universities in Andhra Pradesh)

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PREFACE TO THE 2ND REVISED EDITION

This text book of B.Sc Mathematics, Vol. I for the first year students studying in all universities of Andhra Pradesh was first published in the year 1988 and has undergone several editions and many reprints. The authors are very happy that the earlier editions have been very well used by the students.

This **revised syllabus** is being adopted by all the universities in Andhra Pradesh, following **Common Core model curriculam from the academic year 2015 - 2016**. This book strictly covers the new curriculum for 1st year, 2nd semester of the theory as well as practical.

Objective type covering multiple choice type and fill in the blank type questions which are useful to M. Sc. entrance tests are given with answers at the end of each section.

Detailed solutions for all the problems in the various exercises of different chapters are given at the end.

All suggestions for the further improvement of the book are welcome.

We are thankful to the Management Team and the Editorial Department of S Chand And Company Limited, New Delhi for all help and support in the publication of this book.

AUTHORS

SYLLABUS

AS PER COMMON CORE MODEL CURRICULUM FROM THE ACADEMIC YEAR 2015 - 2016

SEMESTER - II
SOLID GEOMETRY

UNIT - I : (12 hours) **: The Plane :**

Equation of plane in terms of its intercepts on the axis, Equation of the plane through the given points, Length of the perpendicular from a given point to a given plane, Bisectors of angles between two planes, Combined equation of two planes, Orthogonal projection on a plane.

UNIT - II : (12 hours) **: The Line :**

Equations of a line, Angle between a line and a plane, The condition that a given line may lie in a given plane, The condition that two given lines are coplanar, Number of arbitrary constants in the equations of a straight line. Sets of conditions which determine a line, The shortest distance between two lines. The length and equations of the line of shortest distance between two straight lines, Length of the perpendicular from a given point to a given line, Intersection of three planes, Triangular Prism.

UNIT - III : (12 hours) **: The Sphere:**

Definition and equation of the sphere, Equation of the sphere through four given points, Plane sections of a sphere. Intersection of two spheres; Equation of a circle. Sphere through a given circle; Intersection of a sphere and a line. Power of a point; Tangent plane. Plane of contact. Polar plane, Pole of a plane, Conjugate points, Conjugate planes; Angle of intersection of two spheres. Conditions for two spheres to be orthogonal; Radical plane. Coaxial system of spheres; Simplified from of the equation of two spheres.

UNIT - IV: (12 hours) : **Cones**

Definitions of a cone, vertex, guiding curve, generators. Equation of the cone with a given vertex and guiding curve. Enveloping cone of a sphere. Equations of cones with vertex at origin are homogenous. Condition that the general equation of the second degree should represent a cone. Condition that a cone may have three mutually perpendicular generators Intersection of a line and a quadric cone. Tangent lines and tangent plane at a point. Condition that a plane may touch a cone. Reciprocal cones. Intersection of two cones with a common vertex. Right circular cone. Equation of the right circular cone with a given vertex, axis and semi-vertical angle.

UNIT - V: (12 hours) : **Cylinders and conicoids:**

Definition of a cylinder. Equation to the cylinder whose generators intersect a given conic and are parallel to a given line, Enveloping cylinder of a sphere. The right circular cylinder. Equation of the right circular cylinder with a given axis and radius.

The general equation of the second degree and the various surfaces represented by it; Shapes of some surfaces. Nature of Ellipsoid. Nature of Hyperboloid of one sheet.

Reference Text Book : *N. Krishna Murthy & others* ***" A text book of Mathematics for B. A./B. Sc. Vol. 1, Semester 2"***, *S. Chand & Company, New Delhi.*

CONTENTS

UNIT - I

1. **Introduction**
   Axioms, Definitions and Theorems (without proofs) of Euclidean geometry having application in solid analytical geometry.
2. **Coordinates**
   Coordinates, Interpratation of Equations, Concepts of Vector Algebra which are useful in prooving theorems or solving problems, Revision of concepts learnt in earlier class.
3. **The Plane**
   Equation of plane in terms of its intercepts on the axes, Equation of the plane through the three given non-collinear points, Length of the perpendicular from a given point to a given plane, systems of planes, Bisectors of angles between two planes, joint equation of a pair of planes.

1

INTRODUCTION

1.1. Logical development of any branch of mathematics depends on a set. The elements of the set are underfined terms. Associated with them are certain statements which are taken as axioms or postulates for the subject.

In earlier classes the set language was used for the effective understanding of geometrical concepts. It may be recalled that the structure of geometry was developed by taking point, line, plane and space as undefined concepts and that every geometrical figure is a set of points.

Solid analytical geometry (Three dimensional coordinate geometry) is a subject redeveloped on the above lines. The study of quantitative and qualitative nature of the space very much depends upon coordinates and algebraic operations and methods associated with it. Also vectors as ordered triads have an application in the treatment of solid analytical geometry.

1. 2. We take a set S and call it the $3-D$ *space,* R^3 *–space* and its elements the points of the $3-D$ *space.* We take *line L, plane* π, *sphere S, etc. as subsets of S.* If $P \in L$, we say that P is a point on the line L, *i.e.,* line L passes through the point P. If $P \notin L$, we say that the point is not on the line L *i.e.,* line L does not pass through the point P. Points on the same line are said to be *collinear* and points on the same plane are said to be *coplanar.* If $L \subset \pi$, we say that L is a line in the plane π *i.e.,* the plane π contains the line L. Further $L \not\subset \pi$ implies L is not in the plane π.

1. 3. We give below, not all, but certain axioms, definitions and theorems (without proofs) of Euclidean geometry which have wide application in solid analytical geometry. Further some well known terms and concepts which are used in the development of the subject and which have to be redefined are deliberately left out as their definitions and the results involving them given out in earlier classes are true even here.

1. Axiom. One and only one line passes through two distinct points.

If A, B are two distinct points on L, we say that L is the one and only one line which passes through A and B. We write L as $\overleftrightarrow{AB}$. (Fig. 2)

2. Axiom. One and only one plane passes through three non-collinear points. (Fig.1).

If π is a plane passing through three non-collinear points A, B, C we say that π is the one and only one plane determined by A, B, C. We write the plane π as $\overleftrightarrow{ABC}$.

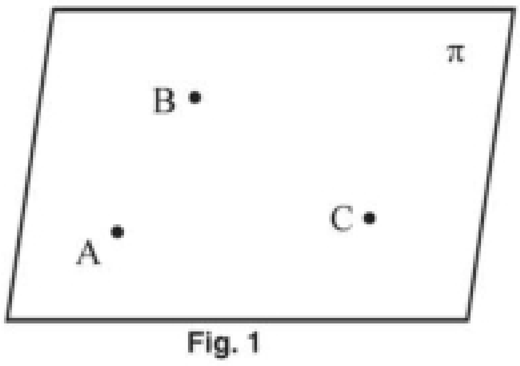


Fig. 1

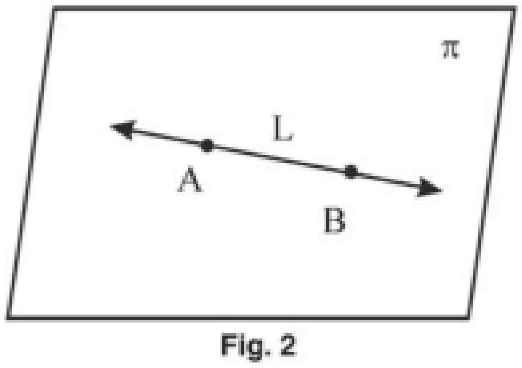


Fig. 2

3. Axiom. If two distinct points lie in a plane then the line through the two points lies in the plane. (Fig. 2).

If A, B are two distinct points in a plane π, we say that $\overleftrightarrow{AB}$ lies in the plane π.

4. Axiom. If two planes intersect, they intersect in one and only one line (Fig.3).

π_1, π_2 are two intersecting planes intersecting in the line L.

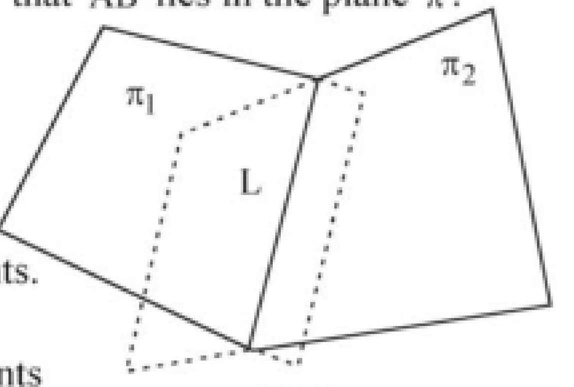


Fig. 3

5. Axiom. Every line consists of at least two points. Every plane consists of at least three non-collinear points. In space there always exist atleast four points which are non-coplanar.

6. Axiom. Corresponding to any two points, there exists always a unique real number called the distance between the points.

If A, B are any two points, there exists a unique real number $d\,(\mathrm{AB})$ or AB called the distance between the points A and B with the following properties *(i)* $\mathrm{AB} \geq 0$, *(ii)* $\mathrm{AB} = 0 \Leftrightarrow \mathrm{A} = \mathrm{B}$ *(iii)* AB = BA *(iv)* $\mathrm{AC} + \mathrm{CB} \geq \mathrm{AB}$ where C is any point.

7. Definition. If A, P, B are three collinear points such that AP + PB = AB then we say that P lies between A and B on the line. We write $\mathrm{A}-\mathrm{P}-\mathrm{B}$. The set of points P is called the line segment between A and B with end points A, B. We denote the line segment as AB.

It is to be understood that the meaning of AB is to be understood depending on the context.

If $\mathrm{A}-\mathrm{P}-\mathrm{B}$ and AP = PB, then P is called the middle point of the line segment AB.

If A, B are different points, then $\mathrm{AB} \cup \{\mathrm{P}/\mathrm{A}-\mathrm{B}-\mathrm{P}\}$ is called the ray from A through B and we write it as $\overrightarrow{AB}$.

8. Definition. *Two lines* L_1, L_2 in a plane π *are said to intersect if* $L_1 \cap L_2 \neq \phi$. Now L_1, L_2 are called intersecting lines.

9. Definition. *Two lines* L_1, L_2 *in a plane* π *are said to be parallel if either* L_1, L_2 *are coincident* $(L_1 = L_2)$ *or* L_1, L_2 *are not intersecting* $(L_1 \cap L_2 \neq \phi)$. We write $\mathrm{L}_1 \parallel \mathrm{L}_2$.

If $\mathrm{A}, \mathrm{B} \in \mathrm{L}_1$ and $\mathrm{C}, \mathrm{D} \in \mathrm{L}_2$ then we write $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ or $\mathrm{AB} \parallel \mathrm{CD}$.

If $\overrightarrow{AB}, \overrightarrow{CD}$ are non-collinear, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and B, D lie on the same side of $\overleftrightarrow{AC}$ we say that $\overrightarrow{AB} \parallel \overrightarrow{CD}$. Also if $\overrightarrow{AB}, \overrightarrow{CD}$ are collinear we say that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

If $\mathrm{L}_1, \mathrm{L}_2$ are not parallel we write $\mathrm{L}_1 \nparallel \mathrm{L}_2$.

10. Axiom. To a given line, through a given point one and only one parallel line exists.

11. Theorem. ***Two distinct lines cannot intersect at more than one point.***

12. Theorem. ***If L is a line not in the plane π and intersecting π, then the line L intersects the plane π in one and only one point.***

If the point is M in π, then M is called the foot of L in π.

13. Theorem. ***Plane containing a line and a point not on the line is unique.***

If $\mathrm{L} \subset \pi$ and $\mathrm{P} \in \pi$ and $\mathrm{P} \notin \mathrm{L}$, then π is unique.

14. Theorem. ***Two intersecting lines determine a unique plane.***

If L_1 and L_2 are two intersecting lines, then L_1 and L_2 determine a plane. (Fig. 4).

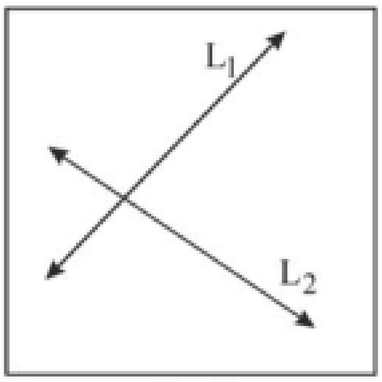


Fig. 4

15. Axiom. For every angle there corresponds a unique real number between 0 and π. This real number is called the measure of the angle.

The measure of an angle when measured by comparison with selected standard - a radial, the measure will be in radians. When the selected standard - a right angle, the measure will be in degrees.

If θ is the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$, we write $\theta = \angle(\overrightarrow{AB}, \overrightarrow{AC}) = (\overrightarrow{AB}, \overrightarrow{AC})\,(\overrightarrow{AC}, \overrightarrow{AB}) = \angle BAC = \angle CAB$ such that $0 \le \theta \le \pi$.

Also $(\overrightarrow{AB}, \overrightarrow{AB}) = 0$ and $(\overrightarrow{AB}, \overrightarrow{BA}) = \pi$.

If $\angle BAC = \dfrac{\pi}{2}$, we say that $\overrightarrow{BA}$ is perpendicular to $\overleftrightarrow{AC}$ and we write $\overleftrightarrow{BA} \perp \overleftrightarrow{AC}$ or $\overleftrightarrow{AB} \perp \overleftrightarrow{AC}$ or $AB \perp AC$.

If $\overrightarrow{OP}, \overrightarrow{OQ}$ are parallel to two lines L_1, L_2 respectively, the angle between L_1 and L_2 is $(\overrightarrow{OP}, \overrightarrow{OQ})$ or $\pi - (\overrightarrow{OP}, \overrightarrow{OQ})$. Angle between parallel lines is 0 or π.

Here π is the measure of an angle and not a symbol used to denote a plane and the meaning of π is to be understood depending on context.

16. Definition. *A line L cuts (or intersects) a plane π in a point P. If all the lines in π through P are perpendicular to L, then the line L is said to be perpendicular to the plane π. We write $L \perp \pi$ or $\pi \perp L$. If $M \in L$, then we also write $\overleftrightarrow{PM} \perp \pi$ or* $PM \perp \pi$.(Fig. 5).

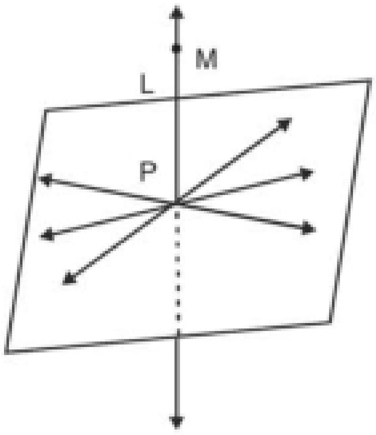


Fig. 5

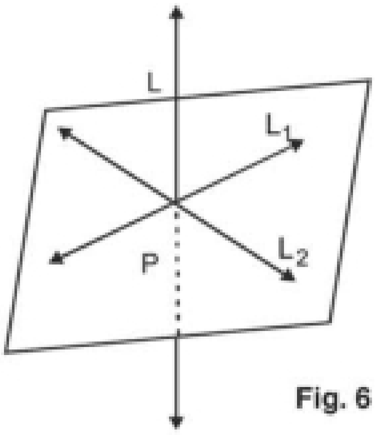


Fig. 6

17. Theorem. L_1 ***and*** L_2 ***are two lines intersecting at P. Then line L perpendicular to the lines*** L_1 ***and*** L_2 ***at P is perpendicular to the plane determined by*** L_1 ***and*** L_2. (Fig. 6.).

18. Theorem. ***P is a point and L is a line. Through P and perpendicular to L one and only one plane*** (π) ***exists.***

If L meets π ***in M, the*** $PM \perp L$. (Fig. 7.).

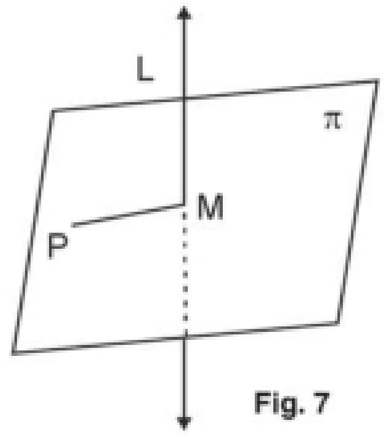


Fig. 7

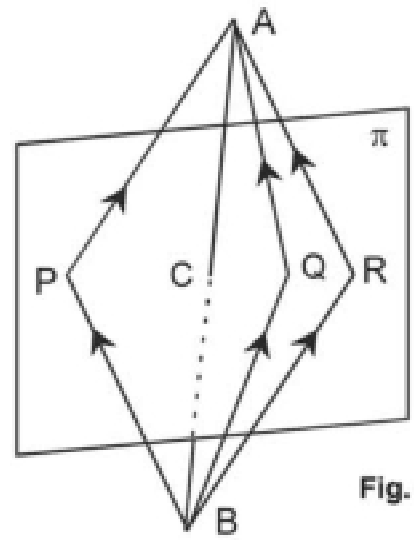


Fig. 8

19. Theorem. ***The set of points so that each point is equidistant from two points A and B determine a unique plane (π) perpendicular to AB and intersecting AB at its middle point (C).***

If P, Q, R .... are points such that AP = PB, AQ = QB, AR = RB, ..., then P, Q, R ... are in the plane π where $\pi \perp AB$ at C (Fig. 8).

20. Theorem. ***If*** L_1, L_2 ***are two distinct lines perpendicular to the plane*** π, ***then*** L_1 ***and*** L_2 ***are parallel and coplanar*** (Fig. 9).

21. Theorem. ***P is a point in the plane*** π. ***One and only one perpendicular line to*** π ***exists through P.*** (Fig. 10)

22. Theorem. L_1, L_2 ***are two parallel lines. If a plane is perpendicular to*** L_1 ***then it is also perpendicular to*** L_2. (Fig. 11).

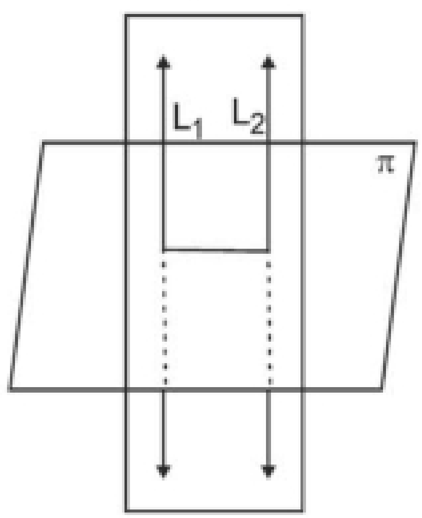


Fig. 9

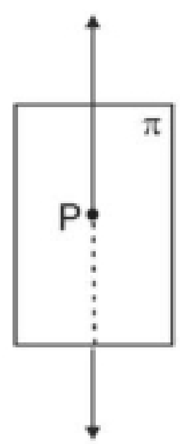


Fig. 10

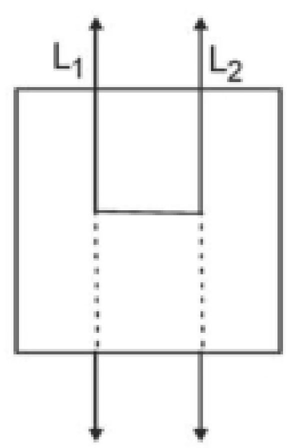


Fig. 11

23. Theorem. ***One and only one perpendicular line can be drawn to a plane from a point not in the plane.***

24. Theorem. ***L is a line and P is a point on L. If*** π ***is the plane perpendicular to L at P, then all the lines perpendicular to L through P lie in the plane*** π. **(Fig.12)**

25. Definition. *L is a line and* π *is a plane. If no point of L is in* π $(L \cap \pi = \phi)$ *or if L is in* π, *then L is said to be parallel to* π *and we write* $L \parallel \pi$.

26. Definition. *π_1, π_2 are two planes which are either coincident $(\pi_1 = \pi_2)$ or parallel $(\pi_1 \cap \pi_2 = \phi)$. Then π_1, π_2 are said to be parallel and we write $\pi_1 \parallel \pi_2$.*

27. Theorem. ***π_1, π_2 are two distinct parallel planes. If a plane π cuts π_1 in a line L_1 and π_2 in a line L_2, then $L_1 \parallel L_2$.*** (Fig. 13).

28. Theorem. ***If L is a line perpendicular to the plane π_1, then L is perpendicular to the planes parallel to π_1*** (Fig. 14).

29. If π_1, π_2 are two planes perpendicular to the lines L, then $\pi_1 \parallel \pi_2$. (Fig. 14).

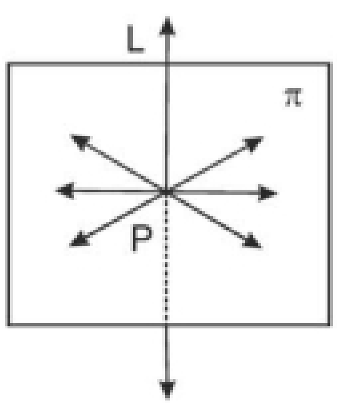


Fig. 12

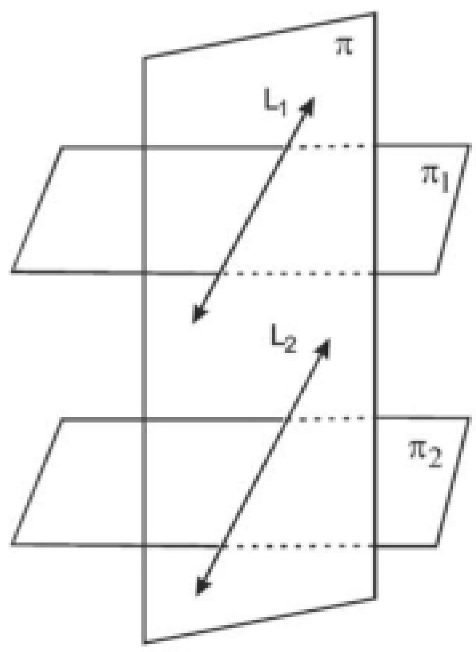


Fig. 13

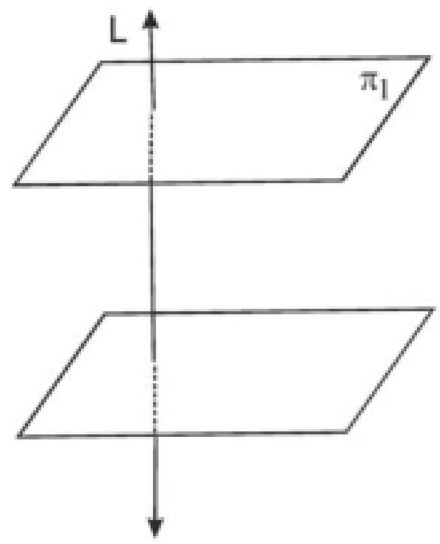


Fig. 14

30. Theorem. ***If L_1 is a line parallel to any line in a plane π then $L_1 \parallel \pi$.***

31. Theorem. ***L_1 is a line parallel to the plane $\pi_1 \cdot \pi_2$ is a plane containing L_1 and intersecting π_1 in L_2. Then $L_1 \parallel L_2$.*** (Fig. 15.).

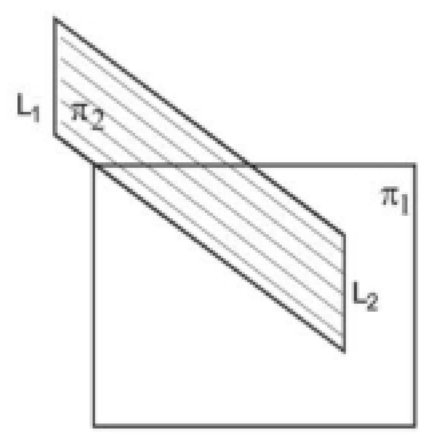


Fig. 15

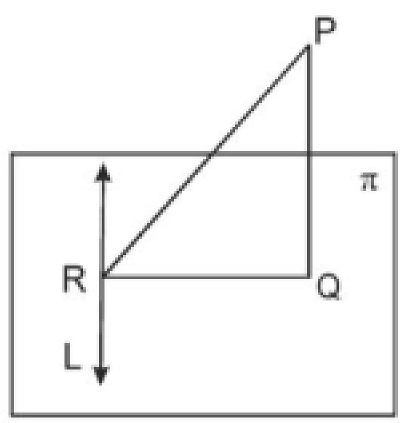


Fig. 16

32. Theorem. ***L is a line in a plane π. R is a point on L. Also Q is a point in π, but not on L. Further P is a point not in π. Then***

(*i*) $QR \perp L$ and $PR \perp L \Rightarrow PQ \perp \pi$

(*ii*) $PR \perp L$ and $PQ \perp \pi \Rightarrow QR \perp L$

(*iii*) $PQ \perp \pi$ and $QR \perp L \Rightarrow PR \perp L$ (Fig. 16).

This is called the theorem of three perpendiculars.

33. Theorem. ***Two parallel planes π_1, π_2 alswys make equal intercepts or lines perpendicular to π_1, π_2.***

Definition. *L is perpendicular to π_1, π_2. The intercept on L by π_1, π_2 is called the distance between π_1 and π_2* (Fig. 17.).

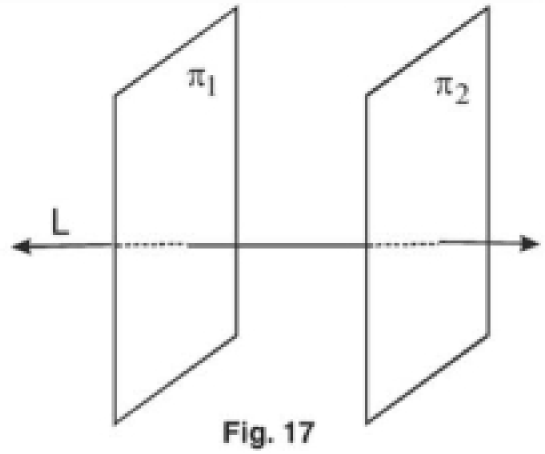


Fig. 17

34. Definition. *If L_1, L_2, L_3 are three lines such that $L_1 \cap L_2 \cap L_3 = P$, then the lines L_1, L_2 and L_3 are said to be concurrent.*

35. Theorem. ***If $L_1 \parallel L_2,\ L_2 \parallel L_3$, then $L_1 \parallel L_3$.***

We say that L_1, L_2, L_3 are parallel and we write $L_1 \parallel L_2 \parallel L_3$.

36. Theorem. ***π_1, π_2, π_3 are three distinct planes so that no two of which are parallel. The three lines of their intersection are either parallel or concurrent.***

(Fig. 18, Fig.19, Fig.20).

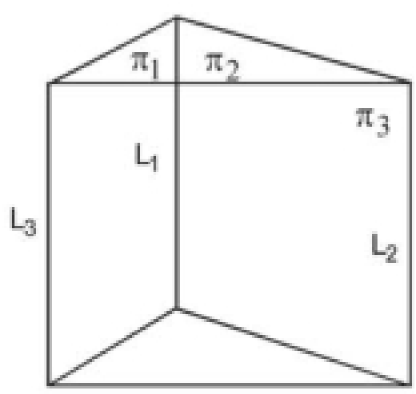


Fig. 18

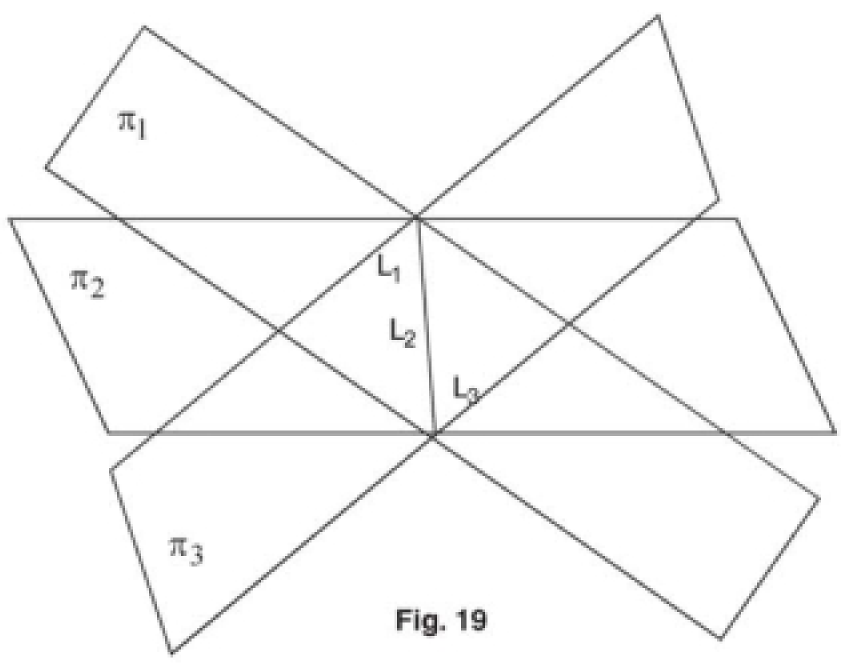


Fig. 19

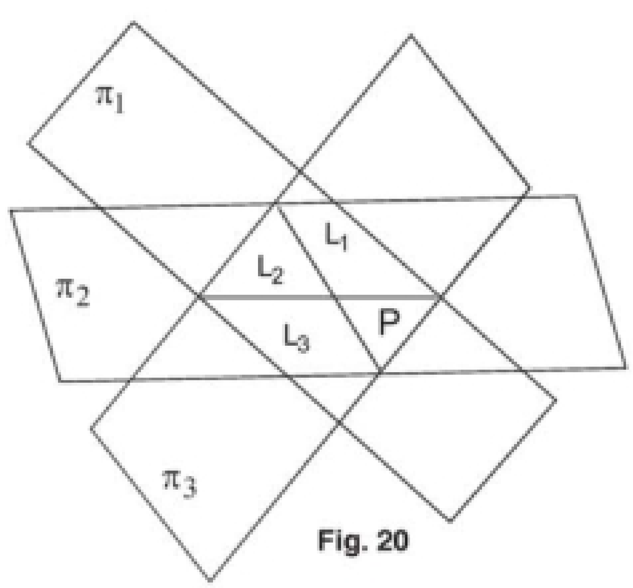


Fig. 20

37. Projections. Here peojection will mean orthogonal projection only.

1. Definition. *P is a point and L is a line. $P \notin L$. If M is the foot of the perpendicular from P on L, then M is called the projection of P in L.*

2. Definition. *P is a point and π is a plane not containing P. If the perpendicular from P to π meets π in the point M, then M is called the projection of P in π.*

3. Definition. *P, Q are two points and L is a line, $P \in L$ and $Q \notin L$. If M is the projection of Q in L, then PM is called the projection of PQ in L. (Fig.21).*

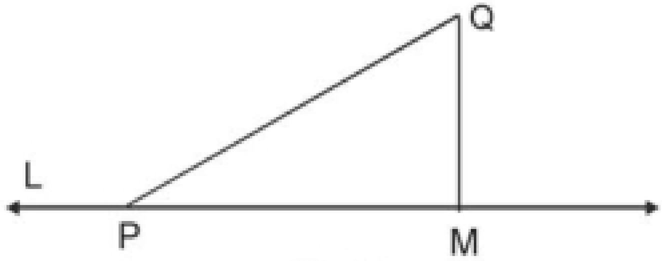


Fig. 21

4. Definition. *P, Q are two points and L is a line. $P, Q \notin L$. If M, N are the respective projections of P, Q in L, then the line segment MN is called the projection of the line segment PQ in L.*

5. Theorem. ***π is a plane and L is a line not in π. If perpendiculars are drawn from every point on L to π, then all the projections in π are either collinear (when L is not perpendicular to π) or coincident in a point (when $L \perp \pi$).***

38. Some useful results

1. L is a line and P is a point on it. The perpendiculars to L at P are coplanar.
2. The planes perpendicular to a line are all parallel.
3. L is a line perpendicular to a plane π.
   All the planes through L are perpendicular to π.
4. L_1, L_2 are a pair of intersecting lines. L_3, L_4 are a second pair of intersecting lines so that $L_3 \parallel L_1$ and $L_4 \parallel L_2$. Then

(*i*) the angle between the first pair = the angle between the second pair.

(*ii*) the plane determined by the first pair is parallel to the plane determined by the second pair.

39. Definition. *L is a line and R is the set of real numbers. $f : L \to R$ is a one-one mapping. If $A, B \in L$ such that $|A, B| = |f(A) - f(B)|$, then f is called a coordinate system for L. The real number $x\,(= f(P))$ is called the coordinate of P w.r.t. the coordinate system f on the line L and we write it as $P(x)$.*

In this context L is called the *coordinate line.* The set $P(x)$ is called the *geometric figure* on L.

Axiom. Every line has a coordinate system.

A coordinate system on the line L depends on an arbitrarily chosen points O and I on L such that

(*i*) O, called the origin, to correspond to the number O, and

(*ii*) I, called the unit point, to correspond to the number 1.

Clearly we can have infinitely many coordinate systems defined on L.

Then for every real number x and for each point P on L one-to-one correspondence exists with the following properties and notation '$P(x)$ denotes the point P with coordinate x' (Fig.22).

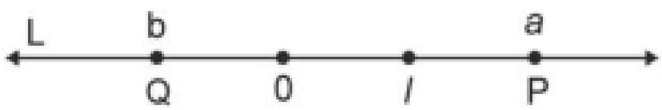


Fig. 22

$P(a), Q(b) \in L \;\Rightarrow$

(*i*) $P(a)=0 \Leftrightarrow P=0$,

(*ii*) $P(a)$ lies to the right of O is $a>0$, and $Q(b)$ lies to the left of O if $b<0$,

(*iii*) Between P and Q, distance $= PQ = |b-a|$ and directed distance $= b-a$,

(*iv*) $P(a)$ lies to the right of $Q(b)$ if $a>b$,

(*v*) $Q(b)$ lies to the left of $P(a)$ if $b<a$,

(*vi*) $P(a)$ coincides with $Q(b)$ *i.e.*, $P=Q$ if $a=b$.

40. Definition. *$A(a), P(x), B(b)$ are points on the coordinate line L. Then*

(i) $A-P-B \Rightarrow P$ is said to divide the line segment AB internally in the ratio $(x-a):(b-x)$ and $x-a, b-x$ are positive.

(ii) $P-A-B$ or $A-B-P \Rightarrow P$ is said to divide the line segment AB externally in the ratio $(x-a):(b-x)$ and $(x-a), (b-x)$ are of opposite signs.

We write $(P\,;A, B)=(x-a):(b-x)$. It is $+ve$ *or* $-ve$ according as P divides AB internally or externally.

If $(P\,;A, B)=m:n$, then $(x-a):(b-x)=m:n \Rightarrow n(x-a)=m(b-x)$

41. Theorem. ***In a given ratio a line segment is divided at one and only the point.***

42. Skew lines.

Definition. *Any two non-parallel and non-intersecting lines are called skew lines.*

Since any two lines in a plane must be either parallel or intersecting, skew lines are non-coplanar. Conversely any two non-coplanar lines are skew lines. ***(N. U. 07)***

43. Theorem. ***L_1, L_2 are two skew lines. Then there exists one and only one plane π through one of the lines and parallel to the second.***

44. Theorem. ***If $\overleftrightarrow{MN}$ is the projection of a line L in a plane π, then $\overleftrightarrow{MN}$, L are coplanar.***

If $\mathbf{L} \parallel \pi$, then $\mathbf{L} \parallel \overleftrightarrow{\mathbf{MN}}$.

45. Theorem. ***If L_1, L_2 are two skew lines, then there exists one and only one line which intersects L_1, L_2 and is perpendicular to L_1, L_2.***

46. Definition. *L is a line and π_1 is a plane not containing L. If L is not parallel to π_1, then the angle between L and π_1 (written as (L, π_1)) is the angle between L and $\overleftrightarrow{MN}$, where $\overleftrightarrow{MN}$ is the projection of L in π_1. We write $(L, \pi_1)=(L, \overleftrightarrow{MN})$.*

We can have $(L, \pi_1)=\dfrac{\pi}{2} \pm \theta$ where θ is the acute angle between L and a normal to π_1.

2

COORDINATES

2.1. COORDINATES OF A POINT IN SPACE

Let O be any point in space S. Let $\overleftrightarrow{X'X}$, $\overleftrightarrow{Y'Y}$ be two perpendicular lines through O. Let the plane determined by the lines be $\overleftrightarrow{XOY}$. Through O and perpendicular to the plane $\overleftrightarrow{XOY}$ let $\overleftrightarrow{Z'Z}$ be a line. Imagine the plane $\overleftrightarrow{XOY}$ as the plane of the paper and the perpendicular $\overleftrightarrow{Z'Z}$ is to be visualized as perpendicular to the plane of the paper at O. $\overleftrightarrow{Z'Z}$ may be regarded as vertical and $\overleftrightarrow{X'X}$, $\overleftrightarrow{Y'Y}$ as horizontal. $\overleftrightarrow{X'X}$, $\overleftrightarrow{Y'Y}$, $\overleftrightarrow{Z'Z}$ are three non-coplanar mutual perpendicular lines through O. The lines $\overleftrightarrow{Y'Y}$ and $\overleftrightarrow{Z'Z}$ determine the plane $\overleftrightarrow{YOZ}$ and the lines $\overleftrightarrow{Z'Z}$, $\overleftrightarrow{X'X}$ determine the plane $\overleftrightarrow{ZOX}$. Also $\overleftrightarrow{XOY}$, $\overleftrightarrow{YOZ}$, $\overleftrightarrow{ZOX}$ are three mutually perpendicular planes through O.

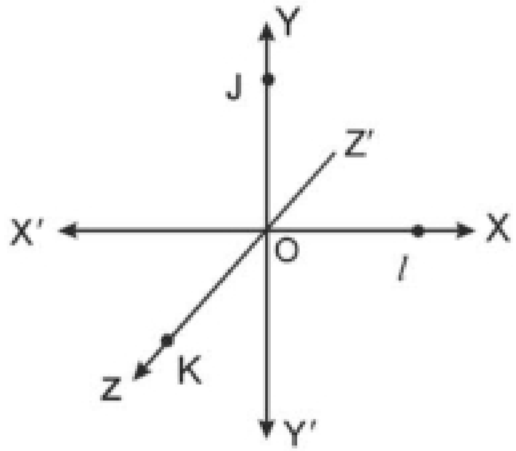


Fig. 23

On $\overleftrightarrow{X'X}$ take O as the origin, $\bar{I}$ as the unit point ;

on $\overleftrightarrow{Y'Y}$ take O as the origin, $\bar{J}$ as the unit point and

on $\overleftrightarrow{Z'Z}$ take O as the origin, $\bar{K}$ as the unit point such that $\overline{OI} = \overline{OJ} = \overline{OK}$ (Fig.23).

The coordinate of a point on $\overleftrightarrow{X'X}$ is called its x - coordinate, the coordinate of a point on $\overleftrightarrow{Y'Y}$ is called its y - coordinate and the coordinate of a point on $\overleftrightarrow{Z'Z}$ is called its z - coordinate. $\overleftrightarrow{XOY}$ (XY plane), $\overleftrightarrow{YOZ}$ (YZ plane), $\overleftrightarrow{ZOX}$ (ZX plane) are called the *rectangular coordinate planes.* $\overleftrightarrow{X'X}$ (x -axis), $\overleftrightarrow{Y'Y}$ (y -axis), $\overleftrightarrow{Z'Z}$ (z -axis) are called the *rectangular coordinate axes.* Such an assigned system of axes is called the *frame of reference or coordinate frame* (denoted by OXYZ) so that the coordinates of a point change with the change in the frame of reference. (Fig.24). P is any point in sapce. L, M, N are respectively the projections on the coordinate axes. Let the X coordinate of L be x, Y coordinate of M be y and Z coordinate of N be z. The numbers x, y, z taken in this order are called the *rectangular coordinates of P.* We write $P = (x, y, z)$. Thus the point P is associated with an *ordered triad of real numbers.*

Let (x, y, z) be an ordered triad. Take the point L of co-ordinate x on $\overleftrightarrow{X'X}$, the point M of coordinate y on $\overleftrightarrow{Y'Y}$ and the point N of coordinate z on $\overleftrightarrow{Z'Z}$. Through the points L, M, N draw planes π_1, π_2, π_3 perpendicular to the coordinate axes $\overleftrightarrow{X'X}$, $\overleftrightarrow{Y'Y}$, $\overleftrightarrow{Z'Z}$ respectively.

The three mutually perpendicular planes π_1, π_2, π_3 intersect at a unique point P.

Since $P \in \pi_1$, the projection of P on $\overleftrightarrow{X'X}$ is L and hence the x - coordinate of P is equal to L. Similarly the y-coordinate of P is equal to the y-coordinate of M and the z -coordinate of P is equal to the z -coordinate of N. The coordinates of P are x, y, z in that order. Thus **for the ordered triad (x, y, z) we have a unique point P in space.**

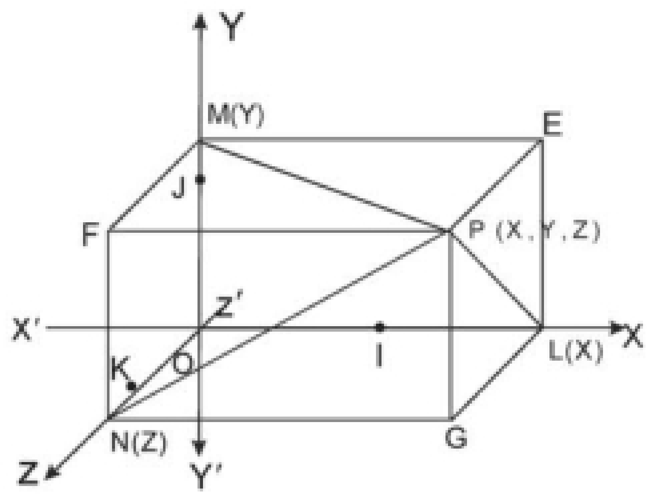


Fig. 24

Hence a one-to-one correspondence is established between the set of points in space and the set of ordered triads of real numbers. This space is called 3D space or R^3 space.

The three coordinate planes divide the space into eight compartments, each of which is called an octant.

Octant with founding lines	$\overrightarrow{OX}$ $\overrightarrow{OY}$ $\overrightarrow{OZ}$	$\overrightarrow{OX'}$ $\overrightarrow{OY}$ $\overrightarrow{OZ}$	$\overrightarrow{OX'}$ $\overrightarrow{OY'}$ $\overrightarrow{OZ}$	$\overrightarrow{OX}$ $\overrightarrow{OY'}$ $\overrightarrow{OZ}$	$\overrightarrow{OX}$ $\overrightarrow{OY}$ $\overrightarrow{OZ'}$	$\overrightarrow{OX'}$ $\overrightarrow{OY}$ $\overrightarrow{OZ'}$	$\overrightarrow{OX'}$ $\overrightarrow{OY'}$ $\overrightarrow{OZ'}$	$\overrightarrow{OX}$ $\overrightarrow{OY'}$ $\overrightarrow{OZ'}$
Sign of the coordinates of any point in the octant	+,+,+	−,+,+	−,−,+	+,−,+	+,+,−	−,+,−	−,−,−	+,−,−

Clearly the three planes through P are respectively parallel to the coordinate planes and the six planes form a rectangular parallelopiped. The three pairs of rectangular faces are

PFNG, EMOL ; PGLE, FNOM ; PEMF, GLON

We have

(*i*) *For any point on X-axis, $y = 0$ and $z = 0$.*
For any point on Y-axis, $x = 0$ and $z = 0$.
For any point on Z-axis, $x = 0$ and $y = 0$.

(*ii*) *For any point on $\overleftrightarrow{XOY}$ plane $z = 0$.*

For any point on $\overleftrightarrow{YOZ}$ plane $x = 0$.

For any point on $\overleftrightarrow{ZOX}$ plane $y = 0$.

(*iii*) *Origin $O = (0, 0, 0)$*

(*iv*) $|x|$ = OL = ME = NG = FP = *Distance of P from YZ plane,*
$|y|$ = OM = LE = NF = GP = *Distance of P from ZX plane,*
$|z|$ = ON = LG = MF = EP = *Distance of P from XY plane.*

(*v*) PL, PM, PN are respectively perpendicular to X axis, Y axis, Z axis.

(*vi*) *Distance of P from the X-axis* = PL = $\sqrt{(LE^2 + EP^2)} = \sqrt{(y^2 + z^2)}$,

$$(\because (\overrightarrow{EL}, \overrightarrow{EP}) = 90^\circ)$$

Distance of P from the Y-axis = PM = $\sqrt{[EP^2 + ME^2]} = \sqrt{(z^2 + x^2)}$,

$$(\because (\overrightarrow{EP}, \overrightarrow{EM}) = 90^\circ)$$

Distance of P from the Z-axis = PN = $\sqrt{[\mathrm{NF}^2 + \mathrm{FP}^2]} = \sqrt{(x^2+y^2)}$,

$(\because (\overrightarrow{\mathrm{FP}}, \overrightarrow{\mathrm{FN}}) = 90°)$

(*vii*) $\mathrm{OP} = \sqrt{[\mathrm{OL}^2 + \mathrm{PL}^2]} = \sqrt{(x^2+y^2+z^2)}$ $\quad(\because (\overrightarrow{\mathrm{LO}}, \overrightarrow{\mathrm{LP}}) = 90°)$

(*viii*) *The projections of* $P(x, y, z)$ *on the coordinate axes are L, M, N where* $L = (x,0,0), M = (0, y,0), N = (0,0, z)$.

The projections of $P(x, y, z)$ *on the coordinate planes* (YZ, ZX, XY) *are F, G, E, where* $F = (0, y, z), \ G = (x,0, z), \ E = (x, y,0)$.

(*ix*) X axis = $\{P(x, y, z) / y = 0, z = 0\}$, YZ plane = $\{P(x, y, z) / x = 0\}$,
Y axis = $\{P(x, y, z) / x = 0, z = 0\}$, XY plane = $\{P(x, y, z) / z = 0\}$,
Z axis = $\{P(x, y, z) / x = 0, y = 0\}$, ZX plane = $\{P(x, y, z) / y = 0\}$.

2. 2. INTERPRETATION OF EQUATIONS

Definition. *A locus is the set of points and only those points satisfying a given condition.*

Definition. *F is a function from* R^3 *into R.*

Then the locus $S = \{(x, y, z) | F(x, y, z) = 0\}$ *is called the surface represented by the equation* $F(x, y, z) = 0$.

$F(x, y, z) = 0$ is called an equation to the surface S.

Definition. *If F is a polynomial (not a zero polynomial) in x, y, z then the locus (surface) represented by* $F(x, y, z) = 0$ *is called an algebraic surface.*

Consider : (*i*) $F(x) = 0$ where $F(x)$ is a polynomial. So $F(x) = 0$ has a finite number of real roots. Let $x_1, x_2, ..., x_n$ be the real roots. The locus (surface) $\pi_1 = \{(x, y, z) / x = x_1\}$ is a plane parallel to YZ plane since every point in π_1 has its x coordinate x_1 (Fig. 25).

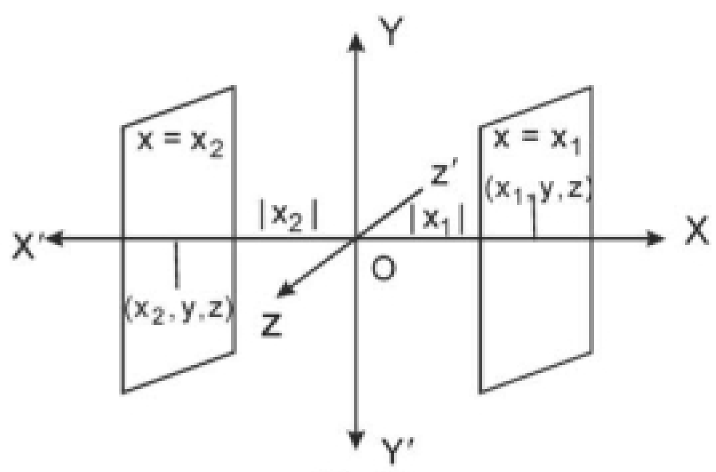


Fig. 25

Similarly the locus (surface) of each of the equations $x = x_2, ..., x = x_n$ is a plane parallel to YZ plane. Thus the locus of the equation $F(x) = 0$ is a system of planes parallel to YZ plane.

Similarly $F(y) = 0$ and $F(z) = 0$ can be interpreted.

(*ii*) $F(x, z) = 0$ where $F(x, z)$ is a first degree polynomial in x, z (say, $3x - 5z = 15$). This is a line in ZX plane (Fig.26). The locus (surface) $\pi = \{(x, y, z) | F(x, z) = 0\}$ is a plane parallel to Y axis since every point $P(x, y, z)$ in π has a point on the line with the same x and z coordinates.

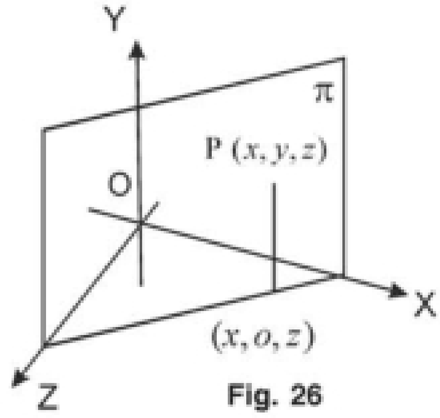


Fig. 26

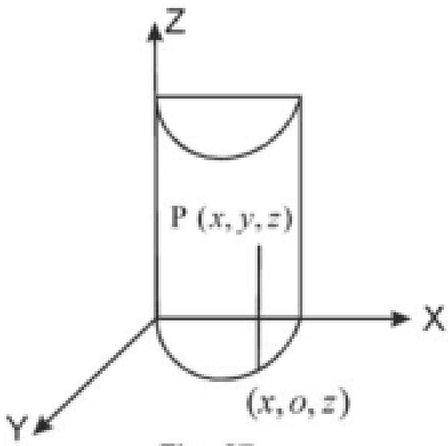


Fig. 27

Similarly $F(x, y) = 0$ where $F(x, y)$ is a first degree polynomial is the equation of a plane parallel to Z axis and $F(y, z) = 0$ where $F(y, z)$ is a first degree polynomial is the equation of a plane parallel to x-axis.

(*iii*) The curve $F(x, z) = 0$ is of second degree in ZX plane in the two dimensional Cartesian system (Fig.27). Let $Q(x, 0, z)$ be any point on $F(x, z) = 0$. Then the point $P(x, y, z)$ where $y \in R$ satisfies the equation $F(x, z) = 0$. $\overleftrightarrow{PQ}$ is a line parallel to Y-axis.

$\therefore$ For all Q on $F(x, z) = 0$, we have a set of lines parallel to Y-axis.

Thus the locus of the equation $F(x, z) = 0$ in 3D-space is a system of lines parallel to y-axis and the locus is a surface called a cylinder. Each of the lines is called a generator of the cylinder.

Similarly $F(x, z) = 0$ and $F(z, x) = 0$ can be interpreted.

A surface generated by a line so that it keeps parallel to a fixed line and intersects a fixed curve in a plane is called a cylindrical surface or a cylinder. In accordance with this definition, a plane in (*i*), or (*ii*) is a special case of a cylinder.

Thus the locus (surface) or an equation in two variables is a cylinder with generators parallel to the axis of the missing variable.

(*iv*) If $F(x, y, z) = 0$ and $\phi(x, y, z) = 0$ separately represent two surfaces then the points satisfying both the equations lie on the curve of intersection of the two surfaces.

EXERCISE 2 (*a*)

1. Find the feet of the perpendiculars from $(-1, 6, -2)$ to XY, YZ, ZX planes.
2. A rectangular parallelopiped with the coordinate planes as adjacent faces is taken. If one vertex is $(-1, 6, -2)$, write down the other vertices.
3. Write the locus of the point whose
   (*i*) x-co-ordinate = 2 (*ii*) y- co-ordinate = -3 (*iii*) z- co-ordinate = 4.
4. Write the locus of the point for which
   (*i*) $x = 0$ (*ii*) $y = 0$ (*iii*) $z = 0$ (*iv*) $x = 0, y = 0$ (*v*) $x = 0, z = 0$
   (*vi*) $x = 3, y = 2$.
5. Write the locus of the point for which
   (*i*) $x^2 + y^2 = 25, z = 0$ (*ii*) $y^2 = 4ax, z = 0$ (*iii*) $\frac{x^2}{9} + \frac{y^2}{4} = 1, z = 0$
   (*iv*) $y^2 = 4x$ (*v*) $x^2 + z^2 = 9$ (*vi*) $4x^2 + 9y^2 = 36$.
6. The end points of a diagonal of a rectangular parallelopiped with faces parallel to the coordinate planes are (2, 3, 5) and (5, 7, 10). Find the lengths of its edges.

ANSWERS

1. $(-1, 6, 0), (0, 6, -2) (-1, 0, -2)$.

2. $(0, 0, 0), (-1, 0, 0), (-1, 0, -2), (0, 0, -2) (0, -6, 0), (-1, 6, 0), (0, 6, -2)$

3. (*i*) Plane parallel to yz plane (*ii*) Plane parallel to zx plane
(*iii*) Plane parallel to xy plane

4. (*i*) yz plane (*ii*) zx plane (*iii*) xy plane
(*iv*) $z-$axis (*v*) $y-$axis (*vi*) Line parallel to $z-$axis

5. (*i*) Circle in xy plane (*ii*) Parabola in xy plane (*iii*) Ellipse in xy plane
(*iv*) Cylinder (*v*) Cylinder (*vi*) Cylinder. **6.** 3, 4, 5.

2.3. The study of 3D-geometry is made through vector methods wherever feasible. The students are already familiar with a detailed study of the geometrical concept of vectors in Vector Algebra. By expressing the equivalence of a vector to an ordered triad, we recapitulate the necessary ideas of Vector Algebra in the ensuing articles.

If felt convenient, methods using concepts of Vector Algebra may be used by students while proving theorems or solving problems.

2.4. VECTOR

Let OXYZ be a frame of reference and $\bar{i}, \bar{j}, \bar{k}$ be a unit orthogonal vector ordered triad (along $\overline{OX}, \overline{OY}, \overline{OZ}$) in the right handed system.

Let $P(x, y, z)$ be any point in space and be determined by its position vector $\overline{OP}$. Then we can have $\overline{OP} = x\bar{i} + y\bar{j} + z\bar{k}$ for unique scalars x, y, z.

In view of this, let the point (x, y, z) be associated with a unique vector $x\bar{i} + y\bar{j} + z\bar{k}$ and the vector $x\bar{i} + y\bar{j} + z\bar{k}$ be associated with a unique point (x, y, z).

Thus in R^3-space a one-to-one correspondence is established between the set of points and the set of position vectors. Hence we write $(x, y, z) = x\bar{i} + y\bar{j} + z\bar{k}$

The co-ordinates of any point P are the rectangular components of its position vector $\overline{OP}$.

Any point on x-axis $= (x, 0, 0) = x\bar{i}$, any point on y-axis $(0, y, 0) = y\bar{j}$ and any point on z-axis $(0, 0, z) = z\bar{k}$.

If $\bar{a}$ is taken to represent the position vector of the point $A(x, y, z)$, then

$A = \bar{a} = x_1\bar{i} + y_1\bar{j} + z_1\bar{k} = (x_1, y_1, z_1)$.

Note. $\bar{O} = 0\bar{i} + 0\bar{j} + 0\bar{k} = (0, 0, 0)$.

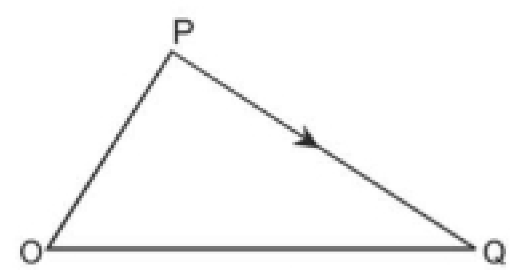


Fig. 28

2.5. $\bar{P} = (x_1, y_1, z_1)$, $\bar{Q} = (x_2, y_2, z_2)$ are any two points (Fig. 28). Then

(*i*) $\overline{OP} + \overline{OQ} = (x_1\bar{i} + y_1\bar{j} + z_1\bar{k}) + (x_2\bar{i} + y_2\bar{j} + z_2\bar{k})$

$= (x_1 + x_2)\bar{i} + (y_1 + y_2)\bar{j} + (z_1 + z_2)\bar{k} = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

(*ii*) $\overline{OP} - \overline{OQ} = (x_1 - x_2, y_1 - y_2, z_1 - z_2)$.

(*iii*) $\lambda\overline{OP} = (\lambda(x\bar{i} + y\bar{j} + z\bar{k}) = \lambda x\bar{i} + \lambda y\bar{j} + \lambda z\bar{k} = (\lambda x, \lambda y, \lambda z)$

where λ is a real number.

(*iv*) $\overline{PQ} = \overline{PO} + \overline{OQ} = \overline{OQ} - \overline{OP} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$

2.6. Definition. *If* $\bar{a} = \overline{AB}, \bar{b} = \overline{CD}$ *such that A, B, C, D are collinear or* $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, *then* $\bar{a}, \bar{b}$ *are said to be parallel or collinear.*

Sometimes we write $\bar{a} \parallel \overleftrightarrow{CD}$ or $\overleftrightarrow{AB} \parallel \bar{b}$. We take that null vector is parallel to every vector.

If $\bar{a}, \bar{b}$ are not parallel or not collinear, then $\bar{a}, \bar{b}$ are called non-parallel or non-collinear vectors.

2.7. $P = (x_1, y_1, z_1), Q = (x_2, y_2, z_2)$ are any two points. O, P, Q are collinear

$\Leftrightarrow \overline{OP} = \lambda \overline{OQ}, \lambda$ is a real number

$\Leftrightarrow (x_1, y_1, z_1) = \lambda (x_2, y_2, z_2) \Leftrightarrow x_1 : x_2 = y_1 : y_2 : z_2 = \lambda : 1$

2.8. LENGTH OR MAGNITUDE OF A VECTOR

Any point $\overline{P} = (x, y, z)$ and $\overline{OP} = (x, y, z)$. The length or magnitude or norm or modulus of the vector $\overline{OP} = \left|\overline{OP}\right| = \overline{OP} = \sqrt{(x^2, y^2, z^2)}$

2.9. Theorem. ***Distance between two points*** (x_1, y_1, z_1) ***and*** (x_2, y_2, z_2) ***is*** $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$.

Proof : Let $A = (x_1, y_1, z_1), B = (x_2, y_2, z_2)$.

Complete the parallelogram OABP so that $OP \parallel AB$ and $OP = AB$. $\therefore \overline{OP} = \overline{AB}$

$\therefore \quad \overline{OP} = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$

$\Rightarrow \left|\overline{OP}\right| = OP = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

$\therefore \quad \left|\overline{AB}\right| = AB = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

$\therefore$ Distance between points $A = (x_1, y_1, z_1), B = (x_2, y_2, z_2)$

$= AB = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

2.10. If $\bar{a}, \bar{b}$ **are two vectors, then there exists a unique vector** $\bar{c}$ **such that**

$\bar{c} + \bar{b} = \bar{a}$ *i.e.,* if $\bar{a} = (x_1, y_1, z_1),\ \bar{b} = (x_2, y_2, z_2), \bar{c} = (x, y, z)$ then

$(x, y, z) + (x_2, y_2, z_2) = (x_1, y_1, z_1) \Rightarrow (x_1, y_1, z_1) = (x_1 - x_2, y_1 - y_2, z_1 - z_2)$

2.11. UNIT VECTOR

If A, B and $A \neq B$, are points, then $\dfrac{\overline{AB}}{|\overline{AB}|}$ is the unit vector along $\overleftrightarrow{AB}$ in the direction from A to B.

If $A=(x_1, y_1, z_1), B=(x_2, y_2, z_2)$ then the unit vector along $\overleftrightarrow{AB}$ in the direction from A to B $= \dfrac{(x_2-x_1, y_2-y_1, z_2-z_1)}{\sqrt{[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2]}}$

2.12. If $\overline{OA}=\bar{a}$, $\overline{OB}=\bar{b}$ are two non-collinear vectors, $\overleftrightarrow{OA}, \overleftrightarrow{OB}$ determine a unique plane denoted by $\overleftrightarrow{AOB}$ and we say that it is the plane containing $\bar{a}, \bar{b}$.

2.13. COPLANAR, NON - COPLANAR VECTORS

Let $\bar{a}, \bar{b}$ be two non-collinear vectors and $\bar{c}$ be a vector. Let O be the origin and A, B, C be three points such that $\overline{OA}=\bar{a}$, $\overline{OB}=\bar{b}, \overline{OC}=\bar{c}$. Since $\overline{OA}, \overline{OB}$ are non-collinear, they determine the plane $\overleftrightarrow{AOB}$.

If $C \in \overleftrightarrow{AOB}$, *then* $\overline{OA}, \overline{OB}, \overline{OC}$ *are said to be coplanar and if* $C \notin \overleftrightarrow{AOB}$, *then* $\overline{OA}, \overline{OB}, \overline{OC}$ *are said to be non-coplanar.* $\bar{i}, \bar{j}, \bar{k}$ are non-coplanar.

2.14. ANGLE BETWEEN VECTORS

If $\bar{a}=\overline{OA}, \bar{b}=\overline{OB}$, then the angle between the vectors $\bar{a}, \bar{b}$ [written as $(\bar{a}, \bar{b})$] is $\left(\overrightarrow{OA}, \overrightarrow{OB}\right)$ such that $0^\circ \le \left(\overrightarrow{OA}, \overrightarrow{OB}\right) \le 180^\circ$.

We write $0^\circ \le (\bar{a}, \bar{b}) \le 180^\circ$. We have $(\bar{a}, -\bar{b}) = (-\bar{a}, \bar{b}) = \left(\overrightarrow{OA}, \overrightarrow{BO}\right) = \left(\overrightarrow{AO}, \overrightarrow{OB}\right)$

If $(\bar{a}, \bar{b}) = 90^\circ$, then we write $\bar{a} \perp \bar{b}$ or $\overleftrightarrow{OA} \perp \bar{b}$ or $\bar{a} \perp \overleftrightarrow{OB}$ or $\overline{OA} \perp \overline{OB}$. **We take that null vector is perpendicular to every vector.**

2.15. *If* $\bar{a}, \bar{b}, \bar{c}$ *are non-coplanar and* $\bar{r}$ *is any vector, then there exist unique real numbers* x, y, z *such that* $\bar{r} = x\bar{a}+y\bar{b}+z\bar{c}$. $\bar{r}$ *is said to be a linear combination of* $\bar{a}, \bar{b}, \bar{c}$.

Since $\bar{i}, \bar{j}, \bar{k}$ are non-coplanar, any vector $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$, for unique scalars x, y, z.

2.16. $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar.

(i) $x_1\bar{a}+y_1\bar{b}+z_1\bar{c} = x_2\bar{a}+y_2\bar{b}+z_2\bar{c} \Rightarrow x_1 = x_2, y_1 = y_2, z_1 = z_2$

(ii) $x_1\bar{a}+y_1\bar{b}+z_1\bar{c} = 0 \qquad \Rightarrow x_1 = 0, y_1 = 0, z_1 = 0$.

2.17. Definition. *A, B are two points. If* $\lambda_1, \lambda_2, (\lambda_1+\lambda_2 \ne 0)$ *are two real numbers such that* $P \in AB$ *and* $\lambda_2 \overline{AP} = \lambda_1 \overline{PB}$, *we say that P divides AB in the ratio* $\lambda_1 : \lambda_2$.

$\lambda_1, \lambda_2 > 0$, $A-P-B$ *i.e.,* P is said to divide the line segment AB *internally* in the $\lambda_1 : \lambda_2$ *and* $\lambda_1, \lambda_2 < 0$, $P-A-B$ or $A-B-P$ *i.e.,* P is said to divide the line segment AB *externally* in the ratio $\lambda_1 : \lambda_2$.

2.18. If $A = \bar{a}$, $B = \bar{b}$ and $(P ; A, B) = \lambda_1 : \lambda_2$ then $\overline{OP} = \dfrac{\lambda_1 \bar{a} + \lambda_2 \bar{b}}{\lambda_1 + \lambda_2} (\lambda_1 + \lambda_2 \neq 0)$.

2.19. $\bar{A}, \bar{B}, \bar{C}$ are three points whose position vectors are $\bar{a}, \bar{b}, \bar{c}$ respectively.

$A(\bar{a}), B(\bar{b}), C(\bar{c})$ are collinear $\Leftrightarrow l\bar{a} + m\bar{b} + n\bar{c} = 0, l + m + n = 0, (l, m, n) \neq (0, 0, 0)$

2.20. DOT PRODUCT OR DIRECT PRODUCT OR SCALAR PRODUCT OR INNER PRODUCT.

Definition. *If $\bar{a}, \bar{b}$ are two non-zero vectors, then $\bar{a} . \bar{b} = |\bar{a}| \; |\bar{b}| \cos (\bar{a}, \bar{b})$ and if one of $\bar{a}, \bar{b}$ is zero then $\bar{a} . \bar{b} = 0$.*

Now $\bar{i} . \bar{i} = 1, \bar{j} . \bar{j} = 1, \bar{k} . \bar{k} = 1, \bar{i} . \bar{j} = 0, \bar{i} . \bar{k} = 0, \bar{j} . \bar{k} = 0$.

If $\bar{a} = (x_1, y_1, z_1), \bar{b} = (x_2, y_2, z_2)$, then $\bar{a} . \bar{b} = (x_1, y_1, z_1) . (x_2, y_2, z_2)$

$= (x_1\bar{i}, y_1\bar{j}, z_1\bar{k}) . (x_2\bar{i}, y_2\bar{j}, z_2\bar{k}) = x_1x_2 + y_1y_2 + z_1z_2$

Also we have $(x_1, y_1, z_1) . (x_2, y_2, z_2) = (x_2, y_2, z_2) . (x_1, y_1, z_1)$

Also if $|\bar{a}| = \bar{a}$, then $\bar{a}^2 = \bar{a}^2 = \bar{a} . \bar{a} = x_1^2 + y_1^2 + z_1^2$

If $\bar{a}$ is a unit vector, then $\bar{a}^2 = |\bar{a}|^2 = 1$.

2.21. If $P = \bar{a} = (a_1, b_1, c_1)$, $Q = \bar{b} = (a_2, b_2, c_2)$, $P \neq Q \neq O$ and $(\overrightarrow{OP}, \overrightarrow{OQ}) = (\bar{a}, \bar{b}) = \theta$, then

$$\cos \theta = \frac{\bar{a} . \bar{b}}{|\bar{a}| \; |\bar{b}|} = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)} . \sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

$\bar{a} . \bar{b}$ are parallel vectors $\Leftrightarrow \bar{a} = \lambda \bar{b}$

$\Leftrightarrow (a_1, b_1, c_1) = \lambda(a_2, b_2, c_2) \Leftrightarrow a_1 = \lambda a_2, b_1 = \lambda b_2, c_1 = \lambda c_2$

$\Leftrightarrow a_1 : b_1 : c_1 = a_2 : b_2 : c_2$ or $a_1 : a_2 = b_1 : b_2 = c_1 : c_2$

$\bar{a} . \bar{b}$ are perpendicular vectors $\Leftrightarrow \bar{a} . \bar{b} = 0 \Leftrightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$

2.22. Projection of $\bar{b}$ on $\bar{a} (\neq 0)$ is $\bar{b} . \dfrac{\bar{a}}{|\bar{a}|} = \dfrac{\bar{b} . \bar{a}}{|\bar{a}|} = \bar{b} . \bar{e}$ where $\bar{e}$ is the unit vector in the direction of $\bar{a}$.

2.23. $\bar{a} . (\bar{b} + \bar{c}) = \bar{a} . \bar{b} + \bar{a} . \bar{c}, \bar{a} . (\bar{b} - \bar{c}) = \bar{a} . \bar{b} - \bar{a} . \bar{c},$

$(\bar{a} - \bar{b})^2 = (\bar{a} - \bar{b}) . (\bar{a} - \bar{b}) = \bar{a}^2 - 2\bar{a} . \bar{b} + \bar{b}^2$

2.24. CROSS PRODUCT OR SKEW PRODUCT OR VECTOR PRODUCT

If $\bar{a}, \bar{b}$ are two non-zero or non-parallel vectors, then

$(\bar{a} \times \bar{b}) = |\bar{a}| \; |\bar{b}| \sin (\bar{a}, \bar{b}) \, \bar{n}$

where $\bar{n}$ is a unit vector perpendicular to the plane containing $\bar{a}, \bar{b}$ so that $\bar{a}, \bar{b}, \bar{n}$ form a right handed system and if at least one of $\bar{a}, \bar{b}$ is a null vector or $\bar{a} \| \bar{b}$, then $(\bar{a} \times \bar{b}) = 0$.

We find that $\bar{a}\times\bar{b}=-\bar{b}\times\bar{a}$ and $|\bar{a}\times\bar{b}|=|\bar{a}|\ |\bar{b}|\ |\sin(\bar{a},\bar{b})|$

Now $\bar{i}\times\bar{i}=0, \bar{j}\times\bar{j}=0, \bar{k}\times\bar{k}=0$, $\quad \bar{i}\times\bar{j}=\bar{k}, \bar{i}\times\bar{k}=-\bar{j}, \bar{j}\times\bar{k}=\bar{i}, \bar{k}\times\bar{j}=-\bar{i}$

$\bar{j}\times\bar{i}=\bar{k},\ \bar{k}\times\bar{i}\times\bar{j}$. If $\bar{a}=(x_1, y_1, z_1),\ \bar{b}=(x_2, y_2, z_2)$, then

$$\bar{a}\times\bar{b}=(x_1, y_1, z_1)\times(x_2, y_2, z_2)=(x_1\bar{i}+y_1\bar{j}+z_1\bar{k})\times(x_2\bar{i}, y_2\bar{j}, z_2\bar{k})$$

$$= x_1y_2\bar{k}-x_1z_2\bar{j}-x_2y_1\bar{k}+y_1z_2\bar{i}+z_1x_2\bar{j}-z_1y_2\bar{i}$$

$$=(y_1z_2-y_2z_1)\bar{i}-(x_1z_2-x_2z_1)\bar{j}+(x_1y_2-x_2y_1)\bar{k}$$

$$=(y_1z_2-y_2z_1, x_2z_1-x_1z_2, x_1y_2-x_2y_1)=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\end{vmatrix}$$

i.e. $(x_1, y_1, z_1)\times(x_2, y_2, z_2)=(y_1z_2-y_2z_1, x_2z_1-x_1z_2, x_1y_2-x_2y_1)$

$(x_2, y_2, z_2)\times(x_1, y_1, z_1)=-(y_1z_2-y_2z_1, x_2z_1-x_1z_2, x_1y_2-x_2y_1)$

and $|(x_1, y_1, z_1)\times(x_2, y_2, z_2)|=\sqrt{\left[\sum(y_1z_2-y_2z_1)^2\right]}$

2.25. If $P=\bar{a}=(a_1, b_1, c_1), Q=\bar{b}=(a_2, b_2, c_2)=(P\neq Q\neq O)$

and $(\overrightarrow{OP}, \overrightarrow{OQ})=(\bar{a}, \bar{b})=\theta$, then $\sin\theta=\dfrac{|\bar{a}\times\bar{b}|}{|\bar{a}||\bar{b}|}=\dfrac{|(b_1c_2-b_2c_1, c_1a_2-c_2a_1, a_1b_2-a_2b_1)|}{\sqrt{(a_1^2+b_1^2+c_1^2)}\cdot\sqrt{(a_2^2+b_2^2+c_2^2)}}$

2.26. If $\overline{ABC}$ is a triangle, then the area of $\Delta\,\overline{ABC}=\frac{1}{2}\left|\overline{AB}\times\overline{AC}\right|$ square units. Also $\overline{AB}\times\overline{AC}$ is a vector perpendicular to the plane of $\Delta\,\overline{ABC}$.

Area of $\Delta\,\overline{ABC}=0\Leftrightarrow$ A,B,C are collinear.

A, B, C, D are coplanar points. If ABCD is a parallelogram then the area of the parallelogram $=\left|\overline{AB}\times\overline{AD}\right|$ or $\frac{1}{2}\left|\overline{AC}\times\overline{BD}\right|$ square units.

If ABCD is a quadrilateral, then the area of the quadrilateral $=\frac{1}{2}\left|\overline{AC}\times\overline{BD}\right|$ sq. units.

2.27. $\bar{a}=(a_1, a_2, a_3), \bar{b}=(b_1, b_2, b_3), \bar{c}=(c_1, c_2, c_3)$

$\bar{a}\times\bar{b}=(a_2b_3-a_3b_2, a_3b_1-a_1b_3, a_1b_2-a_2b_1)$ $\quad\therefore [\bar{a}\,\bar{b}\,\bar{c}]=(\bar{a}\times\bar{b})\cdot\bar{c}$

$$=(a_2b_3-a_3b_2, a_3b_1-a_1b_3, a_1b_2-a_2b_1)\cdot(c_1, c_2, c_3)$$

$$=(a_2b_3c_1-a_3b_1c_1+a_3b_1c_2-a_1b_3c_2+a_1b_2c_3-a_2b_1c_3)=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

i.e. $[\bar{a}\,\bar{b}\,\bar{c}]=[(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3)]=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$

2.28. $\bar{a},\bar{b},\bar{c}$ are three non-coplanar vectors. If V is the volume of the parallelopiped with adjacent sides $\bar{a},\bar{b},\bar{c}$ then $\bar{V} = |[\bar{a},\bar{b},\bar{c}]|$ cubic units.

If V is the volume of the tetrahedron with adjacent sides $\bar{a},\bar{b},\bar{c}$, then $\bar{V} = \frac{1}{6}|[\bar{a},\bar{b},\bar{c}]|$ cubic units. Also if any two of $\bar{a},\bar{b},\bar{c}$ are parallel, $[\bar{a},\bar{b},\bar{c}] = 0$

2.29. One of $\bar{a},\bar{b},\bar{c}$ is $0 \Rightarrow [\bar{a},\bar{b},\bar{c}] = 0$. Also if any two of $\bar{a},\bar{b},\bar{c}$ are parallel, $[\bar{a},\bar{b},\bar{c}] = 0$.

2.30. $\bar{a},\bar{b},\bar{c}$ are three non-zero, non-parallel vectors. $\bar{a},\bar{b},\bar{c}$ are coplanar $\Leftrightarrow [\bar{a},\bar{b},\bar{c}] = 0$.

If $\bar{a},\bar{b},\bar{c}$ are three non-coplanar vectors, then $[\bar{a},\bar{b},\bar{c}]^2 = \begin{vmatrix} \bar{a}.\bar{a} & \bar{a}.\bar{b} & \bar{a}.\bar{c} \\ \bar{b}.\bar{a} & \bar{b}.\bar{b} & \bar{b}.\bar{c} \\ \bar{c}.\bar{a} & \bar{c}.\bar{b} & \bar{c}.\bar{c} \end{vmatrix}$

2.31. A, B are two distinct points. Distance of P from $\overleftrightarrow{AB} = \frac{|\overline{AP} \times \overline{AB}|}{|\overline{AB}|}$

2.32. A, B, C are distinct points.

$\overline{AB}, \overline{AC}$ are parallel $\Leftrightarrow$ A, B, C are collinear.

A, B, C, D are distinct points.

$\overline{AB}, \overline{AC}, \overline{AD}$ are coplanar $\Leftrightarrow$ A, B, C, D are coplanar.

2.33. NOTATION

'$\bar{m} \perp \bar{L}$' means : There exist two points A, B such that $\overline{AB} = \bar{m}$ and $AB \perp L$.

'$\bar{m} \parallel \bar{L}$' means : There eixst two points A, B such that $\overline{AB} = \bar{m}$ and $AB \parallel L$.

'$\bar{m} \subset \pi$' means : There exist two points A, B such that $\overline{AB} = \bar{m}$ and $\overleftrightarrow{AB} \subset \pi$.

'$\bar{m} \in \pi$' means : $\bar{m}$ is a point in the plane π.

SOLVED PROBLEMS

Ex. 1. *Show that the points* $(3, -2, 4), (1, 1, 1), (-1, 4, -2)$ *are collinear. Hence find* $(C; A, B)$.

Sol. Let $A = (3, -2, 4), B = (1, 1, 1)$ and $C = (-1, 4, -2)$

$\therefore AB = |\overline{AB}| = |(1-3, 1+2, 1-4)| = |(-2, 3, -3)| = \sqrt{(4+9+9)} = \sqrt{(22)}$,

$BC = |(-2, 3, -3)| = \sqrt{(4+9+9)} = \sqrt{(22)}$

$AC = |(-4, 6, -6)| = \sqrt{(16+36+36)} = 2\sqrt{(22)}$. $\therefore AB + BC = AC$.

$\therefore$ A, B, C are collinear and $(C ; A, B) = (-1-3) : (1+1) = -2 : 1$ (Art 1.40)

OR : $\overline{AC} = (-4, 6, -6), \overline{CB} = (2, -3, 3)$

Since $\overline{AC} = -2(2, -3, 3) = -2\,\overline{CB}$, A, B, C are collinear. Let $(C; A, B) = \lambda_1 : \lambda_2$

$\therefore \lambda_2 \overline{AC} = \lambda_1 \overline{CB}$

$\Rightarrow \lambda_2(-4, 6, -6) = \lambda_1(2, -3, 3) \Rightarrow 2\lambda_1 = -4\lambda_2 \Rightarrow \lambda_1 : \lambda_2 = -2 : 1 \Rightarrow (C; A, B) = -2 : 1$

Ex. 2. *Show that the points* $(-1,-2,-1)$, $(2,3,2)$, $(4,7,6)$ *and* $(1,2,3)$ *form a parallelogram.*

Sol. Let $A=(-1,-2,-1), B=(2,3,2), C=(4,7,6)$ and $D=(1,2,3)$

$\therefore AB=\sqrt{[(2+1)^2+(3+2)^2+(2+1)^2]}=\sqrt{(43)}$, $BC=6$, $CD=\sqrt{(43)}$, $AD=6$

Also $AC=\sqrt{(155)}$ and $BD=\sqrt{3}$

$\therefore AB=CD, BC=AD$ and $AC\neq BD$. $\therefore ABCD$ is a parallelogram.

Ex. 3. *Find the centre and radius of the sphere determined by the points* $(1,-5,-3),(0,-6,-1),(-2,-2,3),(1,-2,0)$.

Sol. Sphere is the set of points $P=(x,y,z)$ where $A=(a,b,c)$ and $PA=r$ (a non-negative number) A is called the centre and r is called the radius.

Let $P_1=(1,-5,3), P_2=(0,-6,-1), P_3=(-2,-2,3)$ and $P_4=(1,-2,0)$.

$\therefore P_1A=P_2A=P_3A=P_4A\Rightarrow P_1A^2=P_2A^2=P_3A^2=P_4A^2$

$$\Rightarrow \underbrace{(a-1)^2+(b+5)^2+(c-3)^2}_{(1)}=\underbrace{(a-0)^2+(b+6)^2+(c+1)^2}_{(2)}$$

$$=\underbrace{(a+2)^2+(b+2)^2+(c-3)^2}_{(3)}=\underbrace{(a-1)^2+(b+2)^2+(c-0)^2}_{(4)}$$

From (1) and (2) : $-2a-2b-8c=2\Rightarrow a+b+4c=-1$ .... (5)

From (1) and (3) : $-6a+6b=-18\Rightarrow a-b=3$ .... (6)

From (1) and (4) : $6b-6c=-30\Rightarrow b-c=-5$ .... (7)

Solving (5), (6), (7), $a=-1, b=-4, c=1$. $\therefore$ Centre $A=(-1,-4,1)$.

Radius $P_1A=\sqrt{[(a-1)^2+(b+5)^5+(c-3)^2]}=\sqrt{(4+1+4)}=3$

EXERCISE 2 (*b*)

1. If $A=(-1,3,5)$ and $B=(4,-12,-20)$ find whether O, A, B are collinear.
   If O, A, B are collinear, then find (*i*) (A ; O, B) (*ii*) the ratio in which B divides the line segment OA.
2. If $A=(2,-3,2\sqrt{3})$, find unit points on (*i*) $\overrightarrow{OA}$ (*ii*) $\overrightarrow{AO}$
3. Find the distance between the points $(-1,0,6)$ and $(5,3,0)$.
4. Show that the three points (*i*) $(1,1,0),(1,0,1),(0,1,1)$ from an equilateral triangle.
   (*ii*) $(1,1,1),(-2,4,1),(-1,5,5)$ form a right angled isosceles triangle.
   (*iii*) $(2,3,5),(-1,5,-1),(4,-3,2)$ form a right angled isosceles triangle.
5. Which triangle is formed with the vertices $(a,b,c),(b,c,a),(c,a,b)$.
6. Show that the following points are collinear.
   (*i*) $(-1,0,7),(3,2,1),(5,3,-2)$ ***(A. U. A12)*** (*iii*) $(1,2,3),(7,0,1),(-2,3,4)$
7. Show that the following four points
   (*i*) $(-1,-3,4),(5,-1,1)\ (7,-4,7),(1,-6,10)$ form a rhombus.
   (*ii*) $(-2,4,1),(-1,5,5),(2,2,5)\ (1,1,1)$ form a square.

8. Find the centre and radius of the sphere determined by the points.
(*i*) $(a, 0, 0), (0, b, 0), (0, 0, c), (0,0,0)$ (*ii*) $(-1, 1, 3), (2, 1, 2), (0, 5, 6), (3, 2, 2)$

9. Find the point equidistant from the points. $(2, 0, 0), (0, 4, 0), (0, 0, 6), (0, 0, 0)$

10. Find the equation to the locus of $P(x, y, z)$ such that $A = (a, 0, 0), B = (-a, 0, 0)$ and $PA + PB = 2K\ (\neq 0)$

11. Find the equation to the locus of P (x, y, z) such that the sum of its distances from $(4, 0, 0)$ and $(-4, 0, 0)$ is 10.

ANSWERS

1. Yes (*i*) $-1:5$ (*ii*) $-4:5$ **2.** (*i*) $\left(\frac{2}{5}, \frac{-3}{5}, \frac{2\sqrt{3}}{5}\right)$ (*ii*) $\left(\frac{-2}{5}, \frac{3}{5}, \frac{-2\sqrt{3}}{5}\right)$

3. 9. **5.** Equilateral **8.** (*i*) $(a/2, b/2, c/2)$, $\sqrt{a^2+b^2+c^2/2}$ (*ii*) $(1, 3, 4)$; 3.

9. $(1, 2, 3)$ **10.** $\left(1-\frac{a^2}{k^2}\right)x^2+y^2+z^2=k^2-a^2$ **11.** $9x^2+25y^2+25z^2=225$ .c

2.34. Theorem. *If* $\bar{A} = (x_1, y_1, z_1), \bar{B} = (x_2, y_2, z_2)$ ***and P is a point dividing the line segment*** $\overline{AB}$ ***in the ratio*** $\lambda_1 : \lambda_2\ (\lambda_1+\lambda_2 \neq 0)$***, then***

$$P = \left(\frac{\lambda_2\bar{x}_1+\lambda_1\bar{x}_2}{\lambda_1+\lambda_2}, \frac{\lambda_2\bar{y}_1+\lambda_1\bar{y}_1}{\lambda_1+\lambda_2}, \frac{\lambda_2\bar{z}_1+\lambda_1\bar{z}_1}{\lambda_1+\lambda_2}\right)$$

Proof : Let $P = (x, y, z)$. (Fig. 29)
A,P,B are collinear and

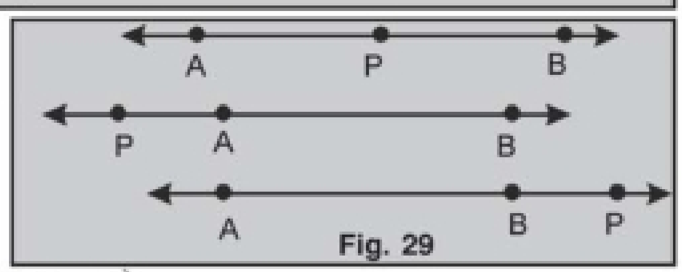


Fig. 29

$(P; A, B) = \lambda_1 : \lambda_2\ (\lambda_1+\lambda_2 \neq 0)$

$\Leftrightarrow \lambda_2\overline{AP} : \lambda_1\overline{PB}$

$\Leftrightarrow \lambda_2(x-x_1, y-y_1, z-z_1) = \lambda_1(x_2-x, y_2-y, z_2-z)$

$\Leftrightarrow \lambda_2(x-x_1) = \lambda_1(x_2-x)$, etc. $\Leftrightarrow (\lambda_1+\lambda_2)\,x = \lambda_2 x_1+\lambda_1 x_2$, etc.

$$\Leftrightarrow x = \frac{\lambda_2 x_1+\lambda_1 x_2}{\lambda_1+\lambda_2}, y = \frac{\lambda_2 y_1+\lambda_1 y_2}{\lambda_1+\lambda_2}, z = \frac{\lambda_2 z_1+\lambda_1 z_2}{\lambda_1+{}^{l_2}}$$

$$\Leftrightarrow P = \left(\frac{\lambda_2 x_1+\lambda_1 x_2}{\lambda_1+\lambda_2}, \frac{\lambda_2 y_1+\lambda_1 y_2}{\lambda_1+\lambda_2}, \frac{\lambda_2 z_1+\lambda_1 z_2}{\lambda_1+{}^{l_2}}\right), (\lambda_1+\lambda_2 \neq 0)$$

Note. 1. If $\lambda_1 = \lambda_2$, P will be the mid - point of AB and

$$P = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

2. $A = (x_1, y_1, z_1), B = (x_2, y_2, z_2)$, $P = \left(\frac{\lambda_2 x_1+\lambda_1 x_2}{\lambda_1+\lambda_2}, \frac{\lambda_2 y_1+\lambda_1 y_2}{\lambda_1+\lambda_2}, \frac{\lambda_2 z_1+\lambda_1 z_2}{\lambda_1+\lambda_2}\right)$

are three points and $\lambda_1, \lambda_2 \in R$ such that $\lambda_1+\lambda_2 \neq 0$.

$\Rightarrow$ A, P, B are collinear.

2.35. THEOREM : ***If (x_r, y_r, z_r), $r = 1, 2, 3$ are the vertices of a triangle, then its medians are concurrent and the point of concurrence trisects any median of the triangle.***

Proof. Let ABC be the triangle (Fig. 30) where $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$, $C = (x_3, y_3, z_3)$

Let D,E,F be the mid - points of the sides.

AD, BE, CF are the medians of Δ ABC.

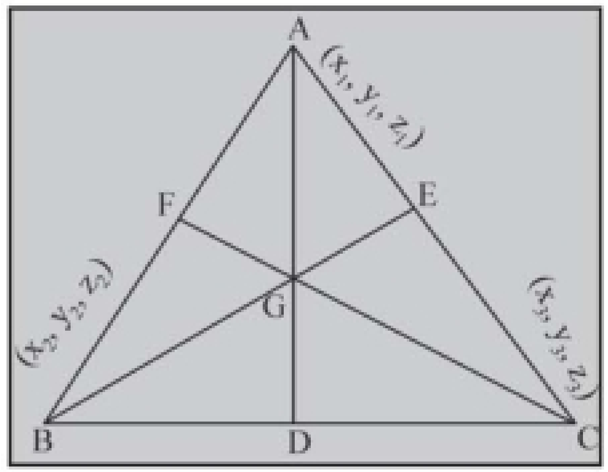


Fig. 30

Now $D = \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right)$

Let (G ; A, D) = 2 : 1.

$$\therefore \quad G = \frac{2\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right) + 1(x_1, y_1)}{2+1}$$

$$= \frac{(x_1+x_2+x_3, y_1+y_2+y_3, z_1+z_2+z_3)}{3} = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

Similarly, the point dividing BE in the ratio 2 : 1 is

$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$ and the point dividing CF in the ratio 2 : 1 is

$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$

$\therefore$ G is the point common to AD, BE, CF.

$\therefore$ Medians in a triangle are concurrent and the point of concurrence trisects each median. This point G is called the centroid.

2. 36. TETRAHEDRON

Let A, B, C, D be four points such that $\overleftrightarrow{ABC}, \overleftrightarrow{ABD}, \overleftrightarrow{ADC}, \overleftrightarrow{BCD}$ are four intersecting planes. Then the points A, B, C, D are said to form a tetrahedron (fig. 31). A, B, C, D are called vertices and the line segments AB, AD, AC, BC, BD, CD are called the edges, AB, CD; AD, BC; AC, BD are called three pairs of opposite edges.

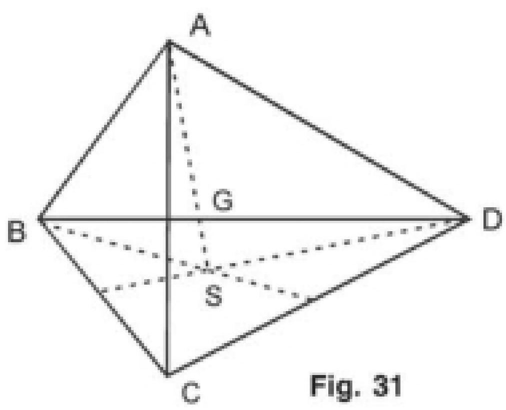


Fig. 31

Observe that for the tetrahedron :

(*i*) Each of the points A, B, C, D is non-coplanar with the remaining three.

(*ii*) Opposite edges form non-coplanar lines *i.e.,* $\overleftrightarrow{AB}, \overleftrightarrow{CD}$ are non-coplanar; $\overleftrightarrow{AD}, \overleftrightarrow{BC}$ are non- coplanar; $\overleftrightarrow{AC}, \overleftrightarrow{BD}$ are non-coplanar.

(*iii*) Four bounding planes $\overleftrightarrow{ABD}, \overleftrightarrow{ADC}, \overleftrightarrow{ABC}, \overleftrightarrow{BCD}$ are triangular faces.

The point of concurrence of the line segments joining the vertices to their respective centroids of opposite triangular faces is called the centroid of the tetrahedron.

If all the edges are of equal length, then it is called a regular tetrahedron.

Theorem. *If* $A=(x_1, y_1, z_1)$, $B=(x_2, y_2, z_2)$, $C=(x_3, y_3, z_3)$, $D=(x_4, y_4, z_4)$ ***are the vertices of the tetrahedron ABCD, then the line segments joining the vertices to their respective centroids of opposite faces are concurrent and the point of concurrence divides each line segment in the ratio 3 : 1.***

Proof : (Fig.31). Let S, P, Q, R be the centroids of $\Delta\,BCD$, $\Delta\,ACD$, $\Delta\,ABD$, $\Delta\,ABC$ respectively.

$$\therefore\quad S=\left(\frac{x_2+x_3+x_4}{3}, \frac{y_2+y_3+y_4}{3}, \frac{z_2+z_3+z_4}{3}\right)$$

Let G divide the line segment AS in the ratio 3 : 1. Since $A=(x_1, y_1, z_1)$,

$$G=\left(\frac{3\left(\frac{x_2+x_3+x_4}{3}\right)+1.x_1}{3+1}, \frac{3\left(\frac{y_2+y_3+y_4}{3}\right)+1.y_1}{3+1}, \frac{3\left(\frac{z_2+z_3+z_4}{3}\right)+1.z_1}{3+1}\right)$$

i.e., $$G=\left(\frac{(x_1+x_2+x_3+x_4)}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$$

Similarly, the points which divide the other line segments BP, CQ, DR in the ratio 3 : 1 can be shown to be G.

$\therefore$ The line segments AS, BP, CQ, DR are concurrent at G and each is divided in the ratio 3 : 1 at G. G is called the centroid of the tetrahedron.

Ex. 1. *Show that the following points are collinear*

$A=(5, 4, 2), B=(8, -2, -7), C=(6, 2, -1)$ *If A, B, C are collinear, find* $(\,C\,;\,A,\,B\,)$

(A. U. M 2014)

Sol. Let P divide AB in the ratio $1:\lambda\ (1+\lambda\neq 0)$.

$$\therefore\quad P=\left(\frac{5\lambda+8}{1+\lambda}, \frac{-2+4\lambda}{1+\lambda}, \frac{-7+2\lambda}{1+\lambda}\right).$$ If possible, let P = C.

$$\text{Then}\quad \left.\begin{aligned} \frac{5\lambda+8}{1+\lambda}&=6,\\ \frac{-2+4\lambda}{1+\lambda}&=2,\\ \frac{-7+2\lambda}{1+\lambda}&=-1 \end{aligned}\right\}\quad \therefore\ \lambda=2.$$

Since $\lambda=2$ satisfies all the three equations, A, B, C are collinear. $\therefore$ (C ; A, B) = 1 : 2.

EXERCISE 2 (*c*)

1. Find the middle point of the line segment with end points $(1, 2, -3)$ and $(-1, 6, 7)$.
2. Find the points dividing the line segment joining $(1, -1, 2)$ and $(2, 3, 7)$ in the ratio (*i*) $2:3$ (*ii*) $-2:3$.
3. Prove that the points (2, –1, 3) (3, –5, 1), (–1, 11, 9) are collinear. ***(O.U.03, M99)***
4. Prove that the ponts A(3, 2, 4), B(5, 4, –6) and C(9, 8, –10) are collinear. Find the ratio in which B divides $\overline{AC}$ ***(N.U.M96)***
5. Find the point of intersection of the line through $(-2, 3, 4), (1, 2, 3)$ with the XZ plane.

6. (*i*) $A = (1, 2, 3)$ and $B = (2, 10, 1)$. If the points A, B, Q are collinear and if the x – coordinate of Q be -1, find the y-coordinate and the z - coordinate of Q.

(*ii*) If $A(-1, 0, 7), B(3, 2, t), C(5, 3, -2)$ are collinear, then show that $t = 1$.

7. (*i*) Find the centroid of the triangle with vertices $(7, -4, 7), (1, -6, 10), (5, -1, 1)$.

(*ii*) A, B, C are the vertices of a triangle, $A = (1, 1, 1), B = (-2, 4, 1)$. If the centroid of the Δ ABC is the origin, then find C.

8. Show that $(5, -1, -1), (-1, 5, -1), (-1, -1, 5), (-3, -3, -3)$ are the vertices of a regular tetrahedron.

9. Show that $(2, 0, 3)$ is the point of intersection of the lines $\overleftrightarrow{AB}, \overleftrightarrow{CD}$ where $A = (3, -4, 11)$, $B = (1, 4, -5), C = (17, -18, -3), D = (7, -6, 1)$.

10. Three vartices of a parallelogram ABCD are A(4, 7, 19), B(1, 4, 7), C(2, 1, –3). Find the fourth vertex D. ***(O.U.2000)***

ANSWERS

1. (0, 4, 2) **2.** (*i*) $(7/5, 3/5, 4)$ (*ii*) $(-1, -9, -8)$ **3.** 1 : 2 **4.** 1 : 2

5. (7, 0, 1) **6.** y coord. $= -14$, z coord. = 7. **7.** (*i*) $(13/3, -11/3, 6)$, (*ii*) $(1, -5, -2)$

10. (5, 4, 3)

2.37. DIRECTION COSINES OF A LINE

$\overrightarrow{PQ}$ is a ray making angles α, β, γ respectively with $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$. Then the ordered triad $(\cos\alpha, \cos\beta, \cos\gamma)$ is called direction cosine triad of $\overrightarrow{PQ}$ *i.e.,* $\cos\alpha, \cos\beta, \cos\gamma$ in that order are called the **direction cosines** (d. cs.) of $\overrightarrow{PQ}$

The direction cosine triad is generally denoted by (l, m, n). Thus $\cos\alpha = l, \cos\beta = m, \cos\gamma = n$ and the d. cs. of $\overleftrightarrow{PQ}$ are l, m, n.

Since the ray $\overrightarrow{QP}$ makes angles $180^\circ - \alpha, 180^\circ - \beta, 180^\circ - \gamma$ respectively with $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ direction cosine triad of $\overrightarrow{QP}$ is $[(\cos(180^\circ - \alpha), \cos(180^\circ - \beta), \cos(180^\circ - \gamma)]$

i.e. $(-\cos\alpha, -\cos\beta, -\cos\gamma)$ *i.e.* $(-l, -m, -n)$

If L is a line parallel to $\overleftrightarrow{PQ}$, then the two ordered triads (l, m, n) and $(-l, -m, -n)$ are defined as direction cosine triads of L *i.e.,* (l, m, n) in that order and $-l, -m, -n$ in that order are defined as the d. cs. of L. Sometimes d. cs. l, m, n are written as (l, m, n).

Theorem. ***If l, m, n are d.cs of a line, then $l^2 + m^2 + n^2 = 1$.***

Proof : (Fig. 32). Let L be the line with d.cs. l, m, n

$\therefore$ $(\cos\alpha, \cos\beta, \cos\gamma) = (l, m, n)$ or $(-l, -m, -n)$.

If $P(x, y, z) (\neq G)$ is a point such that $\overleftrightarrow{OP} \parallel L$ and OP = 1, then :

From Trigonometry

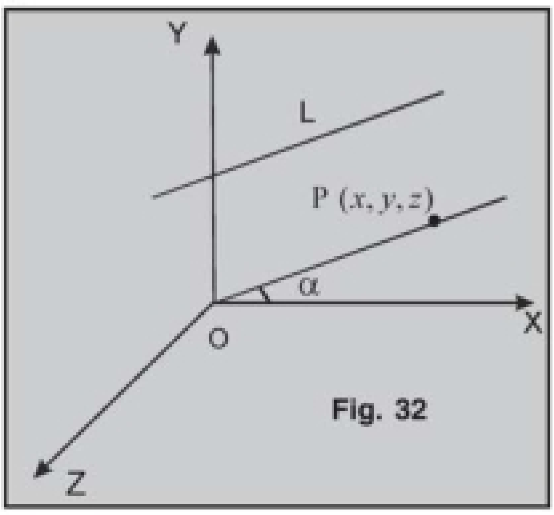


Fig. 32

In $\overleftrightarrow{POX}$ plane, $\cos\alpha = \frac{x}{1} = x;$

In $\overleftrightarrow{POY}$ plane, $\cos\beta = \frac{y}{1} = y;$

In $\overleftrightarrow{POZ}$ plane, $\cos\gamma = \frac{z}{1} = z;$

OR $a = (\overrightarrow{OP}, \overrightarrow{OX}) = (\overline{OP}, \bar{i}) \Rightarrow \cos\alpha = \overline{OP} . \bar{i} = (x, y, z).(1,0,0) = x$, etc.

i.e. $(x, y, z) = (l, m, n)$ or $(-l, -m, -n)$

But $OP^2 = 1 \Rightarrow x^2 + y^2 + z^2 = 1 \Rightarrow l^2 + m^2 + n^2 = 1$ etc.

Note. (l, m, n) and $(-l, -m, -n)$ are the only unit points on $\overleftrightarrow{OP}$ and l, m, n ; $-l, -m, -n$ are the d.cs. of L.

DIRECTION RATIOS OR DIRECTION NUMBERS OF A LINE

L is a line and $P(x, y, z)$ is a point such that $\overleftrightarrow{OP} \parallel L$. Then the coordinates of any point on $\overleftrightarrow{OP}$, other than the origin, are called the direction numbers of L. But any point, other than origin, on $\overleftrightarrow{OP}$ is $(\lambda x, \lambda y, \lambda z)\ (\lambda \neq 0)$. So in that order are called the direction numbers of the line L.

Clearly, the direction numbers for a line L are infinitely many and they are proportional. Further the direction numbers cannot be 0, 0, 0.

The direction numbers are sometimes called as **direction ratios** (d. rs.)

The d.cs. (l, m, n) or $(-l, -m, -n)$ of L are also d.rs. of L since each of (l, m, n) or $(-l, -m, -n)$ is also a point on $\overleftrightarrow{OP}$.

If $P(x, y, z)$ then unit points on $\overleftrightarrow{OP}$ are

$$\left(\frac{x}{\sqrt{(x^2+y^2+z^2)}}, \frac{y}{\sqrt{(x^2+y^2+z^2)}}, \frac{z}{\sqrt{(x^2+y^2+z^2)}}\right),$$
$$\left(\frac{-x}{\sqrt{(x^2+y^2+z^2)}}, \frac{-y}{\sqrt{(x^2+y^2+z^2)}}, \frac{-z}{\sqrt{(x^2+y^2+z^2)}}\right)$$

Hence if d.rs. of L are x, y, z then d.cs. of L are

$$\frac{x}{\sqrt{(x^2+y^2+z^2)}}, \frac{y}{\sqrt{(x^2+y^2+z^2)}}, \frac{z}{\sqrt{(x^2+y^2+z^2)}}$$
$$\frac{-x}{\sqrt{(x^2+y^2+z^2)}}, \frac{-y}{\sqrt{(x^2+y^2+z^2)}}, \frac{-z}{\sqrt{(x^2+y^2+z^2)}}$$

Note. 1. D.rs. of the coordinate axes are respectively
$x, 0, 0;\ 0, y, 0;\ 0, 0, z\ (x \neq 0, y \neq 0, x \neq 0)$

2. If x, y, z are d.rs. of a line L, there exists a point $P(x, y, z)$ such that $\overleftrightarrow{OP} \parallel L$ and $x\bar{i} + y\bar{j} + z\bar{k}$ is a vector along $\overleftrightarrow{OP}$ *i.e.* a vector along L.

3. If l, m, n are the d.cs. of a line L then $l\bar{i} + m\bar{j} + n\bar{k}$ is a unit vector L.

4. l, m, n are d.cs. of a line L. If any two of the d.cs. and the sign of the third is known, then the d.cs. of L can be found.

For example, $\frac{1}{2}, \frac{1}{2}, n$ are d.cs. of L and $n < 0 \Rightarrow n = -\sqrt{\left(1 - \frac{1}{4} - \frac{1}{4}\right)} = \frac{1}{\sqrt{2}}$

$\Rightarrow$ d.cs. of L are $\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}}$

e.g. The d.cs. of the line with d.rs. (3, 2, 6) are

$$\pm\frac{3}{\sqrt{(9+4+36)}}, \pm\frac{2}{(9+4+36)}, \pm\frac{6}{\sqrt{(9+4+36)}} \quad i.e., \quad \frac{3}{7}, \frac{2}{7}, \frac{6}{7}; -\frac{3}{7}, -\frac{2}{7}, -\frac{6}{7}$$

2.38. THEOREM. *If* $P=(x_1, y_1, z_1), Q=(x_2, y_2, z_2)$ *then* $x_2-x_1, y_2-y_1, z_2-z_1$ *are d.rs. of* $\overleftrightarrow{PQ}$.

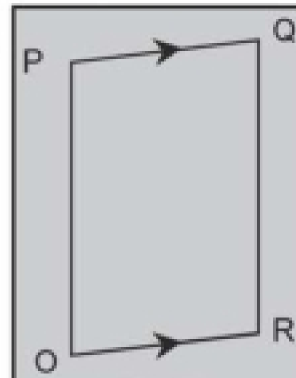


Fig. 33

Proof. Let OPQR be a parallelogram (Fig. 33)

Then $\overline{OR}=\overline{PQ}=(x_2-x_1, y_2-y_1, z_2-z_1)$

$\Rightarrow R=(x_2-x_1, y_2-y_1, z_2-z_1)$

$\therefore$ d.rs. of $\overleftrightarrow{OR}$ are $x_2-x_1, y_2-y_1, z_2-z_1$

$\therefore$ d.rs. of $\overleftrightarrow{PQ}$ are $x_2-x_1, y_2-y_1, z_2-z_1$

2.39. PROJECTION OF A LINE SEGMENT ON ANOTHER LINE.

The projection of a line segment CD on a line $\overleftrightarrow{AB}$ is MN where M, N are the feet of the perpendiculars on $\overleftrightarrow{AB}$. If $\overleftrightarrow{CD}$ make an angle θ with $\overleftrightarrow{AB}$, then the projection of CD on $\overleftrightarrow{AB}$ is $MN = CD\ \cos\theta$. (Fig. 33 (*a*))

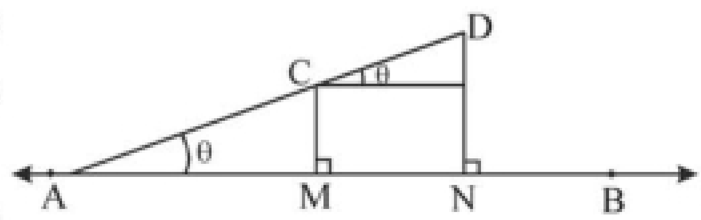


Fig. 33 (*a*)

Let $\overleftrightarrow{AB}$ be a line. On $\overleftrightarrow{AB}$ let the projections of C, D be M, N respectively.

Then the projection of CD on the line $\overleftrightarrow{AB}$ is $\overline{CD}.\dfrac{\overline{MN}}{|\overline{MN}|}$ in the direction $\overline{MN}$.

Theorem. *If* $\overrightarrow{AB}$ *is a ray with d.cs.* l, m, n *and* $P=(x_1, y_1, z_1)$, $Q=(x_2, y_2, z_2)$ *are two points, then the projection of PQ on* $\overrightarrow{AB}$ *in the direction* $\overrightarrow{AB}$ *is* $(x_2-x_1)l+(y_2-y_1)m+(z_2-z_1)n$.

Proof. (Fig. 34). $PQ=(x_2-x_1, y_2-y_1, z_2-z_1)$.

Unit vector along $\overrightarrow{AB}=e=(l, m, n)$

$\therefore$ Projection PQ on $\overleftrightarrow{AB}$ in the direction $\overrightarrow{AB}$

$= PQ.e=(x_2-x_1, y_2-y_1, z_2-z_1)\ (l, m, n)$

$= l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)$

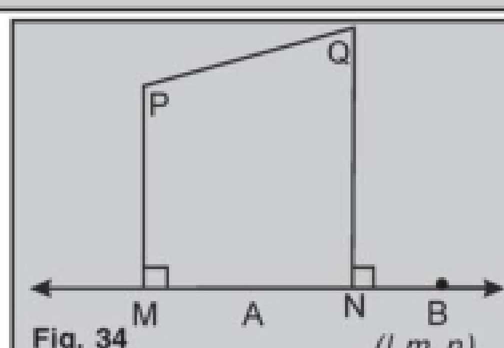


Fig. 34

2.40. ANGLES BETWEEN TWO LINES

Theorem. *If* l_1, m_1, n_1 *and* l_2, m_2, n_2 *are d.cs. of two lines* L_1, L_2, *then an angle* θ *between them is given by* $\cos\theta = l_1l_2+m_1m_2+n_1n_2$

Proof. Let $P=(l_1, m_1, n_1)$ and $Q=(l_2, m_2, n_2)$ and

$\overline{OP}=\overline{e}_1=(l_1, m_1, n_1)$ $\overline{OQ}=\overline{e}_2=(l_2, m_2, n_2)$ $\therefore$ $\overleftrightarrow{OP} \parallel L_1$ and $\overleftrightarrow{OQ} \parallel L_2$

Since θ is one of the angles between L_1, L_2, we take $(\overrightarrow{OP}, \overrightarrow{OQ})=\theta$

$\therefore$ $\cos\theta=\cos(\overrightarrow{OP}, \overrightarrow{OQ})=\cos(OP, OQ)=\cos(e_1, e_2)$

$= e_1.e_2=(l_1, m_1, n_1).(l_2, m_2, n_2)=l_1l_2+m_1m_2+n_1n_2$

$\therefore$ Angles between L_1, L_2 are $\theta, 180^\circ-\theta$.

Note. 1. $\sin\theta = |\bar{e}_1 \times \bar{e}_2| = |(m_1n_2 - m_2n_1, n_1l_2 - n_2l_1, l_1m_2 - l_2m_1)|$

$= \sqrt{[(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2]} = \sqrt{[\sum (m_1n_2 - m_2n_1)^2]}$

OR : $\sin^2\theta = 1 - \cos^2\theta = (l_1{}^2 + m_1{}^2 + n_1{}^2)(l_2{}^2 + m_2{}^2 + n_2{}^2) - \cos^2\theta$

$= (l_1{}^2 + m_1{}^2 + n_1{}^2)(l_2{}^2 + m_2{}^2 + n_2{}^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2$

$= (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$

2. $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\sqrt{\sum (m_1n_2 - m_2n_1)^2}}{l_1l_2 + m_1m_2 + n_1n_2}$ where $l_1l_2 + m_1m_2 + n_1n_2 \neq 0$.

3. (a_1, b_1, c_1) are d.rs. of L_1, (a_2, b_2, c_2) are d.rs. of L_2,

(*i*) $\cos\theta = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1{}^2 + b_1{}^2 + c_1{}^2)}\,.\sqrt{(a_2{}^2 + b_2{}^2 + c_2{}^2)}}$,

$$\sin\theta = \frac{\sqrt{\sum (b_1c_2 - b_2c_1)^2}}{\sqrt{(a_1{}^2 + b_1{}^2 + c_1{}^2)}\,.\sqrt{(a_2{}^2 + b_2{}^2 + c_2{}^2)}}$$

(*ii*) $L_1 \parallel L_2 \Leftrightarrow \sin\theta = 0 \Leftrightarrow \sum (m_1n_2 - m_2n_1)^2 = 0$

$\Leftrightarrow m_1n_2 - m_2n_1 = 0, n_1l_2 - n_2l_1 = 0, l_1m_2 - l_2m_1 = 0 \Leftrightarrow l_1 : l_2 = m_1 : m_2 = n_1 : n_2$

$\Leftrightarrow a_1 : a_2 = b_1 : b_2 = c_1 : c_2 \Leftrightarrow$ d.r. triads are proportional. **(*N. U. S. 98*)**

(*iii*) $L_1 \perp L_2 \Leftrightarrow \cos\theta = 0 \Leftrightarrow l_1l_2 + m_1m_2 + n_1n_2 = 0 \Leftrightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\Leftrightarrow$ dot product of d.r. triads is zero. **(*N. U. S. 98*)**

2.41. LAGRANGE'S IDENTITY

For any real numbers $l_1, m_1, n_1, l_2, m_2, n_2$

$(l_1{}^2 + m_1{}^2 + n_1{}^2)(l_2{}^2 + m_2{}^2 + n_2{}^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2$

$= (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$

By simplifying L.H.S. and regrouping the terms we can show that L.H.S = R.H.S.

SOLVED PROBLEMS

Ex. 1. *Calculate the cosine of the angle A of the triangle with vertices $A(1, -1, 2)$, $B(6, 11, 2)$, $C(1, 2, 6)$.* **(*A. U. M 2014*)**

Sol. Given $A = (1, -1, 2)$, $B = (6, 11, 2)$, $C = (1, 2, 6)$

We have $\overline{AB} = (6-1, 11+1, 2-2)$ and $\overline{AC} = (1-1, 2+1, 6-2)$

$$\cos A = \frac{\overline{AB}\,.\,\overline{AC}}{|\overline{AB}|\,|\overline{AC}|} = \frac{(5, 12, 0)\,.\,(0, 3, 4)}{\sqrt{25+144+0}\,.\sqrt{0+9+16}} = \frac{0+36+0}{13\times 5} = \frac{36}{65}$$

OR: D.rs. of $\overrightarrow{AB}, \overrightarrow{AC}$ are $(6-1, 11+1, 2+2)$ and $(1-1, 2+1, 6-2)$ *i.e.*, $(5, 12, 0), (0, 3, 4)$.

$\therefore$ D.cs. of $\overrightarrow{AB}, \overrightarrow{AC}$ are $\left(\frac{5}{13}, \frac{12}{13}, 0\right), \left(0, \frac{3}{5}, \frac{4}{5}\right)$. $\quad \therefore \cos A = \frac{5}{13}\,.\,0 + \frac{12}{13}\,.\,\frac{3}{5} + 0\,.\,\frac{4}{5} = \frac{36}{65}$.

Ex. 2. *The d.cs. of two rays are connected by the relation*
$l-5m+3n=0$ *and* $7l^2+5m^2-3n^2=0$ *Find them.*

Sol. Given equations are $l-5m+3n=0$ ...(1) $7l^2+5m^2-3n^2=0$ ...(2)

Eliminating l between (1) and (2), $7(5m-3n)^2+5m^2-3n^2=0$

$\Rightarrow 180m^2-210mn+60n^2=0 \Rightarrow (3m-2n)(2m-n)=0 \Rightarrow \frac{m}{n}=\frac{2}{3}, \frac{m}{n}=\frac{1}{2}$

If $\frac{m}{n}=\frac{2}{3}$, from (1), $l-\frac{10n}{3}+3n=0 \Rightarrow l=\frac{n}{3} \Rightarrow 3l=\frac{3m}{2}=n \Rightarrow \frac{l}{1}=\frac{m}{2}=\frac{n}{3}$ .... (3)

If $\frac{m}{n}=\frac{1}{2}$, from (1), $l-\frac{5n}{2}+3n=0 \Rightarrow l=\frac{n}{2} \Rightarrow -2l=2m=n \Rightarrow \frac{l}{-1}=\frac{m}{1}=\frac{n}{2}$ ... (4)

From (3) and (4), d.rs. ofmthe two rays are $1, 2, 3; -1, 1, 2$ and d.cs. of the two lines are

$\pm\frac{1}{\sqrt{(14)}}, \pm\frac{2}{\sqrt{(14)}}, \pm\frac{3}{\sqrt{(14)}}$; $\pm\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{6}}, \pm\frac{2}{\sqrt{6}}$.

Ex. 3. *S.T. the lines with l, m, n as d.c's and satisfying the following equations $2l+2m-n=0; mn+nl+lm=0$ are perpendicular.* **(K. U. 2001, 03, S, S.K.U.98)**

Sol. Given equations are $2l+2m-n=0$ ...(1) $mn+nl+lm=0$ ...(2)

From (1) $n=2l+2m$

Substituting n value in equation (2) we have $m(2l+2m)+(2l+2m)l+lm=0$

$\Rightarrow 2l^2+5lm+2m^2=0 \Rightarrow (l+2m)(2l+m)=0$

$\therefore l=-2m$ or $2l=-m$

Case (*i*) : If $l=-2m \Rightarrow n=-2m$ $\therefore \frac{l}{2}=\frac{m}{-1}=\frac{n}{2}=\frac{\sqrt{\Sigma l^2}}{\sqrt{\Sigma(2)^2}}=\frac{1}{3}$

$\therefore$ DC's of first line are $\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}$

Case (*ii*) : If $2l=-m \Rightarrow n=m$ $\therefore \frac{l}{-1}=\frac{m}{2}=\frac{n}{2}=\frac{1}{3}$

DC's of second line are $\frac{-1}{3}, \frac{2}{3}, \frac{2}{3}$

Now $l_1l_2+m_1m_2+n_1n_2=\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)+\left(\frac{-1}{3}\right)\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)=0$

$\therefore$ Given lines are perpendicular to one another.

Ex.4. *Prove that the two rays whose d.cs. are connected by*
$al+bm+cn=0$ *and* $ul^2+vm^2+wn^2=0$ **(O. U. Oct. 2001, K.U. 2000)**

are perpendicular if $a^2(v+w)+b^2(w+u)+c^2(u+v)=0$ **(S. K. U. 2001 Oct., OU 2001)**

and parallel if $\frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0$ **(S. V. U. A 93, O.U. 01, 2000)**

Sol. Given $al+bm+cn=0$ ....(1) $ul^2+vm^2+wn^2=0$ .....(2)

From (1) and (2)

$ul^2+vm^2+w\left(\frac{al+bm}{-c}\right)^2=0 \Rightarrow c^2ul^2+c^2vm^2+a^2wl^2+2abwlm+b^2wm^2=0$

$$\Rightarrow (a^2w+c^2u)\frac{l^2}{m^2}+2abw.\frac{l}{m}+(b^2w+c^2v)=0 \quad \text{.... (3)}$$

Let $l_1, m_1, n_1;\ l_2, m_2, n_2$ be the d.cs. of the two rays and $m_1 \neq 0,\ m_2 \neq 0$

$\therefore\ \frac{l_1}{m_1}, \frac{l_2}{m_2}$ are the roots of (3) $\quad \therefore\ \frac{l_1}{m_1}.\frac{l_2}{m_2}=\frac{b^2w+c^2v}{a^2w+c^2u} \Rightarrow \frac{l_1l_2}{b^2w+c^2v}=\frac{m_1m_2}{c^2u+a^2w}$

Using symmetry, we can have $\frac{l_1l_2}{b^2w+c^2v}=\frac{m_1m_2}{c^2u+a^2w}=\frac{n_1n_2}{a^2v+b^2u}$

The rays are perpendicular $\Leftrightarrow\ l_1l_2+m_1m_2+n_1n_2=0$

$\Leftrightarrow b^2w+c^2v+c^2u+a^2w+a^2v+b^2u=0 \Leftrightarrow a^2(v+w)+b^2(w+u)+c^2(u+v)=0$.

The rays are parallel $\Leftrightarrow$ the d.cs. are proportional $\Leftrightarrow$ the roots of (3) are equal.
$\Leftrightarrow$ discriminant of (3) = 0.

$\Leftrightarrow 4\,a^2b^2w^2-4\,(c^2u+a^2w)\,(b^2w+c^2v)=0 \Rightarrow a^2c^2vw+b^2c^2wu+c^2uv=0$

$\Leftrightarrow \frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0$.

Similarly, the result can also be obtained when $l_1 \neq 0, l_2 \neq 0$ or $n_1 \neq 0, n_2 \neq 0$

Ex. 5. *Prove that one of the angles between the lines whose d.cs. are determined by the equations* $l+m+n=0$ *and* $2lm+2nl-mn=0$ *is* $2\pi/3$. ***(K.U.99)***

Sol. Let L_1, L_2 be the lines whose d.cs l_1, m_1, n_1 and l_2, m_2, n_2 are determined by $l+m+n=0$ .... (1) $\quad 2lm+2nl-nm=0$ ..... (2)

From (1) and (2) : $-2m\,(m+n)-2n\,(m+n)-mn=0$

$\Rightarrow -2m^2-5mn-2n^2=0 \Rightarrow (2m+n)\,(m+2n)=0 \Rightarrow m=-\frac{n}{2}, m=-2n$

If $m=-\frac{n}{2}, l-\frac{n}{2}+n=0 \Rightarrow l=-\frac{n}{2}$ [from (1)] $\quad \therefore\ 2l=2m=-n \Rightarrow \frac{l}{1}=\frac{m}{1}=\frac{n}{-2}$

If $m=-2n, l-2n+n=0 \Rightarrow l=n$ [from (1)] $\quad \therefore\ l=\frac{m}{-2}=n \Rightarrow \frac{l}{1}=\frac{m}{-2}=\frac{n}{1}$

$\therefore$ d.rs. of L_1, L_2 are $(1,1,-2), (1,-2,1)$. Let θ be one of the angles between L_1, L_2

$\therefore\ \cos\theta=\frac{1.1+1(-2)+(-2).1}{\sqrt{(1+1+4)}.\sqrt{(1+4+1)}}=-\frac{1}{2}$

$\therefore\ \theta=2\pi/3$ (the other angle is $\pi-(2\pi/3)$ *i.e.* $(\pi/3)$)

Ex. 6. *If* $(l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3)$ *are the d.cs. of three mutually perpendicular rays, then find the d.cs. of a ray whose d.rs. are* $l_1+l_2+l_3,\ m_1+m_2+m_3,\ n_1+n_2+n_3$. *Hence show that the ray is equally inclined to the given rays.*

Sol. Let L_1, L_2, L_3 be three mutually perpendicular rays whose d.cs. are $(l_1, m_1, n_1),\ (l_2, m_2, n_2), (l_3, m_3, n_3)$

$\therefore\ l_1^2+m_1^2+n_1^2=1,\ l_2^2+m_2^2+n_2^2=1,\ l_3^2+m_3^2+n_3^2=1,$

$l_1l_2+m_1m_2+n_1n_2=0,\ l_2l_3+m_2m_3+n_2n_3=0,\ l_3l_1+m_3m_1+n_3n_1=0$

Now $(l_1+l_2+l_3)^2+(m_1+m_2+m_3)^2+(n_1+n_2+n_3)^2$

$= (l_1^2 + m_1^2 + n_1^2) + (l_2^2 + m_2^2 + n_2^2) + (l_3^2 + m_3^2 + n_3^2)$
$\qquad + 2\,(l_1 l_2 + m_1 m_2 + n_1 n_2) + 2\,(l_2 l_3 + m_2 m_3 + n_2 n_3) + 2\,(l_3 l_1 + m_3 m_1 + n_3 n_1) = 3$

$\therefore$ d.cs. of the ray L whose d.rs. are $l_1 + l_2 + l_3,\ m_1 + m_2 + m_3$, are $n_1 + n_2 + n_3$

are $\dfrac{l_1 + l_2 + l_3}{\sqrt{3}}, \dfrac{m_1 + m_2 + m_3}{\sqrt{3}}, \dfrac{n_1 + n_2 + n_3}{\sqrt{3}}$

If $(\mathrm{L}, \mathrm{L}_1) = \theta$, then $\cos\theta = \dfrac{l_1 (l_1 + l_2 + l_3) + m_1 (m_1 + m_2 + m_3) + n_1 (n_1 + n_2 + n_3)}{\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

i.e., $\theta = \mathrm{Cos}^{-1}(1/\sqrt{3})$. Similarly, we can have $(\mathrm{L}, \mathrm{L}_2) = (\mathrm{L}, l_3) = \mathrm{Cos}^{-1}(1/\sqrt{3})$.

$\therefore$ L is equally inclined with $\mathrm{L}_1, \mathrm{L}_2, \mathrm{L}_3$.

Ex. 7. *L_1, L_2, L_3 are three concurrent rays whose d.cs. are $(l_1, m_1, n_1), (l_2, m_2, n_2)$, (l_3, m_3, n_3) respectively.*

Prove that L_1, L_2, L_3 are coplanar $\Leftrightarrow \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$

Sol. Rays $\mathrm{L}_1, \mathrm{L}_2, \mathrm{L}_3$ are concurrent and unit vectors along $\mathrm{L}_1, \mathrm{L}_2, \mathrm{L}_3$ are

$l_1 \bar{i} + m_1 \bar{j} + n_1 \bar{k}, l_2 \bar{i} + m_2 \bar{j} + n_2 \bar{k}, l_3 \bar{i} + m_3 \bar{j} + n_3 \bar{k}$ *i.e.* $(l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3)$.

$\mathrm{L}_1, \mathrm{L}_2, \mathrm{L}_3$ are coplanar. $\Leftrightarrow [(l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3)] = 0 \Leftrightarrow \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$

Ex. 8. *The d.cs. of $\overrightarrow{AB}, \overrightarrow{AC}$ are $(l_1, m_1, n_1), (l_2, m_2, n_2)$. Show that d.rs. of the bisectors of $(\overleftrightarrow{AB}, \overleftrightarrow{AC})$ are $l_1 \pm l_2,\ m_1 \pm m_2,\ n_1 \pm n_2$. Hence if $(\overrightarrow{AB}, \overrightarrow{AC}) = \theta$, find the d.cs. of the ray bisecting $(\overrightarrow{AB}, \overrightarrow{AC})$.*

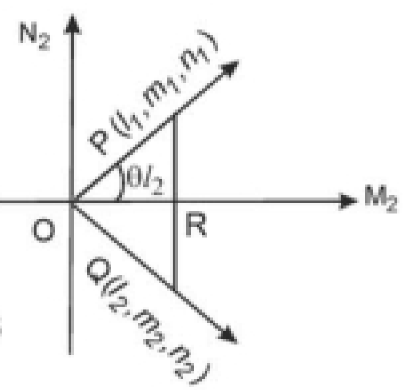


Fig. 35

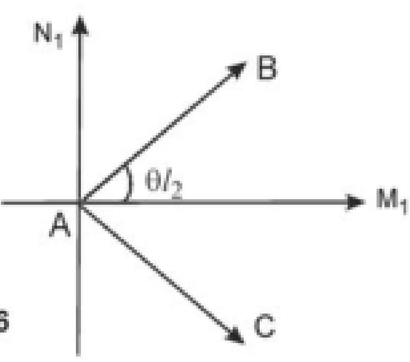


Fig. 36

Sol. O is the origin. Let $\mathrm{P} = (l_1, m_1, n_1)$ and $\mathrm{Q} = (l_2, m_2, n_2)$ such that $\mathrm{OP} = 1$ and $\mathrm{OQ} = 1$

Clearly, $\overrightarrow{\mathrm{AB}} \parallel \overrightarrow{\mathrm{OP}}$ and $\overrightarrow{\mathrm{AC}} \parallel \overrightarrow{\mathrm{OQ}}$.

Let $\overleftrightarrow{\mathrm{AM}_1}$ be a bisector of $(\overleftrightarrow{\mathrm{AB}}, \overleftrightarrow{\mathrm{AC}})$ and $\overleftrightarrow{\mathrm{AN}_1}$ be the other bisector of $(\overleftrightarrow{\mathrm{AB}}, \overleftrightarrow{\mathrm{AC}})$.

$\therefore\ \overleftrightarrow{\mathrm{AM}_1} \perp \overleftrightarrow{\mathrm{AN}_1}$

Let $\overleftrightarrow{\mathrm{OM}_2}$ be a bisector of $(\overrightarrow{\mathrm{OP}}, \overrightarrow{\mathrm{OQ}})$. $\qquad \therefore\ \overrightarrow{\mathrm{OM}_2} \parallel \overrightarrow{\mathrm{AM}_1}$

Let $\overleftrightarrow{\mathrm{OM}_2}$ intersect PQ at R. Since $\mathrm{OP} = \mathrm{OQ}$ in $\Delta\,\mathrm{OPQ}$, R is the mid point of PQ

$\therefore\ \mathrm{R} = \left(\dfrac{l_1 + l_2}{2}, \dfrac{m_1 + m_2}{2}, \dfrac{n_1 + n_2}{2}\right)$. $\quad \therefore$ D. rs. of $\overrightarrow{\mathrm{OR}}$ are $l_1 + l_2,\ m_1 + m_2,\ n_1 + n_2$

Since $\overrightarrow{OM_2} \parallel \overrightarrow{AM_1}$ d.rs. of $\overleftrightarrow{AM_1}$ are $l_1+l_2,\ m_1+m_2,\ n_1+n_2$

Let ON_2 be the other bisector of $(\overleftrightarrow{OP}, \overleftrightarrow{OQ})$. $\therefore\ \overrightarrow{ON_2} \parallel \overrightarrow{AN_1}$

Since $\overleftrightarrow{PQ} \parallel \overleftrightarrow{ON_2}$, $\overleftrightarrow{PQ} \parallel \overleftrightarrow{AN_1}$. Since d.rs. of $\overrightarrow{PQ}$ are $l_1-l_2,\ m_1-m_2,\ n_1-n_2$,

d.rs. of $\overleftrightarrow{AN_1}$ are $l_1-l_2,\ m_1-m_2,\ n_1-n_2$.

$\therefore$ D.rs. of bisectors of $(\overleftrightarrow{AB}, \overleftrightarrow{AC})$ are $l_1 \pm l_2,\ m_1 \pm m_2,\ n_1 \pm n_2$

From Δ ORP, $\angle ORP = \pi/2$ and $\angle POR = \theta/2$

$\therefore\ OR = OP\cos\theta/2 = \cos\theta/2 \qquad \therefore\ |\overline{OR}| = \cos\theta/2$

$$\Rightarrow \sqrt{\left[\left(\frac{l_1+l_2}{2}\right)^2 + \left(\frac{m_1+m_2}{2}\right)^2 + \left(\frac{n_1+n_2}{2}\right)^2\right]} = \cos(\theta/2)$$

$$\sqrt{[(l_1+l_2)^2+(m_1+m_2)^2+(n_1+n_2)^2]} = 2\cos(\theta/2) \Rightarrow \sqrt{[\Sigma(l_1+l_2)^2]} = 2\cos(\theta/2)$$

$\therefore$ D.cs. of OR are $\dfrac{l_1+l_2}{\sqrt{[\Sigma(l_1+l_2)^2]}},\ \dfrac{m_1+m_2}{\sqrt{[\Sigma(l_1+l_2)^2]}},\ \dfrac{n_1+n_2}{\sqrt{[\Sigma(l_1+l_2)^2]}}$

i.e., $\dfrac{l_1+l_2}{2\cos\theta/2},\ \dfrac{m_1+m_2}{2\cos\theta/2},\ \dfrac{n_1+n_2}{2\cos\theta/2}$

Since $\overrightarrow{AM_1} \parallel \overrightarrow{OR}$ d.cs. of $\overrightarrow{AM_1}$ are $\dfrac{l_1+l_2}{2\cos\theta/2},\ \dfrac{m_1+m_2}{2\cos\theta/2},\ \dfrac{n_1+n_2}{2\cos\theta/2}$

Note. The result of the above example may be taken as formula.

Ex. 9. *Find the foot of the perpendicular from* $P(1,8,4)$ *to the line* $\overleftrightarrow{AB}$ *where* $A=(0,-11,4),\ B=(2,-3,1)$.

Sol. Let $(Q;A,B) = 1:\lambda\ (1+\lambda \neq 0)$ $\qquad \therefore\ Q = \left(\dfrac{2}{1+\lambda}, \dfrac{-3-11\lambda}{1+\lambda}, \dfrac{1+4\lambda}{1+\lambda}\right)$

$\overline{PQ} = \left(\dfrac{2}{1+\lambda}-1, \dfrac{-3-11\lambda}{1+\lambda}-8, \dfrac{1+4\lambda}{1+\lambda}-4\right)$ and $\overline{AB} = (2,-3+11,1-4) = (2,8,-3)$

If $PQ \perp AB$, then $\overline{AB}.\overline{PQ} = 0$

$$\therefore\ 2\left(\frac{2-1-\lambda}{1+\lambda}\right) + 8\left(\frac{-3-11\lambda-8-8\lambda}{1+\lambda}\right) - 3\left(\frac{1+4\lambda-4-4\lambda}{1+\lambda}\right) = 0$$

$$\Rightarrow 2-2\lambda-88-152\lambda+9 = 0 \Rightarrow 154\lambda = -77 \Rightarrow \lambda = -\frac{1}{2} \Rightarrow 1:\lambda = 1:-\frac{1}{2}$$

$\therefore$ Foot of the perpendicular from P to $\overleftrightarrow{AB}$ $= \left(\dfrac{2}{(1/2)}, \dfrac{-3+\frac{11}{2}}{(1/2)}, \dfrac{1-2}{(1/2)}\right) = (4,5,-2)$

Note. Length of the perpendicular from P to the line

$$\overleftrightarrow{AB} = \sqrt{(4-1)^2+(5-8)^2+(-2-4)^2} = 3\sqrt{6}$$

Ex. 10. *If* P,Q,R,S *are the points* $(-1,2,4),(1,0,5),(3,4,5),(4,6,3)$, *find the projection of* $\overline{PQ}$ *on* $\overleftrightarrow{RS}$ *in the direction of* $\overline{RS}$.

Sol. $\overline{PQ} = (2,-2,1)$ and $\overline{RS} = (1,2,-2)$

$\therefore$ Projection of $\overline{PQ}$ on $\overleftrightarrow{RS}$ in the direction of $\overline{RS}$

= Projection of $\overline{PQ}$ on $\overline{RS}$ in the direction of $\overline{RS}$

$$= \frac{\overline{PQ}\,.\,\overline{RS}}{|\overline{RS}|} = \frac{(2,-2,1).(1,2,-2)}{|(1,2,-2)|} = \frac{(2)\,1+(-2)\,2+1\,(-2)}{\sqrt{[1+4+4]}} = \frac{-4}{3}$$

Ex. 11. *If the edges of a rectangular parallelopiped are a, b, c prove that the angles between any two diagonals are* $\text{Cos}^{-1}\left[\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right]$

(S.K.U.2002, A.U.2000, O.U.2002, N. U. M. 2001, 03)

Sol. Since a, b, c are the lengths of the edges,

$a > 0, b > 0, c > 0$ (Fig. 37)

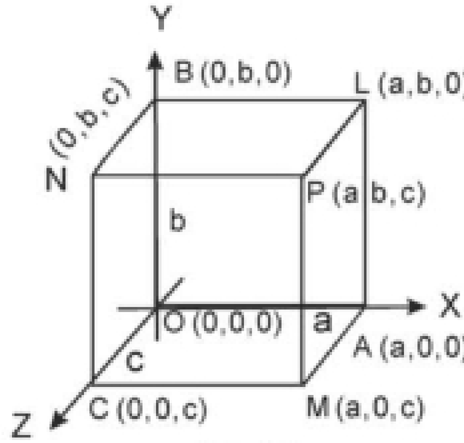


Fig. 37

Let (OALB; CMPN) be the rectangular parallelopiped as shown. Its diagonals are $\overleftrightarrow{OP}, \overleftrightarrow{AN}, \overleftrightarrow{BM}, \overleftrightarrow{CL}$. Direction ratios of $\overrightarrow{OP}, \overrightarrow{AN}, \overrightarrow{BM}, \overrightarrow{CL}$ are respectively $(a,b,c), (-a,b,c), (a,-b,c), (a,b,-c)$. Their d.cs. are respectively

$$\left(\frac{a}{\sqrt{(\Sigma a^2)}}, \frac{b}{\sqrt{(\Sigma a^2)}}, \frac{c}{\sqrt{(\Sigma a^2)}}\right),$$

$$\left(\frac{-a}{\sqrt{(\Sigma a^2)}}, \frac{b}{\sqrt{(\Sigma a^2)}}, \frac{c}{\sqrt{(\Sigma a^2)}}\right),$$

$$\left(\frac{a}{\sqrt{(\Sigma a^2)}}, \frac{-b}{\sqrt{(\Sigma a^2)}}, \frac{c}{\sqrt{(\Sigma a^2)}}\right), \left(\frac{a}{\sqrt{(\Sigma a^2)}}, \frac{b}{\sqrt{(\Sigma a^2)}}, \frac{-c}{\sqrt{(\Sigma a^2)}}\right)$$

Let α be an angle between $\overleftrightarrow{OP}, \overleftrightarrow{AN}$

$$\therefore \cos\alpha = \left(\frac{a}{\sqrt{(\Sigma a^2)}}\right)\left(\frac{-a}{\sqrt{(\Sigma a^2)}}\right) + \left(\frac{b}{\sqrt{(\Sigma a^2)}}\right)\left(\frac{b}{\sqrt{(\Sigma a^2)}}\right) + \left(\frac{c}{\sqrt{(\Sigma a^2)}}\right)\left(\frac{a}{\sqrt{(\Sigma a^2)}}\right)$$

$$= \frac{-a^2+b^2+c^2}{a^2+b^2+c^2} \quad \textit{i.e.}\ \alpha = \text{Cos}^{-1}\left(\frac{-a^2+b^2+c^2}{a^2+b^2+c^2}\right)$$

Similarly, we can find the angles between other pairs of diagonals.

$\therefore$ The angles between any two diagonals are $\text{Cos}^{-1}\left[\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right]$

Note. The signs in the numerator cannot all be + ve or – ve. For the angle become $\text{Cos}^{-1}1$ or $\text{Cos}^{-1}(-1)$ *i.e.,* 0° or 180° which is absurd as diagonals become parallel.

Ex. 12. *Find the area of the* ΔOAB *where O is the origin,* $A = (x_1, y_1, z_1)$ *and* $B = (x_2, y_2, z_2)$.

Sol. $\overline{OA} = (x_1, y_1, z_1)$ and $\overline{OB} = (x_2, y_2, z_2)$. Area of the $\Delta\,OAB = \frac{1}{2}|\overline{OA} \times \overline{OB}|$

But $\overline{OA} \times \overline{OB} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1z_2 - y_2z_1, z_1x_2 - x_1z_2, x_1y_2 - x_2y_1)$

$\therefore$ Area of Δ OAB $= \frac{1}{2}\sqrt{\{(y_1z_2 - y_2z_1)^2 + (z_1x_2 - z_2x_1)^2 + (x_1y_2 - x_2y_1)^2\}}$ sq. units

Ex. 13. *In a tetrahedron* $OABC$. $OA^2 + BC^2 = OB^2 + CA^2 = OC^2 + AB^2$. *Show that opposite edges are perpendicular*

Sol. OABC is a tetrahedron (Fig. 38)

Let O be the origin and

$A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$, $C = (x_3, y_3, z_3)$,

$\therefore\ \overline{OA} = (x_1, y_1, z_1)$

$\overline{BC} = (x_3 - x_2, y_3 - y_2, z_3 - z_2)$,

$\overline{OB} = (x_2, y_2, z_2)$,

$\overline{CA} = (x_1 - x_3, y_1 - y_3, z_1 - z_3)$.

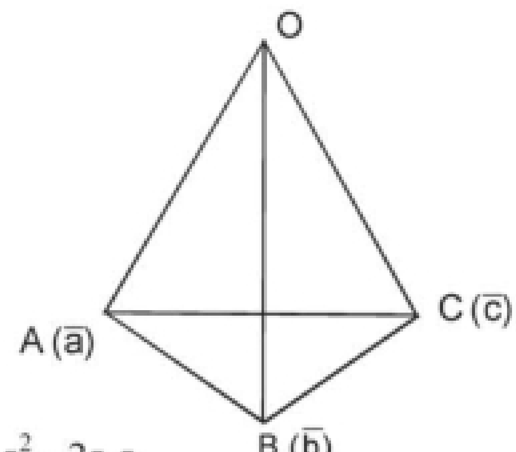


Fig. 38

Now $OA^2 + BC^2 = OB^2 + CA^2$

$\Rightarrow x_1^2 + y_1^2 + z_1^2 + x_3^2 + x_2^2 - 2x_3x_2 + y_3^2 + y_2^2 - 2y_2y_3 + z_3^2 + z_2^2 - 2z_3z_2$

$= x_2^2 + y_2^2 + z_2^2 + x_1^2 + x_3^2 - 2x_1x_3 + y_1^2 + y_3^2 - 2y_1y_3 + z_1^2 + z_3^2 - 2z_1z_3$

$\Rightarrow x_3(x_2 - x_1) + y_3(y_2 - y_1) + z_3(z_2 - z_1) = 0$

$\Rightarrow \overline{OC} . \overline{AB} = 0 \Rightarrow \overline{OC} \perp \overline{AB}$

$\Rightarrow$ opposite edges OC and AB are perpendicular.

Similarly, others follow.

EXERCISE 2 (*d*)

1. O is the origin and P is the following point. Find the d.cs. of $\overrightarrow{OP}$.
   (*i*) (6,2,3) (*ii*) (–3,12,–4)
2. Find the d.cs. of the ray $\overrightarrow{AB}$ if $A = (1, -2, 3), B = (2, 3, -4)$
3. (*i*) If α, β, γ are the angles made by a ray with $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$, find $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$.
   (*ii*) Find the d.cs. of lines which make equal angles with the co-ordinate axes ? How many such lines are there ?
4. (*i*) Prove that the line joining the points $(4,5,7), (1,2,3)$ is parallel to the join of the points $(-1,3,-6), (5,9,2)$.
   (*ii*) Show that the points $(1,-2,3), (2,3,-4), (0,-7,10)$ are collinear.
   (*iii*) Prove that the line joining the points $(2,3,4), (0,1,2)$ is perpendicular to $(2,0,4), (7,-4,3)$.
5. The d.cs. of two rays L_1, L_2 are $(l_1, m_1, n_1), (l_2, m_2, n_2)$. If a ray is perpendicular to both L_1 and L_2, show that $m_1n_2 - m_2n_1,\ n_1l_2 - n_2l_1, l_1m_2 - l_2m_1$ are d.rs. of L.
   If $L_1 \perp L_2$, show that these d.rs. are also one of the d.cs. of L.
6. If direction ratios of $\overleftrightarrow{OP}, \overleftrightarrow{OQ}$ are $(-1,2,3), (-3,4,5)$, find d.rs. of a normal to the plane containing $\overleftrightarrow{OP}, \overleftrightarrow{OQ}$.

7. The projections of $\overline{PQ}$ on the axes are $(12,3,4)$. Find PQ.

8. (*i*) Direction ratios of two lines are determined by $l+5m+3n=0, 7l^2+5m^2-3n^2=0$. Find them.

(*ii*) Direction ratios of two lines are determined by $l+m-n=0, 6ln-12lm+mn=0$ Find d.cs. of the lines.

9. Show that the lines whose d.rs. are determined by the equations $l+m+n=0$ and $2mn+3nl-5lm=0$ are perpendicular to each other.

10. (*i*) θ is one of the angles between the lines whose d.rs. are determined by $3l+m+5n=0, 6mn-2nl+5lm=0$. Show that $\cos\theta=(1/6)$.

(*ii*) Prove that the acute angle between the lines whose d.rs. are determined by $l+m+n=0$ and $l^2+m^2-n^2=0$ is $\pi/3$.

11. L_1, L_2 are two rays whose d.rs. are determined by $al+bm+cn=0$ and $fmn+gnl+hlm=0$ Show that

(*i*) $L_1 \perp L_2 \Rightarrow \frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0$

(K.U.2004, 03, 04, N. U. S. 93, S 98,2001, O. U. O 92, S. V. U. 92, SKU 2000)

(*ii*) $L_1 \| L_2 \Rightarrow \sqrt{(af)}+\sqrt{(bg)}+\sqrt{(ch)}=0$ ***(N. U. S. 98)***

12. $\overrightarrow{OP}, \overrightarrow{OQ}, \overrightarrow{OR}$ have d.rs. $1,-1,1;\ 2,-3,0;\ 1,0,3$ respectively. Prove that $\overrightarrow{OP}, \overrightarrow{OQ}, \overrightarrow{OR}$ are coplanar.

13. Show that a line can be found perpendicular to the three lines with d.rs. $(2,1,5), (4,-2,2), (-6,4,-1)$. Hence show that if these three lines be concurrent, they are also coplanar.

14. $(l_1, m_1, n_1), (l_2, m_2, n_2)$ are d.rs. of two intersecting lines L_1 and L_2. If L_3 is any line passing through the point of intersection of L_1 and L_2 and having d.rs. $l_1+\lambda l_2,\ m_1+\lambda m_2,\ n_1+\lambda n_2$ show that L_1, L_2, L_3 are coplanar.

15. $P=(5,2,4)$. Show that the projection of $\overline{OP}$ on the line having d.cs. $\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}$ is 4.

16. Find the angles of the triangle ABC with vertices $A\ (2,3,5), B\ (-1,3,2)$ and $C\ (3,5,-2)$.

Hence find its area. [**Hint :** Area of $ABC=\frac{1}{2}|\overline{AB}\times\overline{AC}|$]

17. In a cube show that one of the angles between any two diagonals is $\text{Cos}^{-1}(1/3)$

(S. V. U. 2001 S)

18. A ray makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube. Show that

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}$. ***(A. U, A11, S. V. U. 2002A, O. 98, O. U. M. 97, N. U. 93)***

19. Show that the points

(*i*) $A\ (4,7,8), B\ (2,3,4), C\ (-1,-2,1), D\ (1,2,5)$ form a parallelogram.

(*ii*) $A\ (7,-4,7), B\ (1,-6,10), C\ (-1,-3,4), D\ (5,-1,1)$ form a rhombus.

(*iii*) $A\ (2,9,12), B\ (1,8,8), C\ (-2,11,8), D\ (-1,12,12)$ form a square.

20. $A\ (-1,2,-3), B\ (5,0,-6), C\ (0,4,-1)$ are three points. Find d.rs. of the bisectors of the angle $(\overrightarrow{AB}, \overrightarrow{AC})$.

21. Show that the point $(2,5,7)$ is the foot of the perpendicular from A in $\overleftrightarrow{BC}$ where $A=(3,-1,11), B=(0,2,3), C=(4,8,11)$.

22. O, A, B, C are four points in space such that $OA \perp BC$ and $OB \perp AC$ Prove that $OC \perp AB$.

ANSWERS

1. (*i*) $\left(\frac{6}{7},\frac{2}{7},\frac{3}{7}\right)$ (*ii*) $\left(\frac{-3}{13},\frac{12}{13},\frac{-4}{13}\right)$ **2.** $\left(\frac{1}{\sqrt{75}},\frac{5}{\sqrt{75}},\frac{-7}{\sqrt{75}}\right)$ **3.** (*i*) 2.3

(*ii*) $\left(\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}}\right)$; 4 **5.** $-1,-2,1$. **7.** 13. **8.** (*i*) $-1,2,-3;1,1,-2$

(*ii*) $\frac{1}{\sqrt{26}},\frac{3}{\sqrt{26}},\frac{4}{\sqrt{26}};\frac{-1}{\sqrt{26}},\frac{-3}{\sqrt{26}},\frac{4}{\sqrt{26}}$ $\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}},\frac{-1}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}}$

16. $\frac{\pi}{2},\cos^{-1}\sqrt{\frac{2}{3}},\cos^{-1}\frac{1}{\sqrt{3}};9\sqrt{2}$ sq. units. **20.** $24,8,5;-11,20,23$.

3

The Plane

Def : *A **Plane** is a surface such that if any two points are taken on it, the line joinig them lies wholly on the surface.*

3.1. THEOREM. ***Every equation of the first degree in x, y, z represents a plane.***

Proof. Let $ax + bx + cz + d = 0,\ a^2 + b^2 + c^2 \neq 0$ ...(1) **(*A. N. U. A11*)**

be the first degree equation in x, y, z.

If we have to show that (1) represents the equation to the plane, we prove that every point on the line joining any two points on (1) also lies on the locus (1).

Let $P(x, y, z)$ and $Q(x_2, y_2, z_2)$ be any two points on the locus (1).

Then we have $ax_1 + bx_1 + cz_1 + d = 0$ ...(2) and $ax_2 + by_2 + cz_2 + d = 0$ ...(3)

Let R be any point on the line segment joining the points P and Q. Suppose R divides PQ in the ratio K : 1.

then $R = \left(\dfrac{Kx_2 + x_1}{K+1}, \dfrac{Ky_2 + y_1}{K+1}, \dfrac{Kz_2 + z_1}{K+1}\right), K + 1 \neq 0$

We have to show that R lies on the locus (1) for all values of $K(\neq -1)$.

Substititing the coordinates of R in the LHS of (1), we get

$$\frac{a\,(Kx_2 + x_1)}{K+1} + \frac{b\,(Ky_2 + y_1)}{K+1} + \frac{c\,(Kz_2 + z_1)}{K+1} + d$$

$= a\,(Kx_2 + x_1) + b\,(Ky_2 + y_1) + c\,(Kz_2 + z_1) + d\,(K+1)$

$= K(ax_2 + by_2 + cz_2 + d) + (ax_1 + by_1 + cz_1 + d) = K(0) + 0 = 0$ which shows that R lies on the locus (1).

Since R is an arbitrary point on the line joining P and Q, every point on PQ lies on (1)

$\therefore$ The equation $ax + by + cz + d = 0$, $a^2 + b^2 + c^2 \neq 0$ always represents a plane.

3.2. CONVERSE OF THE ABOVE THEOREM.

Theorem. ***The equation to every plane is of the first degree in x, y, z.***

Proof. Let π be the plane and O be the origin.

Case(*i*). Let $O \notin \pi$, and let M be the foot of the perpendicular from O in π.

Let $OM = p\,(>0)$. Let $P(x, y, z)$ be any point in the plane.

Let $[l, m, n]$ be the Dc's of the perpendicular OM.

$P \neq M$. Join OP. OM = Projection of OP along OM

$\Rightarrow p = l\,(x-0) + m\,(y-0) + n\,(z-0) = lx + my + nz$

Case. (*ii*). Let $O \in \pi$, then $p = 0$.

$P \in \pi \Leftrightarrow lx + my + nz = 0 \Leftrightarrow lx + my + nz = p$ where $p = 0$

Since any point P on π satisfies the equation $lx+my+nz=p$ it represents the equation of the plane.

Equation to π is $lx+my+nz=p$ where $p\geq 0$.

Hence the equation to the plane $lx+my+nz=p$ is a first degree equation in x, y, z.

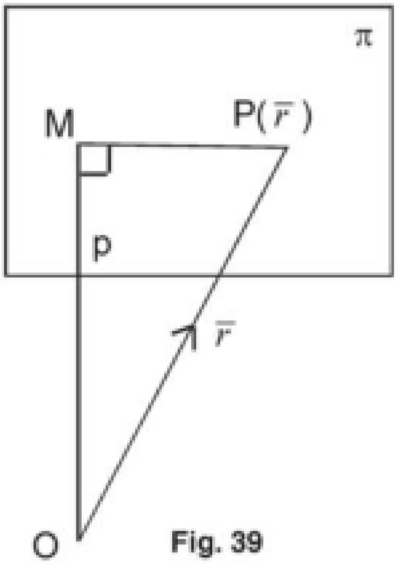


Fig. 39

Note. 1. If $O\in\pi$, equation to the plane is $lx+my+nz=0$.

2. $lx+my+nz=p$ is called the **normal form** of the equation to the plane. Coefficients of x, y, z in the equation are l, m, n and $[l, m, n]$ are Dc's of the normal OM to the plane, where $p\ (\geq 0)$ is the distance of the origin to the plane.

3. An equation to the plane, in general, is taken as $ax+by+cz+d=0$.

3.3. TRANSFORMATION OF THE EQUATION TO THE PLANE INTO THE NORMAL FORM

Let the equation to the plane be $ax+by+cz+d=0,\ a^2+b^2+c^2\neq 0$ ...(1)

We can take $d\geq 0$ or $d\leq 0$. ***(K.U.)***

$ax+by+cz+d=0 \Leftrightarrow ax+by+cz=-d$. Dividing by $\sqrt{a^2+b^2+c^2}$, we get

$$\frac{-a}{\sqrt{a^2+b^2+c^2}}+\frac{-b}{\sqrt{a^2+b^2+c^2}}+\frac{-c}{\sqrt{a^2+b^2+c^2}}=\frac{d}{\sqrt{a^2+b^2+c^2}} \quad ...(2)$$

$$\text{or } \frac{a}{\sqrt{a^2+b^2+c^2}}x+\frac{b}{\sqrt{a^2+b^2+c^2}}y+\frac{c}{\sqrt{a^2+b^2+c^2}}z=-\frac{d}{\sqrt{a^2+b^2+c^2}} \quad ...(3)$$

This ((3) or (2)) is of the form $lx+my+nz=p$ $[p\geq 0$

$$\text{where } l=\pm\frac{a}{\sqrt{\sum a^2}},\ m=\pm\frac{b}{\sqrt{\sum a^2}},\ n=\pm\frac{c}{\sqrt{\sum a^2}},\ p=\mp\frac{d}{\sqrt{\sum a^2}}\ \Big]$$

$\therefore$ The normal form of the equation to the plane (1) is

$$\pm\frac{ax}{\sqrt{\sum a^2}}\pm\frac{by}{\sqrt{\sum a^2}}\pm\frac{cz}{\sqrt{\sum a^2}}=\mp\frac{d}{\sqrt{\sum a^2}} \quad (d\leq 0 \text{ or } d\geq 0)$$

Note. 1. Derection ratios of a normal to the plane $ax+by+cz+d=0$ are (a, b, c). *i.e.,* the coefficients of x, y, z in the equation.

2. Distance of the origin from the plane $ax+by+cz+d=0$ is $\dfrac{|d|}{\sqrt{(a^2+b^2+c^2)}}$

3. First degree equation in x, y, z without constant term $\Leftrightarrow$ plane is passing through the origin.

4. $abc\neq 0$ and $\left(\frac{1}{a}\right)x+\left(\frac{1}{b}\right)y+\left(\frac{1}{c}\right)z+(-1)=0$. This equation represents a plane intersecting $x-$axis in the point $(a, 0, 0)$, intersecting $y-$axis in the point $(0, b, 0)$ and intersecting the $z-$axis in the point $(0, 0, c)$.

3.4.(1). Consider the equation $lx+my=p\ (l\neq 0, m\neq 0)$ of a plane (π), d.cs. of a normal to it being $l, m, 0$. Since 0, 0, 1 are d.cs. of $z-$axis and $l.0+m.0+0.1=0$, the normal to π

is perpendicular to z – axis *i.e.*, π is parallel to z – axis. Hence $lx + my = p$ is the equation to a plane parallel to z – axis.

Similarly, $lx + nz = p$ is the equation to a plane parallel to y – axis and $my + nz = p$ is the equation to a plane parallel to x – axis.

(2) Consider the equation $lx = p\,(l \neq 0)$ of a plane (π), d.cs. of a normal to it being $l, 0, 0$. Since 0, 1, 0 are d.cs. of y – axis and $l.0 + 0.1 + 0.0 = 0$, the normal to π is perpendicular to y – axis *i.e.*, π is parallel to y – axis. Similarly π is also parallel to z – axis. Hence π is a plane parallel to yz plane $(x = 0)$.

Similarly $my = p$ is a plane parallel to zx plane $(y = 0)$ and $nz = p$ is a plane parallel to xy plane $(z = 0)$.

3.5. Theorem. ***If the equations $a_1x+b_1y+c_1z+d_1=0, a_2x+b_2y+c_2z+d_2=0$ represent the same plane, then $a_1 : b_1 : c_1 : d_1 = a_2 : b_2 : c_2 : d_2$.***

Proof. Given equations are $a_1x + b_1y + c_1z + d_1 = 0$ ...(1)

$a_2x + b_2y + c_2z + d_2 = 0$ ...(2)

$\therefore (a_1, b_1, c_1), (a_2, b_2, c_2)$ are d.rs. of normals to the same plane.

Since the normals are either equal (coincident) or parallel,

we have $a_1 : a_2 = b_1 : b_2 = c_1 : c_2 = \lambda$ (say) $(\lambda \neq 0)$ or $(a_1, b_1, c_1) = \lambda\,(a_2, b_2, c_2)$

Let (x_1, y_1, z_1) be any point in the plane represented by (1) and (2).

$\therefore\ d_1 = -(a_1x_1 + b_1y_1 + c_1z_1)$

$= -(a_1, b_1, c_1).(x_1, y_1, z_1) = -\lambda\,(a_2, b_2, c_2).(x_1, y_1, z_1) = -\lambda\,(a_2x_1 + b_2y_1 + c_2z_1) = -\lambda d_2$

$\therefore\ a_1 : a_2 = b_1 : b_2 = c_1 : c_2 = d_1 : d_2$

3.6. ANGLES BETWEEN TWO PLANES

Definition. *Angles between two planes are equal to the angles between their normals.*

Angles between the planes $a_1x + b_1y + c_1z = d_1, a_2x + b_2y + c_2z = d_2$

Let the equation to the planes be

$a_1x + b_1y + c_1z + d_1 = 0$ ...(1) $a_2x + b_2y + c_2z + d_2 = 0$ ...(2)

Dc's of the normal to (1) =

$$m_1 = \left(\frac{a_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}}, \frac{b_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}}, \frac{c_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}}\right) \text{ and}$$

Dc's of the normal to (2) =

$$m_2 = \left(\frac{a_2}{\sqrt{(a_2^2 + b_2^2 + c_2^2)}}, \frac{b_2}{\sqrt{(a_2^2 + b_2^2 + c_2^2)}}, \frac{c_2}{\sqrt{(a_2^2 + b_2^2 + c_2^2)}}\right)$$

Let θ be one of the angles between the planes.

$\therefore\ \theta =$ one of the angles between the normals m_1, m_2

$$= \cos^{-1}\left(\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}\right)$$

The other angle between the planes is $180^\circ - \theta$.

Cor. 1. ***Condition of parallelism.***

Planes are parallel $\Rightarrow \theta = 0^\circ$ or $180^\circ \Rightarrow \pm 1 = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$

$\Rightarrow (a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) = (a_1a_2 + b_1b_2 + c_1c_2)^2$

$\Rightarrow a_1^2b_2^2 + a_1^2c_2^2 + b_1^2a_2^2 + b_1^2c_2^2 + c_1^2a_2^2 + c_1^2b_2^2 - 2a_1a_2b_1b_2 - 2b_1b_2c_1c_2 - 2c_1c_2a_1a_2 = 0$

$\Rightarrow (a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 = 0$

$\Rightarrow a_1b_2 - a_2b_1 = b_1c_2 - b_2c_1 = c_1a_2 - c_2a_1 = 0 \Rightarrow a_1 : a_2 = b_1 : b_2 = c_1 : c_2$

OR : Planes are parallel $\Rightarrow$ their normals are parallel $\Rightarrow$ d.rs of normals are proportional $\Rightarrow a_1 : a_2 = b_1 : b_2 = c_1 : c_2$.

Cor. 2. ***Condition of perpendicularity.***

Planes are perpendicular $\Rightarrow \theta = 90^\circ \Rightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$.

e.g. The plane $x + 2y - 3z + 4 = 0$ is perpendicular to the plane $2x + 5y + 4z + 1 = 0$ since $(1)(2) + (2)(5) + (-3)(4) = 0$

OR : Planes are perpendicular $\Rightarrow$ their normals are perpendicular

$\Rightarrow (a_1, b_1, c_1).(a_2, b_2, c_2) = 0 \Rightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$

Note. 1. The equations $a_1x + b_1y + c_1z + d_1 = 0$, $a_1x + b_1y + c_1z + d_2 = 0$ represent a pair of parallel planes.

2. A plane parallel to $ax + by + cz + d = 0$ is $ax + by + cz + d = k$, where k is an unknown real number.

***e.g.* 1.** The equation of the plane through the point (x_1, y_1, z_1) and parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz = ax_1 + by_1 + cz_1$.

***e.g.* 2.** The normals to the plane as $x - y + z - 1 = 0, 3x + 2y - z + 2 = 0$ are perpendicular since $(1)(3) + (-1)(2) + 1(-1) = 0$.

3.7. DETERMINATION OF A PLANE UNDER GIVEN CONDITIONS

Consider the equation $ax + by + cz + d = 0$ of a plane. Since $(a, b, c) \neq (0, 0, 0)$, without loss of generally we can take $a \neq 0$.

$\therefore$ Equation of the plane is $x + \frac{b}{a}y + \frac{c}{a}z + \frac{d}{a} = 0$

$\therefore$ To know uniquely $\frac{b}{a}, \frac{c}{a}, \frac{d}{a}$ we require three conditions.

For example, we can find the equation to a plane, if (*i*) three non-collinear points in the plane are given. (*ii*) if two points in the plane and a plane perpendicular to the required plane (*iii*) if one point in the plane and two planes perpendicular to the required plane are given.

3.8. Definition. *If a plane* π *intersects the cordinate axes at* $(a, 0, 0), (0, b, 0), (0, 0, c)$ *then* a, b, c *are respectively called the* $x-$*intercept, the* $y-$*intercept, the* $z-$*intercept of the plane* π.

If the plane $lx + my + nz = p$ intersects the $x-$axis at $(a, 0, 0)$, then its $x-$intercept $= a = \frac{p}{l}$.

Similarly its $y-$intercept $= b = \dfrac{p}{m}$, its $z-$intercept $= c = \dfrac{p}{n}$

Note. If $abc \neq 0$ and $ax + by + cz + d = 0$ ..... (1) is a plane,

then its $x-$intercept $= -\dfrac{d}{a}$, (by putting $y = 0, z = 0$ in (1)

$y-$intercept $= -\dfrac{d}{b}$, $z-$intercept $= -\dfrac{d}{c}$.

e.g. The intercepts made by the plane $x - 12y - 2z = 9$ with the axes are $\dfrac{9}{1}, -\dfrac{9}{12}, -\dfrac{9}{2}$, *i.e.*, $9, -\dfrac{3}{4}, -\dfrac{9}{2}$.

3.9. Theorem. ***Equation to the plane making intercepts a, b, c on the coordinate axes is*** $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$.

Proof. Let π be the plane making intercepts,

a, b, c on the coordinate axes (Fig.40)

Let $A = (a, 0, 0)$, $B = (0, b, 0)$ and

$C = (0, 0, c)$. $\therefore\ abc \neq 0$.

Clearly, A, B, C are non-collinear.

Let the equation to the plane π in the normal form be $lx + my + nz = p$ ...(1)

Let M be the foot of the perpndicular from O to π and let $[l, m, n]$ be the Dc's of OM. Let OM $= p$.

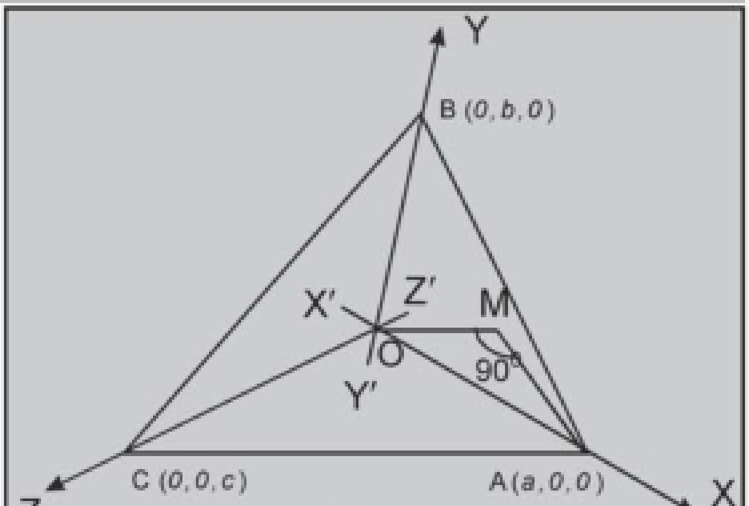


Fig. 40

$p = \text{OM} = $ Projection of OA on OM $= al$.

Similarly $p = bm$, and $p = cn$. $\therefore$ From (1)

equation to the plane π is $\dfrac{p}{a}x + \dfrac{p}{b}y + \dfrac{p}{c}z = p \Rightarrow \dfrac{x}{a} + \dfrac{x}{b} + \dfrac{z}{c} = 1 \quad (\because p \neq 0)$

Note. Equation to the plane ABC is $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$. This is called the intercept form of the equation to the plane and this plane does not pass through the origin.

OR : Let $A = (a, 0, 0)$, $B = (0, b, 0)$, $C = (0, 0, c)$. Let $P = \bar{r} = (x, y, z)$.

Now A, B, C are non-collinear. $\quad P \in \overleftrightarrow{ABC}$ $(P \neq A$ or $P = A)$

$\Leftrightarrow$ $\overline{AP}, \overline{AB}, \overline{AC}$ are coplanar or $\overline{AP}\,(=\bar{0}), \overline{AB}, \overline{AC}$ are three vectors.

$\Leftrightarrow \bar{r} - \bar{a}\,(\neq \bar{0}), \bar{b} - \bar{a}, \bar{c} - \bar{a}$ are coplanar or $\bar{r} - \bar{a}\,(= \bar{0}), \bar{b} - \bar{a}, \bar{c} - \bar{a}$ are three vectors.

$\Leftrightarrow [(x - a, y, z), (-a, b, 0), (-a, 0, c)] = 0$

$$\Leftrightarrow \begin{vmatrix} x-a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = 0 \Leftrightarrow (x-a)\,bc + y\,ac + z\,ab = 0 \Leftrightarrow \frac{x}{a} - 1 + \frac{y}{b} + \frac{z}{c} = 0 \qquad (\because abc \neq 0)$$

$\Leftrightarrow \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. $\therefore$ Equation to $\overleftrightarrow{ABC}$ is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Note. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ *is called the* ***intercept form*** *of the equation of the plane and this plane does not pass through the origin.*

3.10. Theorem. *Equation to the plane determined by three non-collinear points* $A(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3)$ *is* $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0$

Proof. Let the equation of the required plane be $ax+by+cz+d=0$ ...(1)

This passes through the given points if $ax_1+by_1+cz_1+d=0$ ...(2)

$ax_2+by_2+cz_2+d=0$ ...(3) $ax_3+by_3+cz_3+d=0$ ...(4)

Eliminating a, b, c, d from the above equations (1), (2), (3), (4), we have

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0 \quad ...(5)$$

This is the equation to the required plane.

OR :

Proof. Let $A=\overline{a}=(x_1, y_1, z_1)$, $B=\overline{b}=(x_2, y_2, z_2)$, $C=\overline{c}=(x_3, y_3, z_3)$

Let $\overline{r}=(x, y, z)$. $\overline{r}\,(\neq \overline{a}) \in \pi$, $\overline{r}\,(=\overline{a}) \in \pi$

$\Leftrightarrow \overline{r}-\overline{a}, \overline{b}-\overline{a}, \overline{c}-\overline{a}$ are coplanar or $\overline{r}-\overline{a}\,(=\overline{0}), \overline{b}-\overline{a}, \overline{c}-\overline{a}$ are three vectors

$\Leftrightarrow [\overline{r}-\overline{a}, \overline{b}-\overline{a}, \overline{c}-\overline{a}]=0$

$\Leftrightarrow [(x-x_1, y-y_1, z-z_1), (x_2-x_1, y_2-y_1, z_2-z_1), (x_3-x_1, y_3-y_1, z_3-z_1)]=0$

$$\Leftrightarrow \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0 \quad ...(1)$$

$$\Leftrightarrow \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 & 0 \\ x_1 & y_1 & z_1 & 1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 & 0 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 & 0 \end{vmatrix}=0 \quad \Leftrightarrow \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0 \;\begin{vmatrix} R_2+R_1 \\ R_3+R_1 \\ R_4+R_1 \end{vmatrix}$$

Note. Equation (1) may also be taken as the equation of the required plane.

Note. If the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) are such that

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}=0$$, then the points are coplanar.

If $\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \neq 0$, then the points are non-coplanar.

3.11. Theorem. ***Equation to the plane (π) through the point A*** (x_1, y_1, z_1) ***and perpendicular to line (L) with d.rs. (a, b, c) is*** $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$.

Proof : (Fig. 41). Let $P \in \pi$ and $P = (x, y, z)$. $A = (x_1, y_1, z_1)$ and d.rs. of L are (a, b, c)

Now d.rs. of $AP = (x - x_1, y - y_1, z - z_1)$.

$\overleftrightarrow{AP} \in \pi \Leftrightarrow \overleftrightarrow{AP} \perp L$

$\Leftrightarrow a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

$\therefore$ Equation to π is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$.

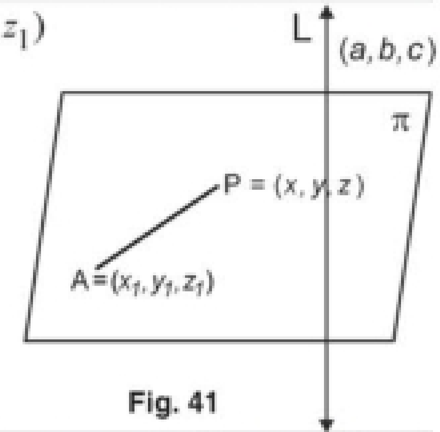


Fig. 41

3.12. PARAMETRIC EQUATION OF A PLANE

Theorem. ***Equation to the plane passing through three non-collinear points A(a), B(b), C(c) is*** $r = (1 - t - s)\ a + sb + tc$, ***where s, t are any scalars (real numbers).***

Proof : $\bar{r} \in \overleftrightarrow{ABC} \Rightarrow \bar{r} - \bar{a}, \bar{b} - \bar{a}, \bar{c} - \bar{a}$ are coplanar ;

or $\bar{r} - \bar{a} (\neq 0), \bar{b} - \bar{a}, \bar{c} - \bar{a}$ are three vectors

$\Leftrightarrow \bar{r} - \bar{a} = s(\bar{b} - \bar{a}) + t(\bar{c} - \bar{a})$ where s, t are any scalars

$\Leftrightarrow \bar{r} = (1 - s - t)\ \bar{a} + s\bar{b} + t\bar{c}$ is the equation of the plane through $\bar{a}, \bar{b}, \bar{c}$

Note. Let $\bar{r} = (x, y, z), \bar{a} = (x_1, y_1, z_1), \bar{b} = (x_2, y_2, z_2), \bar{c} = (x_3, y_3, z_3)$

Then parametic equation to the plane $\overleftrightarrow{ABC}$ is

$(x, y, z) = (1 - s - t)(x_1, y_1, z_1) + s(x_2, y_2, z_2) + t(x_3, y_3, z_3)$

i.e. $x = x_1 + s(x_2 - x_1) + t(x_3 - x_1)$, $\quad y = y_1 + s(y_2 - y_1) + t(y_3 - y_1)$

$z = z_1 + s(z_2 - z_1) + t(z_3 - z_1)$

Cor. Points $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are coplanar

$\Leftrightarrow \bar{d} = (1 - s - t)\ \bar{a} + s\ \bar{b} + t\ \bar{c} \Leftrightarrow (1 - s - t)\ \bar{a} + s\ \bar{b} + t\ \bar{c} + (-1)\ \bar{d} = 0$

$\Leftrightarrow \lambda_1\bar{a} + \lambda_2\bar{b} + \lambda_3\bar{c} + \lambda_4\bar{d} = 0$,

$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 - s - t + s + t - 1 = 0$ and $(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) \neq (0, 0, 0, 0)$

3.13. TWO SIDES OF A PLANE

A plane π divides the space into two half spaces. Let a line $\overleftrightarrow{AB}$ intersect π in C. If A, B are in the same half space, then (C; A, B) is negative and A, B lie on the same side of C. If A, B are in different half spaces, then (C ; A, B) is positive and A, B lie on either side of C.

Theorem. ***If the line through*** $A(x_1, y_1, z_1)$ ***and*** $B(x_2, y_2, z_2)$ ***intersect the plane ax + by + cz + d = 0 in C, then*** $(\mathbf{C; A, B}) = -(ax_1 + by_1 + cz_1 + d) : (ax_2 + by_2 + cz_2 + d)$.

Proof : (Fig.42) Let $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ and C divide AB in the ratio $\lambda_1 : \lambda_2 (\lambda_1 + \lambda_2 \neq 0)$.

$\therefore$ (C; A, B) = $\lambda_1 : \lambda_2 (\lambda_1 + \lambda_2 \neq 0)$ and $C = \left(\frac{\lambda_2 x_1 + \lambda_1 x_2}{\lambda_1 + \lambda_2}, \frac{\lambda_2 y_1 + \lambda_1 y_2}{\lambda_1 + \lambda_2}, \frac{\lambda_2 z_1 + \lambda_1 z_2}{\lambda_1 + \lambda_2}\right)$

$$C \in \pi \Leftrightarrow a\left(\frac{\lambda_2 x_1 + \lambda_1 x_2}{\lambda_1 + \lambda_2}\right) + b\left(\frac{\lambda_2 y_1 + \lambda_1 y_2}{\lambda_1 + \lambda_2}\right) + c\left(\frac{\lambda_2 z_1 + \lambda_1 z_2}{\lambda_1 + \lambda_2}\right) + d = 0$$

$\Leftrightarrow \lambda_2(ax_1 + by_1 + cz_1 + d) + \lambda_1(ax_2 + by_2 + cz_2 + d) = 0$

$\Leftrightarrow \lambda_1(ax_2 + by_2 + cz_2 + d) = -\lambda_2(ax_1 + by_1 + cz_1 + d)$

$\Leftrightarrow \lambda_1 : \lambda_2 = -(ax_1 + by_1 + cz_1 + d) : (ax_2 + by_2 + cz_2 + d)$

$\Leftrightarrow$ (C; A, B) $= -(ax_1 + by_1 + cz_1 + d) : (ax_2 + by_2 + cz_2 + d)$

OR :

Proof : (Fig. 42). Let $A = \bar{a} = (x_1, y_1, z_1)$ and $B = \bar{b} = (x_2, y_2, z_2)$

Let $C = \bar{c}$ divide AB in the ratio $\lambda_1 : \lambda_2$.

$\therefore$ (C; A, B) $= \lambda_1 : \lambda_2 (\lambda_1 + \lambda_2 \neq 0)$

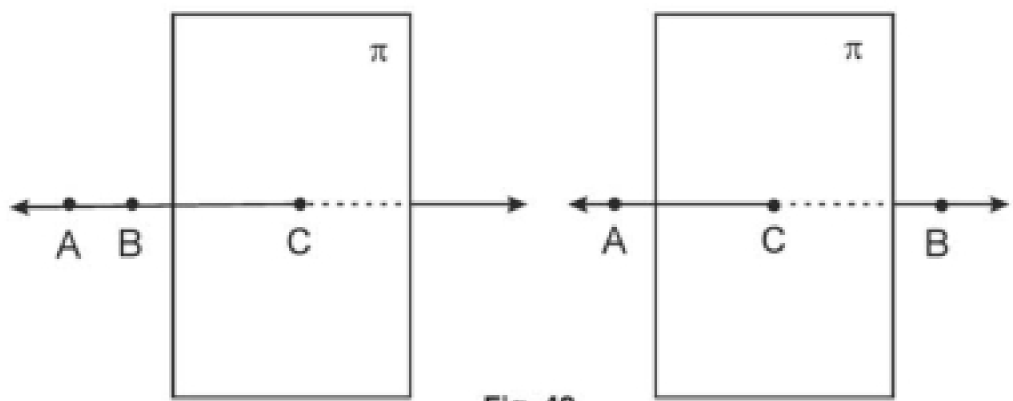


Fig. 42

Then $\bar{c} = \frac{\lambda_2 \bar{a} + \lambda_1 \bar{b}}{\lambda_1 + \lambda_2}$

Let the plane represented by $ax + by + cz + d = 0$ be $\bar{r} . \bar{m} = q$ where

$\bar{r} = (x, y, z), \bar{m} = (a, b, c)$ and $q = -d$

$\therefore C \in \pi \Leftrightarrow \bar{c} . \bar{m} - q = 0 \quad \Leftrightarrow \left(\frac{\lambda_2 \bar{a} + \lambda_1 \bar{b}}{\lambda_1 + \lambda_2}\right) . \bar{m} - q = 0$

$\Leftrightarrow \lambda_2(\bar{a} . \bar{m}) + \lambda_1(\bar{b} . \bar{m}) = q\lambda_1 + q\lambda_2 \Leftrightarrow \lambda_1(\bar{b} . \bar{m} - q) = -\lambda_2(\bar{a} . \bar{m} - q)$

$\Leftrightarrow \lambda_1 : \lambda_2 = -(\bar{a} . \bar{m} - q) : (\bar{b} . \bar{m} - q)$

$\Leftrightarrow \lambda_1 : \lambda_2 = -\{(x_1, y_1, z_1) . (a, b, c) + d\} : \{(x_2, y_2, z_2) . (a, b, c) + d\}$

$\Leftrightarrow \lambda_1 : \lambda_2 = -(ax_1 + by_1 + cz_1 + d) : (ax_2 + by_2 + cz_2 + d)$

$\Leftrightarrow$ (C; A, B) $= -(ax_1 + by_1 + cz_1 + d) : (ax_2 + by_2 + cz_2 + d)$

Note 1. A, B lie in the same half space.

$\Leftrightarrow ax_1 + by_1 + cz_1 + d,\ ax_2 + by_2 + cz_2 + d$ are of the same sign and

A, B lie in the different half spaces.

$\Leftrightarrow ax_1 + by_1 + cz_1 + d,\ ax_2 + by_2 + cz_2 + d$ are of the different signs.

e.g. 1. The points $(2, 3, 5), (0, 4, -7)$ lie in the different half spaces (on the opposite sides) of the plane $x + 2y + 2z - 9 = 0$ since $2 + 2(3) + 2(5) - 9 > 0$ and $0 + 2(4) + 2(-7) - 9 < 0$.

2. The points $(1, 2, -5), (0, 4, -7)$ lie in the same half space (on the same side) of the plane $x + 2y + 2z - 9 = 0$ since $(1) + 2(2) + 2(-5) - 9 < 0$ and $0 + 2(4) + 2(-7) - 9 < 0$.

3. The points $(1, -1, 3)$ and $(3, 3, 3)$ lie on different sides of the plane $x+2y-7z+9=0$ since $5(1)+2(-1)-7(3)+9=-9<0$ and $5(3)+2(3)-7(3)+9=9>0$. ***(N.U.M.98)***

3.14. PERPENDICULAR DISTANCE OF A POINT FROM A PLANE

Theorem. ***The distance of*** $A(x_1, y_1, z_1)$ ***from the plane*** $ax+by+cz+d=0$ ***i.e. length of the perpendicular from the point*** $A(x_1, y_1, z_1)$ ***to the plane*** $ax+by+cz+d=0$ ***is*** $\dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{(a^2+b^2+c^2)}}$. ***(O.U.Oct. 2001)***

Proof : Let π be the plane $ax+by+cz+d=0$ ...(1)

Let $A=(x_1, y_1, z_1)$ be the point $(A \notin \pi)$ from which the perpendicular drawn to the plane π meets it in C.

Let the normal form of π be $lx+my+nz=p$ ...(2)

the equation to the plane parallel to (2) and passing through the point A be

$lx+my+nz=p_1$ ...(3) where $lx_1+my_1+nz_1=p_1$ ...(4)

Let ODE be perpendicular to (2) and (3) as shown.

$\Rightarrow$ AC $= p_1 - p$

$\perp r$ distance of A to the plane π

$=$ AC $=$ OE $-$ OD $= lx_1+my_1+nz_1-p$

$$= +\frac{a}{\sqrt{\Sigma a^2}}x_1 + \frac{b}{\sqrt{\Sigma a^2}}y_1 + \frac{c}{\sqrt{\Sigma a^2}}z_1 \pm \frac{d}{\sqrt{\Sigma a^2}}$$

i.e., $$\pm\frac{(ax_1+by_1+cz_1+d)}{\sqrt{a^2+b^2+c^2}} \text{ or } \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

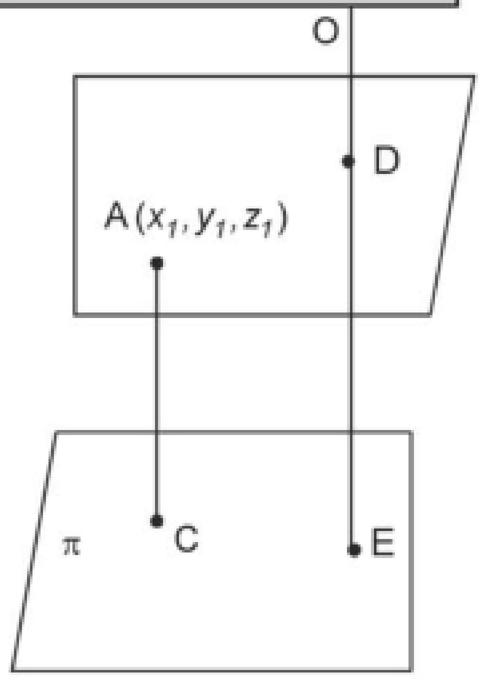


Fig. 43

OR : Proof : (Fig. 44). Let $A=\bar{a}=(x_1, y_1, z_1)$.

Let π be the plane $ax+by+cz+d=0$.

Equation to π can be taken as $\bar{r}.\bar{m}=q$ where $\bar{r}=(x, y, z)$.

$\bar{m}=(a, b, c)$ and $q=-d$.

Now, $|\bar{m}|=\sqrt{(a^2+b^2+c^2)}$.

Let C be the foot of the perpndicular from A to π.

Let $B(\neq C)$ be $\bar{b}$ in π. $\quad\therefore\ \bar{b}.\bar{m}=q$ ...(1)

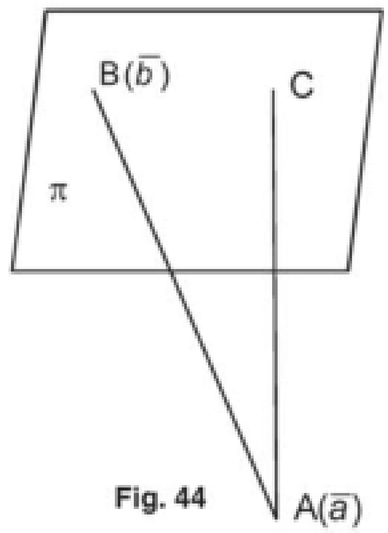


Fig. 44

$$\therefore\ AC = \frac{|\overline{AB}.\overline{AC}|}{|\overline{AC}|} = \frac{|(\bar{b}-\bar{a}).\bar{m}|}{|\bar{m}|} = \frac{|\bar{b}.\bar{m}-\bar{a}.\bar{m}|}{|\bar{m}|} = \frac{|\bar{a}.\bar{m}-\bar{b}.\bar{m}|}{|\bar{m}|} = \frac{|\bar{a}.\bar{m}-\bar{q}|}{|\bar{m}|}$$

$$= \frac{|(x_1, y_1, z_1).(a, b, c)-q|}{\sqrt{a^2+b^2+c^2}} = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

e.g. The distances of the points $(2,3,4)$ and $(1,1,4)$ from the plane $3x-6y+2z+11=0$

$$=\left|\frac{3(2)-6(3)+2(4)+11}{\sqrt{(9+36+4)}}\right|=1 \text{ and } \left|\frac{3-6+8+11}{\sqrt{9+36+4}}\right|=\frac{16}{7} \quad \textbf{(S. V.. U. A12)}$$

3.15. DISTANCE BETWEEN PARALLEL PLANES

Theorem. ***Distance between parallel planes***

$$ax+by+cz+d_1=0,\ ax+by+cz+d_2=0 \ \textit{is}\ \frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}},\quad d_1<0,\ d_2<0.$$

Proof. The equations to the planes are

$ax+by+cz+d_1=0$ ...(1) $\qquad ax+by+cz+d_2=0$ ...(2)

the Dc's of the normal to the planes are $\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$

Now p_1 and p_2 be the perpendicular distances to the planes from the origin

$$\Rightarrow p_1=\frac{-d_1}{\sqrt{a^2+b^2+c^2}},\ p_2=\frac{-d_2}{\sqrt{a^2+b^2+c^2}}$$

$\Rightarrow$ Distance between the parallel planes $=|p_1-p_2|=\dfrac{|d_2-d_1|}{\sqrt{a^2+b^2+c^2}}$.

Note. Equation to the plane parallel to (1) and (2) and midway between (1) and (2)

(*i*) When origin lies on the same side of both (1) and (2) is *i.e.,* $ax+by+cz=\dfrac{-(d_1+d_2)}{2}$

(*ii*) When origin lies in between (1) and (2) is $ax+by+cz=\dfrac{|d_1-d_2|}{2}$

e.g. The distance between the planes $2x-y+3z=6$ and $-6x+3y-9z=5$

= The distance between the planes $2x-y+3z=6$ and $2x-y+3z=-\dfrac{5}{3}$

$$=\frac{\left|6-(-\frac{5}{3})\right|}{\sqrt{(4+1+9)}}=\frac{23}{3\sqrt{14}}$$

SOLVED PROBLEMS

Ex. 1. *Find the point P equidistant from A (4, –3, 7) and B (2, –1, 1) and lying on $y-$axis. Hence find the equation to the plane through P and perpendicular to $\overleftrightarrow{AB}$.*

Sol. Let $P=(0,y,0)$. $\qquad$ Since $PA=PB, PA^2=PB^2$

$\Rightarrow (0-4)^2+(y+3)^2+(0-7)^2=4+(y+1)^2+1 \Rightarrow 4y=-68 \Rightarrow y=-17$

$\therefore$ $P=(0,-17,0)$ and d.rs. of $\overleftrightarrow{AB}$ are $2,-2,6$.

$\therefore$ Equation to the plane through P, and perpendicular $\overleftrightarrow{AB}$ is (Art. 3.11)

$2(x-0)-2(y+17)+6(z-0)=0$ *i.e.,* $2x-2y+6z-34=0$, *i.e.,* $x-y+3z-17=0$

Ex. 2. *Show that the line joining the points* $(6,-4,4),(0,0,-4)$ *intersects the line joining the points* $(-1,-2,-3),(1,2,-5)$. ***(A. U. M 2014)***

Sol. Let $A=(6,-4,4)$, $B=(0,0,-4)$, $C=(-1,-2,-3)$, $D=(1,2,-5)$

$\therefore$ $\overline{AB}=(-6,4,-8)$ and $\overline{CD}=(2,4,-2)$

$\overline{AB}$ is neither perpendicular not parallel to $\overline{CD}$.

$\therefore$ If we prove that A, B, C, D are coplanar, then $\overleftrightarrow{AB}$ intersects $\overleftrightarrow{CD}$.

Now equation to the plane $\overleftrightarrow{ABC}$ is $\begin{vmatrix} x-6 & y+4 & z-4 \\ -6 & 4 & -8 \\ -7 & 2 & -7 \end{vmatrix} = 0$

$\Rightarrow (-28+16)(x-6)-(42-56)(y+4)+(-12+28)(z-4)=0$

$\Rightarrow 6(x-6)-7(y+4)-8(z-4)=0 \Rightarrow 6x-7y-8z-32=0$ ...(1)

Substituting D in the L.H.S. of (1), we get $6-14+40-32=0=$ R.H.S.

$\therefore$ D $\in \overleftrightarrow{ABC}$ $\qquad \therefore$ $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect.

Ex. 3. *Obtain the equation to the plane containing (0, 4, 3) and the line through the points* $(-1,-5,-3), (-2,-2,1)$. *Hence show that* $(0,4,3), (-1,-5,-3), (-2,-2,1)$ *and* $(1,1,-1)$ *are coplanar.* **(*S.V. U.A. 93*)**

Sol. Let π be the required plane. Let a, b, c be d.rs. of a normal to it.

Let $A=(0,4,3),\ B=(-1,-5,-3),\ C=(-2,-2,1)$

D.rs. of $\overleftrightarrow{AB}$ are $-1,-9,-6$ and d.rs. of $\overleftrightarrow{AC}$ are $-2,-6,-2$.

Since $\overleftrightarrow{AB}, \overleftrightarrow{AC}$ are in π,

$\left.\begin{array}{r} -a-9b-6c=0 \\ -2a-6b-2c=0 \end{array}\right\}$. Solving, $\dfrac{a}{-18}=\dfrac{b}{10}=\dfrac{c}{-12}$ *i.e.,* $\dfrac{a}{9}=\dfrac{b}{-5}=\dfrac{c}{6}$

$\therefore$ Equation to π is $9(x-0)-5(y-4)+6(z-3)=0$ *i.e.,* $9x-5y+6z+2=0$ ...(1)

Clearly $(1,1,-1)$ lies on (1). Hence the points are coplanar.

$\therefore$ Equation to the plane containing the points is (1).

Ex. 4. *Find the equation of the plane through* (4, 4, 0) *and perpendicular to the planes* $x+2y+2z=5$ *and* $3x+3y+2z-8=0$.

(*A.N.U. 06, 07, M13, O.U. A12, K.U. A12, A.U. M18, K.U. M18*)

Sol. Let π be the required plane. Let a, b, c be d.rs of normal to π. Since π passes through (4, 4, 0), equation to π is $a(x-4)+b(y-4)+c(z-0)=0$

But π is perpendicular to $x+2y+2z=5$ and $3x+3y+2z-8=0$

$\therefore \left.\begin{array}{r} a+2b+2c=0 \\ 3a+3b+2c=0 \end{array}\right\}$ $\quad \therefore \dfrac{a}{-2}=\dfrac{b}{4}=\dfrac{c}{-3}$.

$\therefore$ Equation to π is $-2(x-4)+4(y-4)-3z=0$ [using (1)] *i.e.,* $2x-4y+3z+8=0$.

Ex. 5. *Find the equation of the plane passing through* $(1,0,-2)$ *and perpendicular to the planes* $2x+y-2=z; x-y-z=3$

(*A.N.U. M15, S.K.U. 2001 S, M13, A.U. M14, S.K.U M18, V.S.P M18*)

Sol. Let π be the required plane. Let a, b, c are the drs. of the above plane.

Equation of the plane passing through $(1,0,-2)$ and having a, b, c as drs. is

$a(x-1)+b(y-0)+c(z+2)=0 \Rightarrow a(x-1)+by+c(z+2)=0$ ...(1)

But the π plane is perpendicular to the planes $2x+y-z=2$ and $x-y-z=3$.

$\therefore 2a+b-c=0$ ...(2), $\quad a-b-c=0$ ...(3) $\quad$ Solving (2) and (3) $\dfrac{a}{2}=\dfrac{b}{-1}=\dfrac{c}{3}$.

Equation of the π plane is $2(x-1)-y+3(z+2)=0$ *i.e.,* $2x-y+3z+4=0$.

Ex. 6. *Find the angles between the planes* $2x-3y-6z=6$ *and* $6x+3y-2z=18$.

(*A. U. M 13, M 14*)

Sol. Let θ be one of the angles between the given planes.

$$\therefore\ \theta = \cos^{-1}\frac{2(6)-3(3)-6(-2)}{\sqrt{(4+9+36)}\sqrt{(36+9+4)}} = \cos^{-1}\left(\frac{15}{49}\right)$$

The other angle between the planes is $180^\circ - \theta$ *i.e.,* $180^\circ - \cos^{-1}\left(\frac{15}{49}\right)$

Ex. 7. *Find the locus of the point whose distance from the origin is three times its distance from the plane* $2x-y+2z=3$. ***(A. U. A11, K.U.)***

Sol. Let O be the origin and P be the point (x_1, y_1, z_1) such that OP is equal to 3 times its distance from the plane $2x-y+2z=3$

$$\therefore\ OP^2 = 9.\frac{(2x_1-y_1+2z_1-3)^2}{4+1+4}$$

$$\Rightarrow x_1^2+y_1^2+z_1^2 = 4x_1^2+y_1^2+4z_1^2+9-4x_1y_1-4y_1z_1+8x_1z_1-12x_1+6y_1-12z_1$$

$$\Rightarrow 3x_1^2+3z_1^2-4x_1y_1-4y_1z_1+8x_1z_1-12x_1+6y_1-12z_1+9=0$$

$\therefore$ Locus of P is $3x^2+3z^2-4xy-4yz+8xy-12x+6y-12z+9=0$

EXERCISE 3 (*a*)

1. Find the intercepts of the plane $2x-3y+4z=12$ on the coordinate axes.
2. What are d.rs. of a normal to the plane $2x-2y+z=5$. Express the equation of the plane in its normal form.
3. (*i*) L, M, are respectively the feet of the perpendiculars from P (a,b,c) to YZ and ZX planes. Find the equation of the plane $\overleftrightarrow{OLM}$.
   (*ii*) Find the equation of the plane containing the lines through the origin with d.c.s proportional to $(1,-2,2)$ and $(2,3,-1)$. ***(O. U. 07)***
4. Foot of the perpendicular for the origin to a plane is $(2,-3,4)$. Find the equation to the plane.
5. O is the origin and A is the point (a,b,c). Find the d.cs. of $\overleftrightarrow{OA}$ and deduce the equation of the plane through A and at right angles to $\overleftrightarrow{OA}$.
6. Find the equation to the plane through the point $(4,0,1)$ and parallel to the plane $4x+3y-12z+8=0$.
7. Determine the constant *k*, so that the planes $x-2y+kz=0$ and $2x+5y-z=0$ are at right angles. Find the equation to the plane through the point $(1,-1,-1)$ and perpendicular to the above planes.
8. (*i*) Find the equation to the plane through the points $(2,2,1), (9,3,6)$ and perpendicular to the plane $2x+6y+6z=9$ ***(K.U. M15, S.K.U. M15, A.U.95, N.U.97, S.V. U. 2002 A, 93)***
   (*ii*) Show that the equation of the plane passing through the points $(1,-2,4), (3,-4,5)$ and perpendicular to XY plane is $x+y+1=0$. ***(S.V. M. M18)***
   (*iii*) Find the equation of the plane through $A=(-1,1,1), B=(1,-1,1)$ and perpendicular to the plane $x+2y+2z=5$ ***(O.U. M14, K.U. M14)***
9. Find the equation of the plane through the point $(-1,3,2)$ and perpendicular to the planes $x+2y+2z=5$ and $3x+3y+2z=8$.
   (K.U. 08, O.U. 07, A.U. 84, S.V. U. 01 S, S.V.U. A 93, 95, A.U.M 18)
10. Prove that the equation of the plane through the points $(1,-2,4)$ and $(3,-4,5)$ and parallel to x – axis is $y+2z=6$.

11. Find the equation to the plane through the point $(2,3,-1)$ and is perpendicular to the line through the points $(3,4,-1)$ and $(2,-1,5)$. ***(O. U. 07, A. U. 95, 12)***

12. Find the equation to the plane through $(-1,6,2)$ and perpendicular to the join of the points $(1,2,3)$ and $(-2,3,4)$.

13. (*i*) Find the equation to the plane bisecting the line segment joining $(2,0,6)$ and $(-6,2,4)$ and perpendicular to the line segment.

(*ii*) Find the equation of the plane which bisects the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2) perpendicularly.

14. (*i*) Show that the equation of the plane through the points $(2,2,-1), (3,4,2), (7,0,6)$ is $5x+2y-3z-17=0$ ***(S. K. U. 12, N. U. O88, O. U. M 13)***

(*ii*) Find the equation to the plane through the points $(1,1,1)$, $(1,-1,1)$ and $(-7,-3,-5)$. Show that it is parallel to y- axis.

15. Show that the following points are coplanar.

(*i*) $(-6,3,2), (-13,17,-1), (3,-2,4), (5,7,3)$ ***(O 97)*** (*ii*) $(2,2,-1), (7,0,6), (3,4,2), (0,4,-3)$

16. Find the equation to the plane containing the points $\text{A} = (3,2,-5)$, $\text{B} = (-3,8,-5)$, $\text{C} = (-3,2,1)$. Show that the point $(-1,4,-3)$ is the circumcentre of the Δ ABC .Also find its centroid.

17. Find the angles between the planes.

(*i*) $x+2y+3z=5, 3x+3y+z=9$ (*ii*) $2x-3y+4z+11=0,\ 3x-2y-3z+27=0$

(*iii*) $2x-y+z=0, x+y+2z=7$ ***(N.U. 96, 99, 2000, 02, 04, V.S.P M18)***

(*iv*) $x+y-5z+6=0,\ 2x+3y+6z+5=0$ (*v*) $3x-4y+5z=0,\ 2x-y-2z=5$ ***(N. U. M 98)***

18. Find whether the following points lie in the same half space or different half spaces of the plane $2x-3y+4z+5=0$. (*i*) $\text{A} = (1,2,0)$, $\text{B} = (1,2,-3)$ (*ii*) $\text{A} = (-2,1,2)$, $\text{B} = (3,1,2)$

If $\overleftrightarrow{\text{AB}}$ meets the plane π in C, find in each case $(\text{C}; \text{A}, \text{B})$.

19. Two systems of rectangular axes have the same origin. If a plane intersects them at distances, a, b, c and a_1, b_1, c_1 respectively from the origin, prove that

$$a^{-2}+b^{-2}+c^{-2}=a_1^{-2}+b_1^{-2}+c_1^{-2}$$ ***(N. U. 90)***

20. A plane meets the coordinate axes in A, B, C. If the centroid of Δ ABC is (a,b,c), show that the equation to the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$

(O.U. M15, S.V.U. 00, S.K.U. 01 O, O.U.M. 98, A. N. U. O 90, M14)

21. (*a*) A variable plane moves so that the sum of the reciprocals of its intercepts on the coordinate axes is a constant. Show that it passes through a fixed point ***(N. U. M. 98, N. U. A 88)***

(*b*) A variable plane makes intercepts on the axes, the sum of whose squares is k^2 (a constant). Show that the locus of the foot of the perpendicular from the origin is $(x^{-2}+y^{-2}+z^{-2})(x^2+y^2+z^2)=k^2$. (Ex. 7 after Art. 10.4) ***(A. U. 12, A. N. U. M13)***

22. A variable plane is at a constant distance $3p$ from the origin and meets the axes in A, B, C. Show that the locus of the centroid of the Δ ABC is $x^{-2}+y^{-2}+z^{-2}=p^{-2}$

(S.V.U. 08, S.K.U. 00, 02, N. U. M. 98, 12, K. U. 12, O. U. O 01, K.U. M18, S.K.U M18)

23. A variable plane is at a constant distance p from the origin and meets the axes in A, B, C. Show that the locus of the centroid of the tetrahedron OABC is $x^{-2}+y^{-2}+z^{-2}=16p^{-2}$ ***(S.V. U. A 93, O 95, A.K.N.U M18)***

24. Prove that the distance between parallel planes
$2x-2y+z+3=0$ and $4x-4y+2z+5=0$ is $1/6$ ***(S.V.U. 08)***

25. If $A=(1,3,2)$, $B=(-5,0,2)$, $C=(1,1,-4)$, find the distance of $(2,3,4)$ from the plane $\overleftrightarrow{ABC}$ without finding the equation to $\overleftrightarrow{ABC}$.

26. Find the equations of the planes through $(6,-4,3)$, $(0,4,-3)$ (other than the plane through the origin) and cutting of intercepts whose sum is zero. ***(A.U. M18)***

27. P is a point such that the sum of the squares of its distances from the planes $x+y+z=0$, $x+y-2z=0$, $x-y=0$ is 5. Show that the locus of P is $x^2+y^2+z^2=5$

(A. U. A10, S. V. U. O 90)

ANSWERS

1. $6, -4, 3$ **2.** $\frac{2x}{3}-\frac{2y}{3}+\frac{z}{3}=\frac{5}{3}$ **3.** (*i*) $bcx+cay-abz=0$ (*ii*) $4x-5y-7z=0$

4. $2x-3y+4z=29$. **5.** $\pm\frac{a}{\sqrt{\Sigma a^2}}, \pm\frac{b}{\sqrt{\Sigma a^2}}, \pm\frac{c}{\sqrt{\Sigma a^2}}$; $ax+by+cz=a^2+b^2+c^2$

6. $4x+3y-12z-4=0$ **7.** $k=-8$, $14x-5y+3z=16$ **8.** (*i*) $3x+4y-5z=9$

9. $2x-4y+3z+8=0$ **10.** $x+5y-6z+19=0$ **11.** $x+5y-6z-23=0$

12. $3x-y-z+11=0$. **13.** (*i*) $4x+y+z+4=0$ (*ii*) $\Sigma(x-x_2)\left(x-\frac{x_1+x_2}{2}\right)=0$

14.(*ii*) $3x-4z+1=0$ **15.** (*i*) $x-y-7z+23=0$ **16.** $x+y+z=0$, $(-1,4,-3)$

17. (*i*) $\cos^{-1}\left(\frac{12}{\sqrt{266}}\right)$, $\pi-\cos^{-1}\left(\frac{12}{\sqrt{266}}\right)$ (*ii*) $\frac{\pi}{2}$ (*iii*) $\frac{\pi}{3}, \frac{2\pi}{3}$

(*iv*) $\cos^{-1}\left(\frac{-25}{21\sqrt{3}}\right)$, $\pi-\cos^{-1}\left(\frac{-25}{21\sqrt{3}}\right)$ (*v*) $\frac{\pi}{2}$

18. (*i*) different half spaces 1 : 11 (*ii*) same half space. $-3:1$, **25.** 1

26. $6x+3y-2z=18$, $2x-3y-6z=6$

3.16. SYSTEMS OF PLANES

Consider the equation $ax+by+cz+d=0$, $(a,b,c)\neq(0,0,0)$, $\lambda_1=\frac{b}{a}, \lambda_2=\frac{c}{a}, \lambda_3=\frac{d}{a}$ of a plane.

When three conditions satisfying the equation are given, $\lambda_1, \lambda_2, \lambda_3$ can be uniquely determined and hence a plane can be uniquely determined. (Art. 3.8, 3.9).

When two conditions satisfying the equation are given, one of $\lambda_1, \lambda_2, \lambda_3$ say, λ_1 cannot be found uniquely and λ_1 is called a parameter. Since λ_1 can be assigned any real value, an infinite number of planes arise and these planes are called a system of planes.

When one condition satisfying the equation is given we have two parameters, say, λ_1, λ_2 giving rise to a system of planes for different values of λ_1, λ_2.

We give below a few systems of planes involving one or two parameters.

(*i*) The equation $ax+by+cz+\lambda=0$ represents the system of planes parallel to a given plane $ax+by+cz+d=0$, λ being the parameter.

(*ii*) The equation $ax+by+cz+\lambda=0$ represents the system of planes perpendicular to lines with d.rs. a, b, c; λ being the parameter.

(*iii*) The equation $a(x-x_1)+b(y-y_1)+c(z-z_1)=0, (a,b,c)\neq(0,0,0)$

i.e., $(x-x_1)+\lambda_1(y-y_1)+\lambda_2(z-z_1)=0$ where $a\neq 0$ (say), $\lambda_1=(b/a), \lambda_2=(c/a)$ represents the system of planes passing through the point (x_1,y_1,z_1); λ_1,λ_2 being the parameters.

(*iv*) The equation $\lambda_1(a_1x+b_1y+c_1z+d_1)+\lambda_2(a_2x+b_2y+c_2z+d_2)=0$

represents the system of planes through the line of intersection of the planes $a_1x+b_1y+c_1z+d_1=0$, $a_2x+b_2y+c_2z+d_2=0$; λ_1,λ_2 being parameters and $(\lambda_1,\lambda_2)\neq(0,0)$.

The truth of the statement can be seen from the theorem proved in the ensuing article.

3.17. Theorem. $\pi_1\equiv a_1x+b_1y+c_1z+d_1=0$, $\pi_2\equiv a_2x+b_2y+c_2z+d_2=0$ ***represent two intersecting planes.***

(A) ***If*** $(\lambda_1,\lambda_2)\neq(0,0)$, ***then*** $\lambda_1\pi_1+\lambda_2\pi_2=0$ ***represents a plane passing through the line L of intersection of*** π_1 ***and*** π_2.

(B) ***Any plane passing through the line L of intersection of*** π_1 ***and*** π_2 ***is given by*** $\lambda_1\pi_1+\lambda_2\pi_2=0$, $(\lambda_1,\lambda_2)\neq(0,0)$

Proof : (A) Let $S\equiv\lambda_1\pi_1+\lambda_2\pi_2=0$, $(\lambda_1,\lambda_2)\neq(0,0)$

S is a first degree equation in x,y,z and hence represents a plane.

Now $\lambda_1=0\Rightarrow S=\pi_2$, $\lambda_2=0\Rightarrow S=\pi_1$. Let $P(x_1,y_1,z_1)\in L$

$\therefore\ a_1x_1+b_1y_1+c_1z_1+d_1=0$ .... (1) $a_2x_1+b_2y_1+c_2z_1+d_2=0$ .... (2)

Also from (1) and (2), $P\in S$, when $(\lambda_1,\lambda_2)\neq(0,0)$

$\therefore$ S represents a plane through the line L of intersection of π_1 and π_2.

If $\lambda_1\neq 0,\lambda_2\neq 0$, for different values of λ_1,λ_2; S represents any plane through the line L of intersection π_1 and π_2 and different from π_1 and π_2.

(B) Let $P(x_1,y_1,z_1), Q(x_2,y_2,z_2)$ be different points on L such that $x_1\neq x_2$ (say).

Let $S\equiv\alpha x+\beta y+\gamma z+\delta=0$ be a plane through L and hence

$\alpha x_1+\beta y_1+\gamma z_1+\delta=0$ .... (3) $\alpha x_2+\beta y_2+\gamma z_2+\delta=0$ .... (4)

Let l,m,n be d.rs. of L, the line of intersection of the planes π_1 and π_2.

$$\therefore\ \left.\begin{array}{l}a_1l+b_1m+c_1n=0\\ a_2l+b_2m+c_2n=0\end{array}\right\}\ \frac{l}{b_1c_2-b_2c_1}=\frac{m}{c_1a_2-c_2a_1}=\frac{n}{a_1b_2-a_2b_1}$$

Since $(b_1c_2-b_2c_1,\ c_1a_2-c_2a_1,\ a_1b_2-a_2b_1)\neq(0,0,0)$ without loss of generality we can take $b_1c_2-b_2c_1\neq 0$.

For λ_1,λ_2 and $(\lambda_1,\lambda_2)\neq(0,0)$ there exist equations $\lambda_1b_1+\lambda_2b_2=\beta$, $\lambda_1c_1+\lambda_2c_2=\gamma$ such that they have a unique solution λ_1 and λ_2.

$$\therefore\ \alpha x+\beta y+\gamma z+\delta\equiv\alpha x+(\lambda_1b_1+\lambda_2b_2)y+(\lambda_1c_1+\lambda_2c_2)z+\delta$$

$$\equiv\lambda_1(a_1x+b_1y+c_1z+d_1)+\lambda_2(a_2x+b_2y+c_2z+d_2)+\alpha x-\lambda_1a_1x-\lambda_2a_2x-\lambda_1d_1-\lambda_2d_2+\delta$$

$$\equiv\lambda_1(a_1x+b_1y+c_1z+d_1)+\lambda_2(a_2x+b_2y+c_2z+d_2)$$
$$+(\alpha-\lambda_1a_1-\lambda_2a_2)x+(\delta-\lambda_1d_1-\lambda_2d_2)$$

$$\equiv\lambda_1(a_1x+b_1y+c_1z+d_1)+\lambda_2(a_2x+b_2y+c_2z+d_2)+\lambda_3x+\lambda_4 \quad \text{.... (5)}$$

where $\lambda_3=\alpha-\lambda_1a_1-\lambda_2a_2$, $\lambda_4=\delta-\lambda_1d_1-\lambda_2d_2$

$\therefore\ P\in L\Rightarrow P\in S\Rightarrow\alpha x_1+\beta y_1+\gamma z_1+\delta=0\Rightarrow\lambda_3x_1+\lambda_4=0$ .... (6)

using (3) and (5), and $Q\in L\Rightarrow Q\in S\Rightarrow\lambda_3x_2+\lambda_4=0$ .... (7)

using (4) and (5),

$\therefore$ $(6)-(7) \Rightarrow \lambda_3(x_1 - x_2) = 0 \Rightarrow \lambda_3 = 0$ $(\because x_1 \neq x_2)$ $\therefore$ From (6), $\lambda_4 = 0$.

$\therefore$ $S \equiv \lambda_1\pi_1 + \lambda_2\pi_2 = 0$ is the plane passing through the line of intersection of π_1 and π_2.

Note. Let $\lambda_1 \neq 0$ (say). Now equation to the plane (distinct from π_1, π_2) passing through the line of intersection of planes π_1 and π_2 can be taken as $\pi_1 + \left(\frac{\lambda_2}{\lambda_1}\right)\pi_2 = 0$ *i.e.* $\pi_1 + \lambda\pi_2 = 0$ where $\lambda = \frac{\lambda_2}{\lambda_1}$. This form of equation might be taken while doing problems.

3.18. PLANES BISECTING THE ANGLES BETWEEN TWO PLANES.

Theorem. *$\pi_1 \equiv a_1x + b_1y + c_1z + d_1 = 0$, $\pi_2 \equiv a_2x + b_2y + c_2z + d_2 = 0$ and $d_1d_2 > 0$. Equation to the plane bisecting the angle containing the origin between the planes*

π_1, π_2 is $\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = +\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$

and to the plane bisecting the other angle between the planes

π_1, π_2 is $\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = -\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$

Proof : Equations to π_1, π_2 are

$a_1x + b_1y + c_1z + d_1 = 0$ .... (1) $a_2x + b_2y + c_2z + d_2 = 0$ .... (2) and $d_1 d_2 > 0$

We know that if P (x, y, z) is any point on one of the planes bisecting the angle between π_1, π_2 then the perpendicular distances of P from π_1, π_2 are equal (in magnitude) so that

$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$ are the equations to the bisecting planes.

$\therefore$ Equation to the plane bisecting the angle containing the origin is

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = +\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \quad \text{if } d_1 > 0,\ d_2 > 0 \qquad .... (7)$$

This plane bisecting the angle containing the origin also bisects the vertically opposite angle.

$\therefore$ Equation to the plane bisecting the other angle and its vertically opposite angle is

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = -\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \quad \text{if } d_1 > 0,\ d_2 > 0 \qquad .... (8)$$

Note 1. In the bisecting planes (7) and (8), one bisects the acute and the other bisects the obtuse angle between the given planes π_1, π_2.

The bisecting plane of the acute angle makes with either of the planes π_1, π_2 an angle less than $45°$ and the bisecting plane of the obtuse angle makes with either of the planes π_1, π_2 an angle greater than $45°$ (of course $< 90°$). This gives a test for determining which angle (acute or obtuse) each bisecting plane bisects.

2. Even if $d_1 < 0, d_2 < 0$, the theorem holds.

But if $d_1 > 0, d_2 < 0$ or $d_1 < 0, d_2 > 0$, equation to the plane bisecting the angle containing the origin is $\frac{a_1x + b_1y + c_1z + d_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = -\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$ and the other equation gives the other bisecting plane.

3. $l_1x + m_1y + n_1z = q_1,\ l_2x + m_2y + n_2z = q_2$ are two intersecting planes such that $(l_1, m_1, n_1), (l_2, m_2, n_2)$ are unit points and $q_1\, q_2 > 0$ (q_1, q_2 are of the same sign).

$\therefore$ Equation to the plane bisecting the angle containing the origin is

$(l_1 - l_2)\, x + (m_1 - m_2)\, y + (n_1 - n_2)\, z = q_1 - q_2$ and equation to the plane bisecting the other angle is $(l_1 + l_2)\, x + (m_1 + m_2)\, y + (n_1 + n_2)\, z = q_1 + q_2$

SOLVED PROBLEMS

Ex. 1. *Find the equation to the plane through the point* (x_1, y_1, z_1) *and parallel to the plane* $ax + by + cz + d = 0$

Sol. Let $ax + by + cz + \lambda = 0$ .... (1) be the plane parallel to $ax + by + cz + d = 0$ .... (2) for all values of λ.

If (1) passes through (x_1, y_1, z_1) then $ax_1 + by_1 + cz_1 + \lambda = 0$ *i.e.,* $\lambda = -ax_1 - by_1 - cz_1$

$\therefore$ Required plane is $ax + by + cz - ax_1 - by_1 - cz_1 = 0$

i.e., $a\,(x - x_1) + b\,(y - y_1) + c\,(z - z_1) = 0$

Ex. 2. *Find the equations of the planes through the intersection of the planes* $x + 3y + 6 = 0$ *and* $3x - y - 4z = 0$ *such that the perpendicular distances of each from the origin is unity.* ***(A.K.N.U. M18)***

Sol. Let the plane passing through the intersection of the planes

$x + 3y + 6 = 0,\ 3x - y - 4z = 0$ be $(x + 3y + 6) + \lambda\,(3x - y - 4z) = 0$

$\Rightarrow (1 + 3\lambda)\, x + (3 - \lambda)\, y - 4\lambda z + 6 = 0$ .... (1)

Perpendicular distance of origin from (1) = 1

$$\Rightarrow \frac{6}{\sqrt{(1 + 3\lambda)^2 + (3 - \lambda)^2 + 16\lambda^2}} = 1 \Rightarrow 26\lambda^2 = 26 \Rightarrow \lambda = \pm 1$$

$\therefore$ Required planes are $4x + 2y - 4z + 6 = 0,\ -2x + 4y + 4z + 6 = 0$

i.e., $2x + y - 2z + 3 = 0,\ x - 2y - 2z - 3 = 0$

Ex. 3. *Find the equation to the plane through the intersection of the planes* $x + 2y + 3z + 4 = 0$ *and* $4x + 3y + 3z + 1 = 0$ *and perpendicuar to the plane* $x + y + z + 9 = 0$ ***(N. U. S. 98)***

Sol. Let the plane through the intersection of the planes

$x + 2y + 3z + 4 = 0,\ 4x + 3y + 3z + 1 = 0$ be $(x + 2y + 3z + 4) + \lambda\,(4x + 3y + 3z + 1) = 0$

$\Rightarrow (1 + 4\lambda)\, x + (2 + 3\lambda)\, y + (3 + 3\lambda)\, z + (4 + \lambda) = 0$ .... (1)

If (1) is perpendicular to $x + y + z + 9 = 0$, then $(1 + 4\lambda).1 + (2 + 3\lambda).1 + (3 + 3\lambda).1 = 0$

i.e., $10\lambda = -6$ *i.e.,* $\lambda = -3/5$. $\therefore$ Required plane is $7x - y - 6z - 17 = 0$

Ex. 4. *Find the equation to the plane through the line of intersection of* $x - y + 3z + 5 = 0$ *and* $2x + y - 2z + 6 = 0$ *and passing through* $(-3, 1, 1)$ ***(A. U. A10, S.V. U. M13)***

Sol. Let the equation to the plane through the intersection of the planes $x - y + 3z + 5 = 0$, $x - y + 3z + 5 = 0$, be $(x - y + 3z + 5) + \lambda\,(2x + y - 2z + 6) = 0$ .... (1), for any λ.

Let (1) pass through the point $(-3,1,1)$. $\therefore \quad -3-1+3+5+\lambda(-6+1-2+6)=0$

i.e., $-\lambda+4=0$ *i.e.,* $\lambda=4$

$\therefore$ Equation to the required plane is $9x+3y-5z+29=0$

Ex. 5. *Find the equation to the plane through* $(2,-3,1)$ *and is normal to the line joining* $(3,4,-1)$ *and* $(2,-1,5)$.

Sol. Let the plane through $(2,-3,1)$ and perpendicular to the join of $P(3,4,-1)$ and $Q(2,-1,5)$ be $a(x-2)+b(y+3)+c(z-1)=0$.

Since d.rs. of $\overleftrightarrow{PQ}$ are $(3-2,4+1,-1-5)$ *i.e.,* $(1,5,-6)$

We have $\frac{a}{1}=\frac{b}{5}=\frac{c}{-6}=\lambda$, say . $\therefore$ Required plane is $\lambda(x-2)+5\lambda(y+3)-6\lambda(z-1)=0$

i.e., $x-2+5(y+3)-6(z-1)=0$ *i.e.,* $x+5y-6z=-19$

Ex. 6. *A variable plane passes through a fixed point (a, b, c). It meets the axes of reference in A, B and C. Show that the locus of the point of intersection of the planes through A, B, C and parallel to the coordinate planes is* $ax^{-1}+by^{-1}+cz^{-1}=1$

(O. U. M. 98, N. U. 93, S.K.V. M18)

Sol. Let the variable plane meeting the coordinate axes in A, B, C be

$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$ .... (1) $\therefore A=(\alpha,0,0),\ B=(0,\beta,0),\ C=(0,0,\gamma)$

Also (1) passes through the fixed point (a,b,c) $\therefore \frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=1$

But equations to the planes through A, B, C and parallel to the coordinate planes are $x=\alpha,\ y=\beta,\ z=\gamma$. Clearly they intersect at $P=(\alpha,\beta,\gamma)$

$\therefore$ Locus of P from (2) is $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$ *i.e.,* $ax^{-1}+by^{-1}+cz^{-1}=1$

Ex. 7. *Find the bisecting plane of the acute angle between the planes* $3x-2y-6z+2=0, -2x+y-2z-2=0$ ***(S.V. U. M15, N. U. S. 98)***

Sol. Equations to the given planes are taken as $3x-2y-6z+2=0$ .... (1)

$2x-y+2z+2=0$ .... (2)

(constant terms are taken as + ve)

$\therefore$ Equations to the bisecting planes between the given planes are

$$\frac{3x-2y+6z+2}{\sqrt{(9+4+36)}}=\pm\frac{2x-y+2z+2}{\sqrt{(4+1+4)}}$$

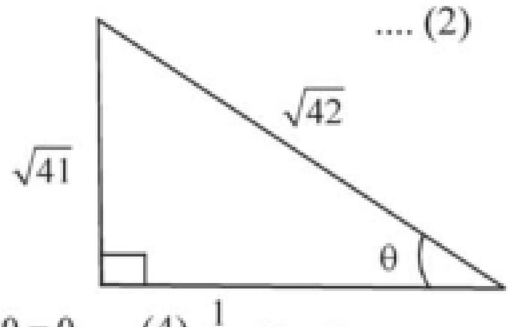


Fig. 45

i.e., $5x-y-4z+8=0$ .... (3), $23x-13y+32z+20=0$ .... (4)

Let θ be the acute angle between (2) and (3)

$\therefore \cos\theta=\left|\frac{10+1-8}{\sqrt{9}.\sqrt{(25+1+16)}}\right|=\frac{1}{(\sqrt{42})}$ (Fig. 45) $\therefore \tan\theta=\sqrt{(41)}>1 \quad \therefore \theta>(\pi/4)$

Hence 2θ, the angle between the planes (1) and (2) is greater than $90°$ *i.e.,* obtuse.

$\therefore$ (3) is the equation to the plane bisecting the obtuse angle between (1) and (2).

$\therefore$ (4) is the equation to the plane bisecting the acute angle between (1) and (2).

Note. $5x-y-4z+8=0$ is the plane bisecting the angle containing the origin between (1) and (2).

EXERCISE 3 (*b*)

1. Find the equation to the plane through (*i*) $(1,-2,-3)$ and parallel to $3x-y+z=10$
(*ii*) $(2,3,4)$ and parallel to $5x-6y+7z-3=0$

2. Find the equations to the planes through the intersection of $2x+y+3z-2=0$, $x-y+z+4=0$ such that each plane is at a distance of 2 units from the origin.

3. Find the equation of the plane through the line of intersection of the planes
(*i*) $x-3y+2z+3=0$ and $3x-y-2z-5=0$ and through the origin. **(*S. V. U. M06, A10*)**
(*ii*) $x+y+z-6=0$ and $2x+3y+4z+5=0$ through the point $(1,1,1)$. **(*S. V. U. M 13*)**
(*iii*) $x+y+z=1$ and $2x+3y-z=-4$ and is parallel to x - axis.

4. Find the equation to the plane through the line of intersection of $ax+by+cz+d=0$, $a_1x+b_1y+c_1z+d_1=0$ and perpendicular to the xy plane.

5. (*i*) Find the equation of the plane through the intersection of the planes $x+y+z=1$, $2x+3y+4z=5$ and perpendicular to the plane $x-y+z=0$ **(*S.K.U. 2001 Oct.*)**
(*ii*) Find the equation of the plane passing through the line of intersection of the planes $2x-y=0$ and $y-3z=0$ and perpendicular to the plane $3z-4x-5y+8=0$ **(*O.U. 07*)**

6. Find the equation of the plane passing through the intersection of the planes $x+2y+3z=4$, $2x+y-z+5=0$ and perpendicular to the plane $6z+5x+3y+8=0$.
(*A. U. A11, A.U. M18*)

7. Find the equation to the plane through the line of intersection of $x-2y-z+3=0$, $-3x-5y+2z+1=0$ and perpendicular to yz plane. **(*O.U. 07*)**

8. A variable plane is at a constant distance $p\ (\neq 0)$ from the origin. It meets the coordinate axes in A, B, C. Through A, B, C planes are drawn parallel to the coordinate planes. Show that the locus of their point of intersection is $x^{-2}+y^{-2}+z^{-2}=p^{-2}$ **(*O.U. 01, N.U.07*)**

9. Find the equations of the planes bisecting the angles between the planes.
(*i*) $x+2y+2z=19$, $4x-3y+12z+3=0$
(*ii*) $3x-6y+2z+5=0$, $4x-12y+3z-3=0$ **(*S. V. U. A12, 13, O.U. 2001M, A. N. U. M13*)**
(*iii*) $2x-y-2z+3=0$, $3x-2y+6z+8=0$
(*K.U., S.V.U. A 97, S. K. U. A11, K.U. M18, V.S.P.V M18*)
Point out which plane bisects the acute angle. Also point out the plane bisecting the angle containing the origin.

10. Find the equation of the plane bisecting the obtuse angle between the planes $3x+4y-5z+1=0$ and $5x+12y-13z=0$ **(*S.V.M. A18*)**

11. Determine the planes through the intersection of the planes $2x+3y-z+4=0$, $x+y+z-1=0$ and which are parallel to the coordinate axes.

ANSWERS

1. (*i*) $3x-y+z=2$ (*ii*) $5x-6y+7z-20=0$

2. $x+2y+2z-6=0,\ 15x-12y+16z+50=0$

3. (*i*) $7x-9y+2z=0$ (*ii*) $20x+23y+26z=69$ (*iii*) $y-3z+6=0$.

4. $(ac_1-a_1c)\,x+(bc_1-b_1c)\,y+(dc_1-d_1c)=0$ **5.** (*i*) $x-z+2=0$ (*ii*) $28x-17y+9z=0$

6. $51x+15y-50z+173=0$ **7.** $11y+z=10$.

9. (*i*) $x+35y-10z=256,\ 25x+17y+62z=238$ (bisector of acute angle and bisector of the angle containing the origin)

(*ii*) $11x+6y+5z+86=0, 67x-162y+47z+44=0$ (bisector of acute angle and bisector of the angle contining the origin)

(*iii*) $5x-y-4z=3, 23x-13y+32z+45=0$ (bisector the acute angle)

11. $y-3z+6=0, x+4z-7=0, 3x+4y+3=0$

3.19. JOINT EQUATION OF A PAIR OF PLANES

Consider the pair of planes π_1, π_2 whose respective equations are

$l_1x+m_1y+n_1z+d_1=0$ .... (1) $l_2x+m_2y+n_2z+d_2=0$ .... (2)

Consider the equation $(l_1x+m_1y+n_1z+d_1)(l_2x+m_2y+n_2z+d_2)=0$ ..... (3)

Let $P=(x_1, y_1, z_1)$.

$P\in\pi_1 \Rightarrow l_1x_1+m_1y_1+n_1z_1+d_1=0 \Rightarrow (l_1x_1+m_1y_1+n_1z_1+d_1)(l_2x_1+m_2y_1+n_2z_1+d_2)=0$

$P\in\pi_2 \Rightarrow l_2x_1+m_2y_1+n_2z_1+d_2=0 \Rightarrow (l_1x_1+m_1y_1+n_1z_1+d_1)(l_2x_1+m_2y_1+n_2z_1+d_2)=0$

P lies on (3) $\Rightarrow (l_1x_1+m_1y_1+n_1z_1+d_1)(l_2x_1+m_2y_1+n_2z_1+d_2)=0$

$\Rightarrow l_1x_1+m_1y_1+n_1z_1+d_1=0$ or $\Rightarrow l_2x_1+m_2y_1+n_2z_1+d_2=0$ $\Rightarrow P\in\pi_1$ or $\Rightarrow P\in\pi_2$

$\therefore$ We have $P\in\pi_1$ or $P\in\pi_2$: P lies on (3)

i.e., an equation is satisfied if and only if a point lies on the one plane or the other plane or both.

$\therefore$ (3) represents the joint or combined equation to the planes π_1 and π_2.

Note 1. $a_1x+b_1y+c_1z+d_1=0, a_2x+b_2y+c_2z+d_2=0$ are two intersecting planes. The combined equation to the pair of planes bsecting the angles between them is

$$\left[\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}}-\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{(a_2^2+b_2^2+c_2^2)}}\right]\left[\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}}+\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{(a_2^2+b_2^2+c_2^2)}}\right]=0$$

i.e., $$\frac{(a_1x+b_1y+c_1z+d_1)^2}{a_1^2+b_1^2+c_1^2}-\frac{(a_2x+b_2y+c_2z+d_2)^2}{a_2^2+b_2^2+c_2^2}=0$$

Theorem. ***If*** $S\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$ ***is a second degree equation representing a locus of which the plane*** $L=0$ ***is a part, then S can be expressed as a product of two linear factors in x, y, z.***

(Proof is left out)

3.20. Theorem. $S\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$ ***represents a pair of planes*** π_1, π_2.

Then $H\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ ***represents a pair of planes through the origin and parallel to the planes*** π_1, π_2.

Proof. Let the planes π_1, π_2 represented by S = 0 be respectively $l_1x+m_1y+n_1z+d_1=0$, $l_2x+m_2y+n_2z+d_2=0$

$\therefore \quad ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d$

$\equiv (l_1x+m_1y+n_1z+d_1)\ (l_2x+m_2y+n_2z+d_2)$

$\Rightarrow l_1l_2=a, m_1m_2=b, n_1n_2=c,\ l_1m_2+l_2m_1=2h,\ m_1n_2+m_2n_1=2f, n_1l_2+n_2l_1=2g.$

Joint equation of the planes passing through the origin and parallel to π_1, π_2 is

$(l_1x+m_1y+n_1z)(l_2x+m_2y+n_2z)=0$

i.e., $l_1l_2x^2 + m_1m_2y^2 + n_1n_2z^2 + (l_1m_2 + l_2m_1)xy + (m_1n_2 + m_2n_1)yz + (n_1l_2 + n_2l_1)\,zx = 0$

i.e., $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ *i.e.,* $\mathrm{H} = 0$.

3.21. Definition. *If* $H \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ *and* $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, *then D is called the determinant of H.*

Theorem. $H \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ ***represents the equation of a pair of planes or a plane if*** $D = 0, f^2 \geq bc, g^2 \geq ac, h^2 \geq ab$.

Proof. H = 0 represents the equation to a pair of planes or a plane $\Rightarrow$ H can be expressed as a product of two linear factors in *x, y, z.*

Let the factors be $l_1x + m_1y + n_1z + d_1,\ l_2x + m_2y + n_2z + d_2$ where $(l_1, m_1, n_1) \neq (0,0,0)$, $(l_2, m_2, n_2) \neq (0,0,0)$

$\therefore\ ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \equiv (l_1x + m_1y + n_1z + d_1)\ (l_2x + m_2y + n_2z + d_2)$

$\Rightarrow l_1l_2 = a,\ m_2m_2 = b,\ n_1n_2 = c,\ l_1m_2 + l_2m_1 = 2h,\ m_1n_2 + m_2n_1 = 2f,\ n_1l_2 + n_2l_1 = 2g.$

$l_1d_2 + l_2d_1 = 0,\ m_1d_2 + m_2d_1 = 0,\ n_1d_2 + n_2d_1 = 0,\ d_1d_2 = 0$

Now $d_1d_2 = 0 \Rightarrow d_1 = 0$ or $d_2 = 0$. $d_2 = 0 \Rightarrow l_2d_1 = 0,\ m_2d_1 = 0,\ n_2d_1 = 0$

$\Rightarrow d_1 = 0$ ($\because$ at least one of l_2, m_2, n_2 is not equal to zero)

Similarly $d_1 = 0 \Rightarrow d_2 = 0.$ $\therefore\ d_1 = d_2 = 0$

We know that $\begin{bmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{bmatrix} \begin{bmatrix} l_2 & m_2 & n_2 \\ l_1 & m_1 & n_1 \\ 0 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 2l_1l_2 & l_1m_2 + l_2m_1 & l_1n_2 + l_2n_1 \\ l_2m_1 + l_1m_2 & 2m_1m_2 & m_1n_2 + m_2n_1 \\ n_1l_2 + l_1n_2 & n_1m_2 + n_2m_1 & 2n_1n_2 \end{bmatrix} = \begin{bmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{bmatrix}$$

$$\therefore\ \det \left\{ \begin{bmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{bmatrix} \begin{bmatrix} l_2 & m_2 & n_2 \\ l_1 & m_1 & n_1 \\ 0 & 0 & 0 \end{bmatrix} \right\} = \det \begin{bmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{bmatrix}$$

$$\Rightarrow \begin{vmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} \begin{vmatrix} l_1 & m_2 & n_2 \\ l_1 & m_1 & n_1 \\ 0 & 0 & 0 \end{vmatrix} = 8 \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & e \end{vmatrix} \Rightarrow 0 \times 0 = 8\mathrm{D} \Rightarrow \mathrm{D} = 0.$$

Also $4f^2 - 4bc = (m_1n_2 + m_2n_1)^2 - 4m_1n_2m_2n_1 = (m_1n_2 - m_2n_1)^2 \geq 0 \Rightarrow f^2 \geq bc$.

Similarly we can prove that $g^2 \geq ac, h^2 \geq ab$.

We give below two theorems, for which proofs may be supplied by the readers if needed.

1. H = 0 represents a pair of planes if (*i*) D = 0 and (*ii*) at least one of $h^2 - ab,\ f^2 - bc,\ g^2 - ac$ is + ve and the remaining two are non-negative.

2. H = 0 represents a plane if D = 0, $h^2 = ab,\ f^2 = bc,\ g^2 = ac$. In this case H takes the form $(a_1x + b_1y + c_1z)^2$.

3.22. Theorem. *If $\theta (\le \pi/2)$ is the angle between the pair of planes $H = 0$, then* $\cos\theta = \left| \dfrac{a+b+c}{\sqrt{\{(a+b+c)^2 + 4\,(f^2+g^2+h^2-ab-bc-ca)\}}} \right|$ **(S.V. M. M18)**

Proof. Let the pair of planes represented by H = 0 be

$$l_1x + m_1y + n_1z = 0, \quad l_2x + m_2y + n_2z = 0 .$$

$$\therefore\ ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \equiv (l_1x + m_1y + n_1z)\ (l_2x + m_2y + n_2z)$$

$$\Rightarrow l_1l_2 = a,\ m_1m_2 = b,\ n_1n_2 = c. \qquad l_1m_2 + l_2m_1 = 2h,\ m_1n_2 + m_2n_1 = 2f,\ n_1l_2 + n_2l_1 = 2g$$

Since $\theta\,(\le (\pi/2))$ is the angle between the planes,

$$\cos\theta = \left| \frac{l_1l_2 + m_1m_2 + n_1n_2}{\sqrt{(l_1^2+m_1^2+n_1^2)}\,.\sqrt{(l_2^2+m_2^2+n_2^2)}} \right|$$

$$= \left| \frac{a+b+c}{\sqrt{\begin{array}{l}(\{l_1^2l_2^2 + m_1^2m_2^2 + n_1^2n_2^2 + (l_1m_2+l_2m_2)^2 - 2l_1l_2m_1m_2 \\ +(m_1n_2+m_2n_1)^2 - 2m_1m_2n_1n_2 + (n_1l_2+n_2l_1)^2 - 2l_1l_2n_1n_2\}\end{array}}} \right|$$

$$= \left| \frac{a+b+c}{\sqrt{\{(a^2+b^2+c^2+4h^2-2ab+4f^2-2bc+4g^2-2ac)\}}} \right|$$

$$= \left| \frac{a+b+c}{\sqrt{\{(a+b+c)^2 + 4\,(f^2+g^2+h^2-bc-ca-ab)\}}} \right|$$

Cor. 1. Planes are perpendicular $\Leftrightarrow\ \theta = 90°$

$\Leftrightarrow \cos\theta = 0 \ \Leftrightarrow\ a+b+c = 0$

$\Leftrightarrow$ Coeff. of x^2 + Coeff. of y^2 + Coeff. of $z^2 = 0$

2. Planes are identical (coincident) $\Leftrightarrow\ \theta = 0°$

$$\Leftrightarrow \cos\theta = 1 \Leftrightarrow \left| \frac{a+b+c}{\sqrt{\{(a+b+c)^2 + 4(f^2+g^2+h^2-ab-bc-ca)\}}} \right| = 1$$

$$\Leftrightarrow (a+b+c)^2 = (a+b+c)^2 + 4\,(f^2+g^2+h^2-ab-bc-ca)$$

$$\Leftrightarrow f^2+g^2+h^2-bc-ca-ab = 0 \Leftrightarrow (f^2-bc)+(g^2-ac)+(h^2-ab) = 0$$

$$\Leftrightarrow f^2 = bc,\ g^2 = ac,\ h^2 = ab \qquad (\because\ f^2 \ge bc, g^2 \ge ac,\ h^2 \ge ab)$$

Note. 1. $l_1x + m_1y + n_1z = 0,\ l_2x + m_2y + n_2z = 0$ are two planes intersecting in a line with d.rs., l, m, n.

$$\Rightarrow l_1l + m_1m + n_1n = 0\ ,\ \ l_2l + m_2m + n_2n = 0 \Rightarrow \frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$$

$$\Rightarrow \frac{l}{[(m_1n_2 - m_2n_1)^2 - 4m_1n_2n_1m_2]} = \frac{m}{\sqrt{[(n_1l_2 - n_2l_1)^2 - 4n_1l_2n_2l_1]}}$$

$$= \frac{n}{\sqrt{[(l_1m_2 - l_2m_1)^2 - 4l_1m_2l_2m_1]}}$$

$$\Rightarrow \frac{l}{\sqrt{(4f^2-4bc)}}=\frac{m}{(\sqrt{4g^2-4ac})}=\frac{n}{\sqrt{(4h^2-4ab)}}\Rightarrow \frac{l}{\sqrt{(f^2-bc)}}=\frac{m}{(\sqrt{g^2-ac})}=\frac{n}{\sqrt{(h^2-ab)}}$$

$\Rightarrow$ d.rs. of the common line are $\sqrt{(f^2-bc)}, \sqrt{(g^2-ac)}, \sqrt{(h^2-ab)}$

2. If $\theta\,(\le \pi/2)$ is the angle between the pair of intersecting planes given by $ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$, then

$$\cos\theta=\left|\frac{a+b+c}{\sqrt{\{(a+b+c)^2+4(f^2+g^2+h^2-ab-bc-ca)\}}}\right|.$$

d.rs. of the line of intersection are $\sqrt{(f^2-bc)}, \sqrt{(g^2-ac)}, \sqrt{(h^2-ab)}$

SOLVED PROBLEMS

Ex. 1. *Prove that the equation $2x^2-6y^2-12z^2+18yz+2zx+xy=0$ represents a pair of planes, and find the angle between them.* ***(S.V. U. M15, A.N.U. M15, S.K.U. M13, S.V.U M18)***

Sol. Let the given equation be $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ .... (1)

Comparing the given equation to (1), $a=2, b=-6, c=-12, f=9, g=1, h=1/2$

$$\therefore\ D=\begin{vmatrix} a & h & g\\ h & b & f\\ g & f & c\end{vmatrix}=\begin{vmatrix} 2 & 1/2 & 1\\ 1/2 & -6 & 9\\ 1 & 9 & -12\end{vmatrix}$$

$$=2\,(72-81)-\frac{1}{2}(-6-9)+1\left(\frac{9}{2}+6\right)=-18+\frac{15}{2}+\frac{21}{2}=0$$

$f^2=81, bc=72\Rightarrow f^2>bc,\ \ g^2=1, ac=-24\Rightarrow g^2>ac,\ h^2=1/4, ab=-12\Rightarrow h^2>ab$.

$\therefore$ Given equation represents a pair of planes through the origin.

Let θ be the acute angle between the planes.

$$\therefore\ \cos\theta=\left|\frac{a+b+c}{\sqrt{[(a+b+c)^2+4(f^2+g^2+h^2-ab-bc-ca)]}}\right|$$

$$=\left|\frac{2-6-12}{\sqrt{\left\{(-16)^2+4\left(81+1+\frac{1}{4}+12-72+24\right)\right\}}}\right|=\frac{16}{21}.\qquad \therefore\ \theta=\cos^{-1}\left(\frac{16}{21}\right)$$

OR : $2x^2-6y^2-12z^2+18yz+2zx+xy=0 \Rightarrow 2x^2+x\,(y+2z)-(6y^2+12z^2-18yz)=0$

$$\Rightarrow x=\frac{-(y+2z)\pm\sqrt{[(y+2z)^2+8\,(6y^2+12z^2-18yz)]}}{4}$$

$\Rightarrow 4x=-(y+2z)\pm\sqrt{(49y^2-140yz+100z^2)}\ \Rightarrow 4x=-y-2z\pm(7y-10z)$

$\Rightarrow 4x-6y+12z=0, 4x+8y-8z=0\Rightarrow 2x-3y+6z=0, x+2y-2z=0$

$\therefore$ Given equation represents a pair of planes through the origin.

$$\cos\theta=\frac{2\,(1)-3\,(2)+6\,(-2)}{\sqrt{(4+9+36)}\,.\sqrt{(1+4+4)}}=\frac{+16}{21}\qquad \therefore\ \theta=\cos^{-1}\left(\frac{16}{21}\right)$$

Ex. 2. *If* $a^2+b^2+c^2>2ab+2bc+2ca$, *show that the equation* $ax^2+by^2+cz^2-(a+b-c)\,xy-(b+c-a)\,yz-(c+a-b)\,zx=0$ *represents a pair of planes. Also show that the line of intersection of the planes make equal angles with the coordinate axes.*

Sol. Let $\mathrm{H}\equiv ax^2+by^2+cz^2-(a+b-c)\,xy-(b+c-a)\,yz-(c+a-b)\,zx=0$ ... (1)

$\therefore$ Determinant of H = D

$$=\begin{vmatrix} a & \frac{-(a+b-c)}{2} & \frac{-(c+a-b)}{2} \\ \frac{-(a+b-c)}{2} & b & \frac{-(b+c-a)}{2} \\ \frac{-(c+a-b)}{2} & \frac{-(b+c-a)}{2} & c \end{vmatrix} = -\frac{1}{8}\begin{vmatrix} 2a & (a+b-c) & (c+a-b) \\ (a+b-c) & -2b & (b+c-a) \\ (c+a-b) & (b+c-a) & -2c \end{vmatrix}$$

$R_1=R_1+R_2+R_3$.
$$=-\frac{1}{8}\begin{vmatrix} 0 & 0 & 0 \\ a+b-c & -2b & b+c-a \\ c+a-b & b+c-a & -2c \end{vmatrix}=0$$

[$f^2>bc$ be condition] : $\left(\frac{b+c-a}{2}\right)^2-bc=\frac{a^2+b^2+c^2-(2ab+2bc+2ca)}{4}>0$ (by hyp.)

Similarly [$g^2>ac, h^2\ge ab$ conditions] $\left(\frac{c+a-b}{2}\right)^2-ac>0$, $\left(\frac{a+b-c}{2}\right)^2-ab>0$

$\therefore$ (1) represents a pair of intersecting planes.

For the common line $\overrightarrow{\mathrm{PQ}}$ of intersection of the planes, d.rs. are

$\sqrt{\left[\left(\frac{b+c-a}{2}\right)^2-bc\right]}, \sqrt{\left[\left(\frac{c+a-b}{2}\right)^2-ca\right]}, \sqrt{\left[\left(\frac{a+b-c}{2}\right)^2-ab\right]}$

i.e. $\frac{\sum a^2-2ab}{4}, \frac{\sum a^2-2ab}{4}, \frac{\sum a^2-ab}{4}$ (By Note 1, Art. 3.22) *i.e.,* $1,1,1$

But d.cs. of $\overrightarrow{\mathrm{OX}}, \overrightarrow{\mathrm{OY}}, \overrightarrow{\mathrm{OZ}}$ are $1,0,0;\ 0,1,0;\ 0,0,1$. Let $\theta=(\overrightarrow{\mathrm{PQ}}\ \ \overrightarrow{\mathrm{OX}})$

$\therefore\ \cos\theta=\frac{1.1+1.0+1.0}{\sqrt{3}.\sqrt{1}}=\frac{1}{\sqrt{3}}$ *i.e.* $\theta=\mathrm{Cos}^{-1}\frac{1}{\sqrt{3}}$

Similarly we can observe that $\overrightarrow{\mathrm{PQ}}\ \ \overrightarrow{\mathrm{OY}}=\mathrm{Cos}^{-1}(1/\sqrt{3}), (\overrightarrow{\mathrm{PQ}}\ \ \overrightarrow{\mathrm{OZ}})=(1/\sqrt{3})$

$\therefore$ The common line $\overleftrightarrow{\mathrm{PQ}}$ makes equal angles with the axes.

Ex. 3. *Show that the equation* $x^2+4y^2+9z^2-12yz-6zx+4xy+5x+10y-15z+6=0$ *represents a pair of parallel planes and find the distance between them.*

(S.V.U. A11, S.K.U. A11, S.V.M M18)

Sol. $x^2+4y^2+9z^2-12yz-6zx+4xy=(x+2y-3z)^2$

$\therefore\ x^2+4y^2+9z^2-12yz-6zx+4xy+5x+10y-15z+6$

$\equiv(x+2y-3z+k)\,(x+2y-3z+l)$ where

$k+l=5, 2k+2l=10, -3k-3l=-15, kl=6$ *i.e.,* $k=3, l=2$

$\therefore$ Given equation represents the planes $x+2y-3z+3=0, x+2y-3z+2=0$ which are parallel.

$\therefore$ Distance between the parallel planes $= \dfrac{|3-2|}{\sqrt{(1+4+9)}} = \dfrac{1}{\sqrt{(14)}}$

EXERCISE 3 (*c*)

1. Show the following equations represent pairs of planes. Also find the angles between them.

(*i*) $2x^2-3y^2+4z^2+xy+6zx-yz=0$ ***(O.U. Oct 2001, A.N.U. M13, K.U. M18, V.S.P.U M18)***

(*ii*) $2x^2-2y^2+4z^2+2yz+6zx+3xy=0$ ***(K. U. 08)***

(*iii*) $12x^2-2y^2-6z^2+7yz+6zx-2xy=0$ ***(S.K.U. M18)***

(*iv*) $6x^2+4y^2-10z^2-11xy+3yz+4zx=0$ ***(O.U.M. 97, S.V.U, K.U. A11, A12, N.U.12)***

(*v*) $x^2+4y^2-z^2+4xy=0$ ***(A.N.U. M06, M14)***

2. If the equation $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ represents a pair of intersecting planes, show that $abc+2fgh-af^2-bg^2-ch^2=0$. Also if $a+b+c \neq 0$, then an angle between the planes is $\operatorname{Tan}^{-1}\left[\dfrac{2(f^2+g^2+h^2-bc-ca-ab)^{1/2}}{a+b+c}\right]$

3. If $\phi(x,y,z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ represents a pair of intersecting planes through the origin, show that the product of the perpendiculars from to the two planes is $\dfrac{|\phi(\alpha,\beta,\gamma)|}{\sqrt{\{(a+b+c)^2+4(f^2+g^2+h^2-ab-bc-(a)\}}}$

4. If $a^2+b^2+c^2-2ab-2bc-2ca>0$, show that $\dfrac{a}{y-z}+\dfrac{b}{z-x}+\dfrac{c}{x-y}=0$ represents a pair of intersecting planes.

5. Find the planes represented by the equation.

$x^2-2y^2-z^2-xy+3yz-6x+3y+9=0$ and hence find the angle between them.

(S.V.U. M06)

6. Show that the equation $x^2+4y^2+4z^2+4xy+8yz+4zx-9x-18y-18z+18=0$ represents a pair of parallel planes. Hence find the distance between them. ***(S.V. U. A 93)***

ANSWERS

1. (*i*) $\cos^{-1}\left(\dfrac{\sqrt{3}}{\sqrt{29}}\right)$ (*ii*) $\cos^{-1}\left(\dfrac{4}{9}\right)$ (*iii*) $\cos^{-1}\left(\dfrac{4}{21}\right)$ (*iv*) $\dfrac{\pi}{2}$ (*v*) $\cos^{-1}\dfrac{\sqrt{2}}{3}$

5. $\operatorname{Cos}^{-1}\left(\dfrac{\sqrt{2}}{3}\right)$ **6.** 1

UNIT - II

4. **The Line (Right Line)**

   Equations of a line, Angle between a line and a plane, Conditions for a line to lie in a plane, coplanarity of lines, Number of arbitrary constants in the equations of a line, sets of conditions which determine a line, shortest distance between two skew lines, length of the perpendicular from a point to a line, Area of a triangle, Orthogonal Projection on a plane, volume of a tetrahedron, Intersection of three plane - Triangular prison.

5. **Change of Axes**

   Translation of Axes, Rotation of Axes.

4

RIGHT LINE

4.1. REPRESENTATION OF LINE

1. Consider XZ and XY planes. Their common line of intersection is X - axis.

P $(x, y, z) \in$ X $-$ axis $\quad \Leftrightarrow$ P $\in$ XY plane and P $\in$ XZ plane $\quad \Leftrightarrow z = 0$ and $y = 0$

$\therefore$ Equations to the line X - axis are $y = 0, z = 0$ *i.e.,* equations to the planes passing through the X - axis.

Similarly equations to Y - axis are $x = 0, z = 0$ *i.e.,* equations to the planes through the Y - axis and equations to Z - axis are $y = 0, x = 0$ *i.e.,* equations to the planes through the Z - axis.

2. Consider any line L and two planes π_1, π_2 whose line of intersection is L.

Let the equations to π_1, π_2 be respectively

$a_1x + b_1y + c_1z + d_1 = 0$ .....(1) $\qquad a_2x + b_2y + c_2z + d_2 = 0$ ..... (2)

P $(x, y, z) \in$ L $\Leftrightarrow$ P $\in \pi_1$ and P $\in \pi_2$

$\Leftrightarrow a_1x + b_1y + c_1z + d_1 = 0\ ,\ a_2x + b_2y + c_2z + d_2 = 0$

$\therefore$ Equations to the line L are $a_1x + b_1y + c_1z + d_1 = 0\ ,\ a_2x + b_2y + c_2z + d_2 = 0$

Thus : A line is represented by the equations of two planes through the line. Since any pair of planes can be taken through the line, the pairs of equations of the line are infinitely many.

4. 2. PARAMETRIC FORM

Theorem. ***Equations to the line passing through the point*** (x_1, y_1, z_1) ***and having d.cs.*** l, m, n***, are*** $x = x_1 + lr,\ y = y_1 + mr,\ z = z_1 + nr$***,*** r ***being any real number.***

Proof. (Fig. 46). Let L be the required line and A (x_1, y_1, z_1). d.cs. of L are l, m, n.

Let P $= (x, y, z) \in$ L. Let AP $= |r|$.

The projection of AP on the x - axis $= x - x_1 = lr$

Similarly $y - y_1 = mr, z - z_1 = nr$

$\Rightarrow x = x_1 + lr,\ y = y_1 + mr,\ z = z_1 + nr$

r being any real number.

$\therefore$ Equations to L are

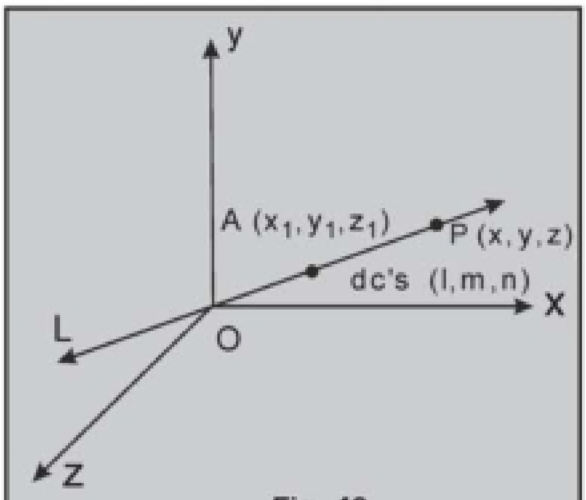


Fig. 46

$x = x_1 + lr,\ y = y_1 + mr,\ z = z_1 + nr$

i.e., $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$

Note.1. $\frac{a_1}{l_1}=\frac{a_2}{l_2}=\frac{a_3}{l_3} \Leftrightarrow a_1:l_1=a_2:l_2=a_3:l_3$ and if any of $l's$ is zero, the corresponding a is also zero.

2. Equations of the line in the parametric form $x=x_1+lr,\ y=y_1+mr,\ z=z_1+nr$ can be written as $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}\ (=r)$.

This form of equations of the line are called the equations of the line in the 'Symmetric form'.

3. Equations of the line through the point (x_1,y_1,z_1) and with d.rs. l,m,n are in

(i) ***Parametric form*** : $x=x_1+lr,\ y=y_1+mr,\ z=z_1+nr$

(ii) ***Symmetric form*** : $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$

(iii) ***Vector form*** : $\bar{r}=\bar{a}+t\bar{d}$ where $\bar{a}=(x_1,y_1,z_1)$, $\bar{d}=(l,m,n)$ and t being any real number.

4. Any point on the line through the point (x_1,y_1,z_1) and having d.rs. l,m,n is $(x_1+lr,y_1+mr,z_1+nr), r$ being any real number or $\bar{a}+t\bar{d}$, t any real number.

5. $\frac{x-x_1}{l}=\frac{y-y_1}{m}$ represents the plane through the line perpendicular to XY plane; $\frac{y-y_1}{m}=\frac{z-z_1}{n}$ represents the plane through the line perpendicular to YZ plane; $\frac{z-z_1}{n}=\frac{x-x_1}{l}$ represents the plane through the line perpendicular to ZX plane.

6. Let $n=0,\ l\neq 0,\ m\neq 0$. Equations to the line L are $\frac{x-x_1}{l}=\frac{y-y_1}{m}, z-z_1=0$

Then L represents a line parallel to Z - axis.

7. Let $m=0=n, l\neq 0$. Equations to the line L are $y-y_1=0,\ z-z_1=0$. Then L represents a line parallel to x-axis.

8. $(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=r^2(l^2+m^2+n^2)=r^2$

$\therefore\ |r|=\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}$

i.e. $|r|$ is the distance of any point (x,y,z) from the given point (x_1,y_1,z_1).

In this context the equations of L are called equations in 'Distance form'.

4. 3. Theorem. ***Equations of a line through two distinct points*** (x_1,y_1,z_1) ***and*** (x_2,y_2,z_2) ***are*** $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$ ***or*** $\frac{x-x_2}{x_2-x_1}=\frac{y-y_2}{y_2-y_1}=\frac{z-z_2}{z_2-z_1}$

Proof. Let L be the required line. Since $(x_1,y_1,z_1),(x_2,y_2,z_2)\in$ L , d.rs. of L are $x_2-x_1, y_2-y_1, z_2-z_1$.

$\therefore$ Equations to L are $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$ or $\frac{x-x_2}{x_2-x_1}=\frac{y-y_2}{y_2-y_1}=\frac{z-z_2}{z_2-z_1}$

4. 4. Theorem. ***Transform the equations $a_1x+b_1y+c_1z+d_1=0$ $a_2x+b_2y+c_2z+d_2=0$ of the line to symmetrical form. (O. U. A12)***

Proof. Let L be the line of intersection of the planes

$a_1x+b_1y+c_1z+d_1=0$ ... (1) $a_2x+b_2y+c_2z+d_2=0$ .... (2)

Let $[l,m,n]$ be the D.cs. of L.

L lies in both the planes (1) and (2). Since the d.rs. of the normals to the planes are (a_1,b_1,c_1) and (a_2,b_2,c_2) we have $a_1l+b_1m+c_1n=0$, $a_2l+b_2m+c_2n=0$

$$\Rightarrow \frac{l}{b_1c_2-b_2c_1}=\frac{m}{c_1a_2-c_2a_1}=\frac{n}{a_1b_2-a_2b_1}$$

Without loss of generality we can take $a_1b_2-a_2b_1\neq 0$.

Now to find the equations to L, we require a point on L.

Let L intersect, say, the XY plane *i.e.* $Z=0$ at P.

$\therefore$ From (1) and (2) : $a_1x+b_1y+d_1=0$, $a_2x+b_2y+d_2=0$.

Solving : $x=\dfrac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}, y=\dfrac{d_1a_2-d_2a_1}{a_1b_2-a_2b_1}$

$\therefore$ $P=\left(\dfrac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1},\dfrac{d_1a_2-d_2a_1}{a_1b_2-a_2b_1},0\right)$ is a point on L.

$\therefore$ Equations to L in the symmetric form are $\dfrac{x-\dfrac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}}{b_1c_2-b_2c_1}=\dfrac{y-\dfrac{d_1a_2-d_2a_1}{a_1b_2-a_2b_1}}{c_1a_2-c_2a_1}=\dfrac{z-0}{a_1b_2-a_2b_1}$

Ex. *Write the equations of the line $x=ay+b$ and $z=cy+d$ in the symmetrical form.* **(N. U. 98)**

Sol. Equations of the line L are $1x+(-a)y+0.z=b$ and $0x+cy+(-1)z=-d$.

Let d.rs. of the line L be l,m,n.

$\therefore$ $1.l+(-a)m+0(n)=0,\ 0.l+cm+(-1)n=0$ *i.e.,* $\dfrac{l}{a}=\dfrac{m}{1}=\dfrac{n}{c}$

A point on L is $(b,0,d)$. $\therefore$ L is $\dfrac{x-b}{a}=\dfrac{y-0}{1}=\dfrac{z-d}{c}$

OR : Equations to the line are $x=ay+b, z=cy+d$.

They can be written as $\dfrac{x-b}{a}=\dfrac{y}{1},\dfrac{z-d}{c}=\dfrac{y}{1}$. *i.e.* $\dfrac{x-b}{a}=\dfrac{y}{1}=\dfrac{z-d}{c}$ which form is the symmetrical form of the equations of the line.

Note. If the equations of a line are given as $a_1x+b_1y+c_1z+d_1=0$, $a_2x+b_2y+c_2z+d_2=0$, then the equations of the line are said to be in unsymmetrical form.

SOLVED PROBLEMS

Ex. 1. *Find the distance of the point $(1,-2,3)$ from the plane $x-y+z=5$ measured parallel to the line whose d.cs. are proportional to $2,3,-6$.*

Sol. Let L be the line through the point $P\,(1,-2,3)$ and parallel to the line with d.rs. $2,3,-6$.

$\therefore$ Equations to L are $\dfrac{x-1}{2}=\dfrac{y+2}{3}=\dfrac{z-3}{-6}$ $(=t$ say)

Any point on L is $Q\,(2t+1, 3t-2, -6t+3)$. Let π be the plane $x-y+z=5$.

$Q\in\pi \Rightarrow 2t+1-3t+2-6t+3=5 \Rightarrow -7t=1 \Rightarrow t=(1/7)$

$\therefore\ Q=\left(\dfrac{9}{7},\dfrac{-11}{7},\dfrac{15}{7}\right)$ $\quad\therefore\ PQ^2=\left(\dfrac{9}{7}-1\right)^2+\left(\dfrac{-11}{7}+2\right)^2+\left(\dfrac{15}{7}-3\right)^2=\dfrac{4+9+36}{49}=1$

$\therefore$ Required distance = 1

Ex. 2. *Find the image of the point* $(2,-1,3)$ *in the plane* $3x-2y+z=9$.

(O.U. 07, A.N.U. M15, 06, M14, S.V.U. A10, S.V. M. M18)

Sol. Let $P=(2,-1,3)$. Let π be the plane $3x-2y+z=9$.

Let $Q=(x_1,y_1,z_1)$ be the image of P in π. $\therefore\ \overleftrightarrow{PQ}\perp\pi$. $\therefore$ D.rs. of $\overleftrightarrow{PQ}$ are $3,-2,1$.

$\therefore$ Equation to $\overleftrightarrow{PQ}$ are $\dfrac{x-2}{3}=\dfrac{y+1}{-2}=\dfrac{z-3}{1}$ $(=t$ say)

Let R be the mid point of PQ. Let $R=(3t+2,-2t-1,t+3)$

$R\in\pi \Rightarrow 9t+6+4t+2+t+3=9 \Rightarrow t=-\dfrac{1}{7}$

$\therefore\ R=\left(\dfrac{-3}{7}+2,\dfrac{2}{7}-1,\dfrac{-1}{7}+3\right)=\left(\dfrac{11}{7},\dfrac{-5}{7},\dfrac{20}{7}\right)$. R is the mid point of PQ

$\Rightarrow 2+x_1=\dfrac{22}{7}$ *i.e.,* $x_1=\dfrac{8}{7}$; $-1+y_1=\dfrac{-10}{7}$ *i.e.,* $y_1=\dfrac{-3}{7}$; $3+z_1=\dfrac{40}{7}$ *i.e.,* $z_1=\dfrac{19}{7}$

$\therefore$ Q = Image of P in $\pi=\left(\dfrac{8}{7},\dfrac{-3}{7},\dfrac{19}{7}\right)$

Ex. 3. *Find the foot of the perpendicular from* $(1,2,3)$ *to the plane* $x+2y+3z+4=0$ ***(A.U. M13)***

Sol. Let the given plane is $\pi=x+2y+3z+4=0$ ... (1)

Given point is $P=(1,2,3)$

Let the foot of the perpendicular from P to the plane is $Q=(x_1,y_1,z_1)$

The Dr. of PQ are x_1-1, y_1-2, z_1-3

Equations of $\overline{PQ}$, $\dfrac{x_1-1}{1}=\dfrac{y_1-2}{2}=\dfrac{z_1-3}{3}=t$ (say)

$\therefore\ x_1=t+1, y_1=2t+2, z_1=3t+3$

Since Q lies on the plane, we have

$(t+1)+2(2t+2)+3(3t+3)+4=0$

$14t+18=0\Rightarrow t=\dfrac{-9}{7}$

$$x_1 = \frac{-9}{7} + 1 = \frac{-2}{7},\ y_1 = 2t + 2 = \frac{-18}{7} + 2 = -\frac{4}{7},\ z_1 = 3t + 3 = \frac{-27}{7} + 3 = \frac{-6}{7}$$

$\therefore$ Foot of the perpendicular $= P = (x_1, y_1, z_1) = \left(\frac{-2}{7}, \frac{-4}{7}, \frac{-6}{7}\right)$

Ex. 4. *Find the angles between the lines* $x - 2y + z = 0;\ x + y - z = 3; \ldots\ L_1$

$x + 2y + z = 5,\ 8x + 12y + 5z = 0 \ldots\ L_2$ ***(A. N. U. 07, M 13)***

Sol. Let the planes be $x - 2y + z = 0$ .... (1) $x + y - z = 3$ .... (2)

$x + 2y + z = 5$ ..... (3) $8x + 12y + 5z = 0$ .... (4)

(1), (2) represent the line L_1 and (3), (4) represent the line L_2.

The d.rs. of the normals to the planes (1), (2), (3), (4) are $(1, -2, 1)$, $(1, 1, -1)$, $(1, 2, 1)$ and $(8, 12, 5)$ respectively.

If (l_1, m_1, n_1) and (l_2, m_2, n_2) and the d.rs. of the lines L_1 and L_2 respectively,

$(l_1, m_1, n_1) = (2 - 1, 1 + 1, 1 + 2) = (1, 2, 3)$; $(l_2, m_2, n_2) = (10 - 12, 8 - 5, 12 - 16) = (-2, 3, -4)$

$\therefore$ D. rs. of L_1 are $1, 2, 3$ and d.rs. of L_2 are $-2, 3, -4$

If θ is one of the angles between L_1, L_2 then $\cos\theta = \frac{-2 + 6 - 12}{\sqrt{1 + 4 + 9}\,.\,\sqrt{4 + 9 + 16}} = \frac{-8}{\sqrt{406}}$

$\therefore$ $(L_1, L_2) = \mathrm{Cos}^{-1}\left(\frac{-8}{\sqrt{(406)}}\right), \pi - \mathrm{Cos}^{-1}\left(\frac{-8}{\sqrt{(406)}}\right)$

Ex. 5. *Find the d.rs. of the projection of the line* $\frac{x-4}{3} = \frac{y-2}{4} = \frac{z-1}{2}$ *in the plane* $9x + 8y + 2z - 1 = 0$.

Sol. Let L_1 be the line $\frac{x-4}{3} = \frac{y-2}{4} = \frac{z-1}{2}$. Let π_1 be the plane $9x + 8y + 2z - 1 = 0$.

Let π_2 be the plane through L_1 and perpendicular to π_1

Let (a, b, c) be the d.rs. of the normal to π_2.

$L_1 \parallel \pi_2 \Rightarrow 3a + 4b + 2c = 0$; $\pi_1 \perp \pi_2 \Rightarrow 9a + 8b + 2c = 0$

$\therefore$ $\frac{a}{-8} = \frac{b}{12} = \frac{c}{-12}$ *i.e.* $\frac{a}{2} = \frac{b}{-3} = \frac{c}{3}$. $\therefore$ D.rs. of the normal to π_2 are $2, -3, 3$

Let L_2 be the projection of L_1 in π_1. Let l, m, n be the d.rs. of L_2.

$L_2 \subset \pi_2 \Rightarrow 2l - 3m + 3n = 0$, $L_2 \subset \pi_1 \Rightarrow 9l + 8m + 2n = 0$

$\therefore$ $\frac{l}{-30} = \frac{m}{23} = \frac{n}{43}$. $\therefore$ d.rs. of L_2 are $-30, 23, 43$.

Note. Equation to the plane π_2 containing L_1 is

$2(x - 4) - 3(y - 2) + 3(z - 1) = 0$ *i.e.* $2x - 3y + 3z - 5 = 0$

$\therefore$ Equations to the line of projection of L_1 in π_1 are $9x + 8y + 2z - 1 = 0 = 2x - 3y + 3z - 5$

Ex. 6. *Find the image of the line* $\frac{x-1}{9}=\frac{y-2}{1}=\frac{z+3}{-3}$ *in the plane* $3x-3y+10z-26=0$ ***(N. U. A 93, S.V.U. A11, A.U M18, S.V. M. M18)***

Sol. Let L_1 be the line $\frac{x-1}{9}=\frac{y-2}{1}=\frac{z+3}{-3}$ $(=t \text{ say})$

Let π be the plane $3x-3y+10z-26=0$

Any point on L_1 is $(9t+1, t+2, -3t-3)$, If this point lies on π,

then $3(9t+1)-3(t+2)+10(-3t-3)-26=0 \Rightarrow -6t-59=0 \Rightarrow t=-\frac{59}{6}$

$\therefore$ L_1 cuts the plane π in the point

$\left(9\left(-\frac{59}{6}\right)+1, -\frac{59}{6}+2, -3\left(-\frac{59}{6}\right)-3\right)$ *i.e.,* $\left(-\frac{525}{6}, -\frac{47}{6}, \frac{159}{6}\right)$

$\therefore$ Image of $\left(-\frac{525}{6}, -\frac{47}{6}, \frac{159}{6}\right)$ in π is itself. Clearly $(1,2,-3)$ is a point on L_1.

$\therefore$ Equation to the line L_2 through $(1,2,-3)$ and perpendicular to the plane π is

$\frac{x-1}{3}=\frac{y-2}{-3}=\frac{z+3}{10}$ $(=r \text{ say})$

A point on this line L_2 is $(3r+1, -3r+2, 10r-3)$. If this point lies on π,

then $3(3r+1)-3(-3r+2)+10(10r-3)-26=0 \Rightarrow r=(1/2)$

$\therefore$ The foot of $(1,2,-3)$ in π is $\left(\frac{3}{2}+1, \frac{-3}{2}+2, 10\times\frac{1}{2}-3\right)$ *i.e.* $\left(\frac{5}{2}, \frac{1}{2}, 2\right)$

If (x_1, y_1, z_1) is the image of $(1,2,-3)$ in π, then

$\frac{1+x_1}{2}=\frac{5}{2}, \ \frac{2+y_1}{2}=\frac{1}{2}, \ \frac{-3+z_1}{2}=2 \Rightarrow x_1=4, \ y_1=-1, \ z_1=7$

$\therefore$ Image of $(1,2,-3)$ in π is $(4,-1,7)$.

$\therefore$ The line through the points $\left(-\frac{525}{6}, -\frac{47}{6}, \frac{159}{6}\right)$, $(4,-1,7)$ is the image of the line L_1 and its equation is *i.e.* $\frac{x-4}{549}=\frac{y+1}{41}=\frac{z-7}{-117}$

Ex. 7. *Find the equations to the line* L_1 *passing through the origin and perpendicular to the* L_2 *whose equations are* $\frac{x-2}{1}=\frac{y+3}{-2}=\frac{z}{1}$. *Also find the foot of the perpendicular from the origin to* L_2. ***(S.V.U.M06)***

Sol. Equations to L_2 are $\frac{x-2}{1}=\frac{y+3}{-2}=\frac{z}{1}$ $(=t \text{ say})$

Any point L_2 is $(t+2, -2t-3, t)$.

Let P be the foot of the perpendicular from the origin to the line L_2.

If $P=(t+2, -2t-3, t)$ then d.rs. of $(\because \ L_1=\overleftrightarrow{OP})$ are $t+2, -2t-3, t$.

$L_1 \perp L_2 \Rightarrow 1(t+2)-2(-2t-3)+1t=0 \Rightarrow 6t=-8 \Rightarrow t=-\frac{4}{3}$

$\therefore$ Foot of the perpendicular from the origin to $L_2 = \left(\frac{-4}{3}+2, \frac{8}{3}-3, \frac{-4}{3}\right) = \left(\frac{2}{3}, \frac{-1}{3}, \frac{-4}{3}\right)$

$\therefore$ Equations to L_1 are $\frac{x-0}{(2/3)} = \frac{y-0}{(-1/3)} = \frac{z-0}{(-4/3)}$ *i.e.,* $\frac{x}{2} = \frac{-y}{1} = \frac{-z}{4}$

Ex. 8. *A variable plane makes intercepts on the axes, the sum of whose squares is k^2 (a constant). Show that the locus of the foot of the perpendicular from origin to the plane is* $(x^{-2}+y^{-2}+z^{-2})\ (x^2+y^2+z^2)^2 = k^2$

(S. V. U., N. U. 92, A. U. 12, (Refer to Ex. 21 (b) of Exercise 9(a))

Sol. π is a variable plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ $(abc \neq 0)$.

$\Rightarrow$ intercepts of π on the axes are a, b, c $\Rightarrow a^2+b^2+c^2=k^2$ (given) ..... (1)

D.rs. of normal to π are $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.

Let L be the line through O and perpendicular to π.

$\therefore$ Equations to L are $\frac{x}{1/a} = \frac{y}{1/b} = \frac{z}{1/c}$ $(=t$ say).

$\therefore$ Any point P on L is $\left(\frac{t}{a}, \frac{t}{b}, \frac{t}{c}\right)$. P is the foot of the perpendicular from O to π

$\Rightarrow P \in \pi \Rightarrow \frac{t}{a^2}+\frac{t}{b^2}+\frac{t}{c^2}=1 \Rightarrow t=(a^{-2}+b^{-2}+c^{-2})^{-1}$

$\therefore$ $P=(a^{-1}(a^{-2}+b^{-2}+c^{-2})^{-1},\ b^{-1}(a^{-2}+b^{-2}+c^{-2})^{-1},\ c^{-1}(a^{-2}+b^{-2}+c^{-2})^{-1})$.

$P=(x_1, y_1, z_1) \Rightarrow x_1 = a^{-1}(a^{-2}+b^{-2}+c^{-2})^{-1};\ y_1 = b^{-1}(a^{-2}+b^{-2}+c^{-2})^{-1};$

$z_1 = c^{-1}(a^{-2}+b^{-2}+c^{-2})^{-1}$.

$x_1^2+y_1^2+z_1^2 = [(a^{-2}+b^{-2}+c^{-2})^{-1}]^2(a^{-2}+b^{-2}+c^{-2}) = (a^{-2}+b^{-2}+c^{-2})^{-1}$ and

$x_1^{-2}+y_1^{-2}+z_1^{-2} = (a^{-2}+b^{-2}+c^{-2})^2(a^2+b^2+c^2)$

$\Rightarrow (x_1^{-2}+y_1^{-2}+z_1^{-2}) = \frac{1}{(x_1^2+y_1^2+z_1^2)^2}.k^2$ [using (1)]

$\Rightarrow (x_1^{-2}+y_1^{-2}+z_1^{-2})\ (x_1^2+y_1^2+z_1^2)^2 = k^2$

$\therefore$ Locus of P is $(x^{-2}+y^{-2}+z^{-2})\ (x^2+y^2+z^2)^2 = k^2$

Ex. 9. *The plane $lx+my+nz=p,\ l^2+m^2+n^2=1,\ p>0$ meets the axes in P, Q, R and G is the centroid of the ΔPQR, If the perpendicular line to the plane at G meets the coordinate planes in A, B, C then prove that* $\frac{1}{GA}+\frac{1}{GB}+\frac{1}{GC}=\frac{3}{p}$

Sol. Let π be the plane $lx+my+nz=p, l^2+m^2+n^2=1$. π meets the axes in P, Q, R.

$\therefore$ $P=\left(\frac{p}{l},0,0\right), Q=\left(0,\frac{p}{m},0\right), R=\left(0,0,\frac{p}{n}\right)$

$\therefore$ Centroid G of Δ PQR $= (p/3l,\ p/3m,\ p/3n)$

Let L be the line perpendicular to π at G.

$\therefore$ Equations to L are $\dfrac{x-(p/3l)}{l} = \dfrac{y-(p/3m)}{m} = \dfrac{z-(p/3n)}{n}$ $\quad (= t$ say)

L meets the YZ plane *i.e.* $x = 0$ in A.

$\therefore |t| = \text{GA} = \left|\dfrac{0-(p/3l)}{l}\right| = \left|\dfrac{-p}{3l^2}\right|$ *i.e.,* $\dfrac{1}{\text{GA}} = \dfrac{3l^2}{p}$ $(p > 0)$

Similarly $\dfrac{1}{\text{GB}} = \dfrac{3m^2}{p}$ and $\dfrac{1}{\text{GC}} = \dfrac{3n^2}{p}$ $\quad \therefore \dfrac{1}{\text{GA}} + \dfrac{1}{\text{GB}} + \dfrac{1}{\text{GC}} = \dfrac{3l^2+3m^2+3n^2}{p} = \dfrac{3}{p}$

EXERCISE 4 (*a*)

1. (*a*) Find the equations of the line through (α, β, γ) and parallel to

(*i*) $\overleftrightarrow{X'X}$ $\quad$ (*ii*) $\overleftrightarrow{Y'Y}$ $\quad$ (*iii*) $\overleftrightarrow{Z'Z}$

(*b*) Show that the line $m(x-a) = l(z-b), y = c$ is perpendicular to the y- axis.

2. Find the equations of the line through $(3,1,2)$ and equally inclined to the axes.

3. Find the equations of the line joining $(-2,1,3)$ and $(1,1,4)$.

4. Prove that the points $(1,2,3), (4,0,4), (-2,4,2), (7,-2,5)$ are collinear.

5. Show that the four points $(1,1,1), (-2,4,1), (-1,5,5), (2,2,5)$ are the vertices of a square. Find the equations to its diagonals.

6. Show that the points A $(2,-1,1)$, B $(1,2,-3)$, C $(1,2,-1)$, D $(2,-1,3)$ form a parallelogram. Find the equations to the side BC.

7. (*i*) Show that $(1,3,-2)$ is the point of intersection of the line $\dfrac{x+1}{1} = \dfrac{y+3}{3} = \dfrac{z-2}{2}$ with the plane $3x+4y+5z=5$. **(*K. U. 08, A. N. U. M13*)**

(*ii*) Show that the line joining $(2,-3,1), (3,-4,-5)$ intersects the plane $2x+y+z=7$ in the point $(1,-2,7)$.

8. Show that the line $\dfrac{x-3}{3} = \dfrac{2-y}{4} = \dfrac{z+1}{1}$ intersects the line $2x+4y+3z+3=0$, $x+2y+3z=0$ in the point $(9,-6,1)$. **(*S.V.U, A.K.N.V M18*)**

9. Find the point of intersection of the lines $\dfrac{x-1}{-3} = \dfrac{y-2}{2} = \dfrac{z-3}{2}$ and $\dfrac{x-1}{3} = \dfrac{y-5}{1} = \dfrac{z}{-5}$ **(*V.S.P.V M18*)**

10. Find the angles between the lines $\dfrac{x-2}{1} = \dfrac{y-4}{0} = \dfrac{z-5}{-1}$ and $\dfrac{x}{3} = \dfrac{y}{4} = \dfrac{z}{5}$.

11. Find k so that the lines

(*i*) $\dfrac{x+1}{-3} = \dfrac{y+2}{2k} = \dfrac{z-3}{2}$ and $\dfrac{x-1}{3k} = \dfrac{y+5}{1} = \dfrac{z+6}{7}$ are perpendicular. **(*K. U. A12*)**

(*ii*) $\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$ and $\frac{x-1}{3k}=\frac{y-5}{1}=\frac{z-6}{-5}$ are perpendicular. ***(N.U.S. 98, S.V.U.08)***

12. Find the equations of the lines bisecting the angles between the lines

$$\frac{x-1}{1}=\frac{y+4}{2}=\frac{z-5}{-2},\ \frac{x-1}{4}=\frac{y+4}{3}=\frac{z-5}{12}.$$

13. Find the distance of the point $(3,-4,5)$ from the plane $2x+5y-6z=16$ measured along a line with d.cs. proportional to $(2,1,-2)$.

14. Find the equations of the planes passing through the line $2x+3y-5z-4=0=3x-4y+5z-6$ and parallel to the coordinate axes.

15. Find the equation of the plane through the point $(1,1,1)$ and perpendicular to the line $x-2y+z-2=0=4x+3y-z+1$.

16. Find in symmetrical form the equation of the line $x+y+z+1=0=4x+y-2z+2$. ***(K.U. M18)***

17. Prove that the equations of the line through (x_1,y_1,z_1) and perpendicular to the lines $\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}$ and $\frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}$ are $\frac{x-x_1}{m_1n_2-m_2n_1}=\frac{y-y_1}{n_1l_2-n_2l_1}=\frac{z-z_1}{l_1m_2-l_2m_1}$.

18. Find the equation to the plane through (x_1,y_1,z_1) and parallel to the lines $\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}$ and $\frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}$.

19. Find the equation of the line through the points (a,b,c) and (a',b',c') and prove that it passes through the origin if

$$(a^2+b^2+c^2)(a'^2+b'^2+c'^2)-(aa'+bb'+cc')^2=(ab'-a'b)^2+(bc'-b'c)^2+(ca'-c'a)^2.$$

20. Find the image of the

(*i*) point $(1,3,4)$ in the plane $2x-y+z+3=0$,
(Kr. U. 12, S. V. U. O 90, N. U. A12, K. U. M14, S.K.U M18, S.V.M M18, V.S.P.V M18)

(*ii*) point $(1,6,3)$ in the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$, ***(S. K. U. A11)***

(*iii*) line $\frac{x-1}{9}=\frac{y-2}{1}=\frac{z+3}{-3}$ in the plane $3x-3y+10z-26=0$,

(*iv*) line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ in the plane $x+y+z=1$

21. Find the equations of the line through the point $(1,2,4)$ and parallel to the line $3x+2y-z=4,\ x-2y-2z=5$. ***(A. K.N.U. M18)***

22. (*i*) Find the angle between the lines $x+2y-2z=0, x-2y+z=7;\ \frac{x-1}{1}=\frac{y+2}{-2}=\frac{z}{2}$.

(*ii*) Show that the lines $2x+3y-4z=0=3x-4y+z-7,\ 5x-y-3z+12=0,$
$x-7y+5z-6=0$ are parallel.

(*iii*) Find an angle between the lines

$$3x+2y+z-5=0=x+y-2z-3,\ \ 2x-y-z=0=7x+10y-8z$$

23. Show that the condition for the lines $x = az + b, y = cz + d$; $x = a_1 z + b_1, y = c_1 z + d_1$ to be perpendicular is $aa_1 + cc_1 + 1 = 0$ ***(O. U. M. 97, O 97, N. U. 88)***

24. Find a, b, c, d so that the line $x = ay + b, y = cz + d$, may pass through the points $(3, 1, -3)$, $(4, 2, -4)$ and hence show that the given points and $(5, 3, -5)$ are collinear.

25. Show that the equations of the perpendicular from the point $(1, 6, 3)$ to the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ are $x - 1 = 0, \frac{y-6}{-3} = \frac{z-3}{2}$ and the foot of the perpendicular is $(1, 3, 5)$ and the length of the perpendicular is $\sqrt{(13)}$.

26. P is a point on the plane $lx + my + nz = p$. If Q is a point on $\overleftrightarrow{OP}$ such that OP . OQ $= p^2$, then show that the locus of Q is $p\,(lx + my + nz) = x^2 + y^2 + z^2$.

27. Find the equation of the line of projection of the line $\frac{x-1}{2} = \frac{y}{-1} = \frac{z+2}{1}$ in the plane $2x + y - 3z - 4 = 0$.

ANSWERS

1. (*i*) $y - \beta = 0, z - \gamma = 0$ (*ii*) $x - \alpha = 0, z - \gamma = 0$. (*iii*) $x - \alpha = 0, y - \beta = 0$

2. $\frac{x-3}{1} = \frac{y-1}{1} = \frac{z-2}{1}$ **3.** $\frac{x+2}{-3} = \frac{z-3}{-1}, y - 1 = 0$

5. $\frac{x-1}{1} = \frac{y-1}{-2} = \frac{z-1}{-2}, \frac{x+2}{-2} = \frac{y-4}{1} = \frac{z-1}{-2}$ **6.** $x - 1 = 0, y - 2 = 0$. **9.** $(-2, 4, 5)$

10. $\cos^{-1}(-1/5), \pi - \cos^{-1}(-1/5)$ **11.** (*i*) $k = 2$ (*ii*) $k = -10/7$

12. $\frac{x-1}{5} = \frac{y+4}{7} = \frac{z-5}{2}; \frac{x-1}{-1} = \frac{y+4}{-17} = \frac{z-5}{62}$ **13.** $\frac{60}{7}$

14. $17y - 25z = 0, 17x - 5z - 34 = 0, 5x - y = 10$ **15.** $x - 5y - 11z + 15 = 0$

16. $\frac{x+1/3}{-1} = \frac{y+2/3}{2} = \frac{z}{-1}$ **18.** $\Sigma\,(x - x_1)(m_1 n_2 - m_2 n_1) = 0$

20. (*i*) $(-3, 5, 2)$ (*ii*) $(1, 0, 7)$ (*iii*) $\frac{x-4}{9} = \frac{y+1}{-1} = \frac{z-7}{-3}$

(*iv*) $\frac{x+7/3}{4} = \frac{y+4/3}{3} = \frac{z+1/3}{2}$ **21.** $\frac{x-1}{6} = \frac{2-y}{5} = \frac{z-4}{8}$

22. (*i*) $\cos^{-1}\left(\frac{8}{\sqrt{406}}\right), \pi - \cos^{-1}\left(\frac{8}{\sqrt{406}}\right)$ (*iii*) $\frac{\pi}{2}$

24. $a = 1, b = 2, c = 1, d = -2$ **27.** $x + 4y + 2z + 3 = 0 = 2x + y - 3z - 4$

4.5. ANGLE BETWEEN A LINE AND A PLANE

Theorem. *If θ is the acute angle between the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ and the plane $ax+by+cz+d=0$, then* $\sin\theta=\pm\frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$

Proof. Let L be the given line and E be the given plane.

The d.rs. of L are (l, m, n) and the d.rs. of the normal to E are (a, b, c).

Since θ is the acute angle between L and E,

the angles between L and normal to E are

$90°\pm\theta$ (vide 46 of Art. 1.3)

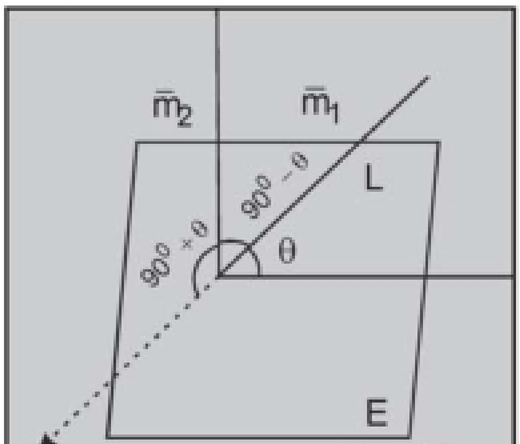


Fig. 47

$$\Rightarrow \cos(90\pm\theta)=\pm\sin\theta=\frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$$

(vide Art. 2.41)

$$\Rightarrow \sin\theta=\pm\frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$$

Note. 1. $al+bm+cn=0 \Leftrightarrow L \parallel E$. **2.** $\frac{a}{l}=\frac{b}{m}=\frac{c}{n}\Leftrightarrow L\perp E$

4. 6. CONDITIONS FOR A LINE TO LIE IN A PLANE

Theorem. *If L is the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ and π is the plane $ax+by+cz+d=0$, then* $L\subset\pi \Leftrightarrow ax_1+by_1+cz_1+d=0,\ al+bm+cn=0$ *(O. U. M15)*

Proof. Any point P on L is (x_1+lr, y_1+mr, z_1+nr) where r is any real number.

$L\subset\pi \Leftrightarrow P\in\pi \Leftrightarrow a(x_1+lr)+b(y_1+mr)+c(z_1+nr)+d=0$ for any real number

$\Leftrightarrow (ax_1+by_1+cz_1+d)+r(al+bm+cn)=0$ for any real number r

$\Leftrightarrow ax_1+by_1+cz_1+d=0,\ al+bm+cn=0$

Note. 1. A line lies in a plane if (*i*) any point on the line lies in the plane, and

(*ii*) the normal to the plane is perpendicular to the line.

2. Equation to a plane containing the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ can be taken as $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ where a, b, c are parameters such that $al+bm+cn=0$.

3. The line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ is parallel to the plane $ax+by+cz+d=0$ and not contained in the plane $\Rightarrow al+bm+cn=0$ and $ax_1+by_1+cz_1+c\neq 0$.

4. The line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ is perpendicular to the plane

$ax+by+cz+d=0\Rightarrow\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$

SOLVED PROBLEMS

Ex. 1. *Find the equation to a plane through the line* $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ *and parallel to another line with d.cs.* l_2, m_2, n_2.

Sol. Equation to the plane through the line $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ can be taken as

$l(x-x_1)+m(y-y_1)+n(z-z_1)=0$ .... (1) where $l l_1+mm_1+nn_1=0$ .... (2)

If (1) is parallel to another line with d.cs. l_2, m_2, n_2 then $l l_2+mm_2+nn_2=0$ .... (3)

$\therefore$ From (2) and (3), $\frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_1}$ (= Ksay)

$\therefore$ Equation to the required plane is

$(m_1n_2-m_2n_1)(x-x_1)+(n_1l_2-n_2l_1)(y-y_1)+(l_1m_2-l_2m_1)(z-z_1)=0$

i.e. $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$ (may also be obtained by eliminating *l, m, n* from (1), (2), (3))

Ex. 2. *Find the equation to the plane through the line* $\frac{x-x_2}{l}=\frac{y-y_2}{m}=\frac{z-z_2}{n}$ *and through the point* (x_1, y_1, z_1) ***(K. U. M. 11, S. V. U. 12)***

Sol. Let L be the line $\frac{x-x_2}{l}=\frac{y-y_2}{m}=\frac{z-z_2}{n}$. $\therefore$ L passes through the point (x_2, y_2, z_2).

Let π be the plane containing the line L and passing through the point (x_1, y_1, z_1).

Let the equation to the plane π be $a(x-x_2)+b(y-y_2)+c(z-z_2)=0$ .... (1)

where $al+bm+cn=0$ .... (2) Also $a(x_1-x_2)+b(y_1-y_2)+c(z_1-z_2)=0$ ..... (3)

Eliminating *a, b, c* from (1), (3) and (2), equation to the plane π

is $\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l & m & n \end{vmatrix}=0$

Ex. 3. *Find the equation of the plane through the origin and containing the line* $x-3y+2z+3=0=3x-y+2z-5$.

Sol. Given line is $x-3y+2z+3=0=3x-y+2z-5$.

Any plane through the given line is $x-3y+2z+3+\lambda(3x-y+2z-5)=0$

If it passes through the origin, then $0+3+\lambda(0-5)=0 \Rightarrow \lambda=3/5$

$\therefore$ Equation to the required plane is $x-3y+2z+3+(3/5)(3x-y+2z-5)=0$

i.e. $7x-9y+8z=0$.

Ex. 4. *Find the equation of the plane which passes through the line* $a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2$ *and is parallel to the line* $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.

Sol. Let the plane through the line $a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2$ and parallel to $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be $a_1x+b_1y+c_1z+d_1+\lambda\,(a_2x+b_2y+c_2z+d_2)=0$ where

$$(a_1+\lambda a_2)\,l+(b_1+\lambda b_2)\,m+(c_1+\lambda c_2)\,n=0$$

i.e., $\lambda=-\dfrac{la_1+mb_1+nc_1}{la_2+mb_2+nc_2}$, $(la_2+mb_2+nc^2\neq 0)$

$\therefore$ Equation to the required plane is

$$(la_2+mb_2+nc_2)\,(a_1x+b_1y+c_1z+d_1)-(la_1+mb_1+nc_1)\,(a_2x+b_2y+c_2z+d_2)=0$$

Ex. 5. $A=(a,0,0), B=(0,b,0), C=(0,0,c)$ *and* $P=(a,b,c)$ *are points distinct from O such that* $p^2=a^2+b^2+c^2\ (p>0)$ *and* $q^{-2}=a^{-2}+b^2+c^{-2}\ (q>0)$. *If* θ *is the acute angle between* $\overleftrightarrow{OP}, \overleftrightarrow{ABC}$ *show that* $\sin\theta=\frac{3q}{p}$.

Sol. Equation to the plane $\overleftrightarrow{ABC}$ is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

D.rs. of its normal are $(1/a),(1/b),(1/c)$.

D.rs. of $\overleftrightarrow{OP}$ are a,b,c. $\qquad\therefore\ \theta=(\overleftrightarrow{OP},\overleftrightarrow{ABC})$

$$\Rightarrow \sin\theta=\frac{a.\frac{1}{a}+b.\frac{1}{b}+c.\frac{1}{c}}{\sqrt{a^2+b^2+c^2}\,.\sqrt{(1/a^2)+(1/b^2)+(1/c^2)}}\Rightarrow \sin\theta=\frac{3}{p.\frac{1}{q}}=\frac{3q}{p}$$

EXERCISE 4 (*b*)

1. Show that the line $\frac{x+1}{-1}=\frac{y+2}{3}=\frac{z+5}{5}$ lies in the plane $x+2y-z=0$
2. Show that the line $\frac{x-2}{1}=\frac{y+3}{-2}=\frac{z+4}{5}$ is parallel to $3x+4y+z=4$
3. Show that the line $\frac{x+3}{-2}=\frac{y+4}{3}=\frac{z+4}{5}$ is perpendicular to the plane $-2x+3y+5z=0$
4. Interpret the nature of the equations $l_1\,(x-x_1)+m_1\,(y-y_1)+n_1\,(z-z_1)=0$;
   $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$
5. Find the equation to the plane containing the parallel lines
   $\frac{x-3}{4}=\frac{y-2}{-5}=\frac{z-4}{-1}$ and $\frac{x+2}{-4}=\frac{y}{5}=\frac{z-3}{1}$ ***(N. U. A10)***
6. Find the equation to the plane containing the line $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z-3}{4}$ and is perpendicular to the plane $x+2y+z-12=0$. ***(A. U. A11, O. U. M14, S.U.M M18)***

7. Find the equation to the plane through the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and parallel to (*i*) the x-axis, (*ii*) the y-axis (*iii*) the z-axis

8. Find the equation of the plane through the z-axis and is perpendicular to the line $\frac{x+5}{\sin\theta}=\frac{y+2}{\cos\theta}, z-7=0$

9. (*i*) Find the equation of the plane through the points $(1,0,-1),(0,-8,1)$ and parallel to the line $6(x+1)=3-3y=2z+4$.

(*ii*) Obtain the condition for a line to be parallel to the given plane and hence write the equation of the plane through the points $(2,-1,0),(3,-4,5)$ and parallel to the line $3x=2y=z$ **(*O. U. M. 98*)**

10. Prove that the plane through the point (x_1, y_1, z_1) and the line $x = py+q = rz+s$ is
$$\begin{vmatrix} x & py+q & rz+s \\ x_1 & py_1+q & rz_1+s \\ 1 & 1 & 1 \end{vmatrix}=0$$

11. (*i*) Find the equation of the plane through the line $x-y+3z+5=0=2x+y-2z+6$ and passing through the point $(3,1,1)$.

(*ii*) Find the equation of the plane containing the line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ and the point $(0,7,-7)$. Also show that the line $6x=14-2y=3z+21$ lies in the same plane. **(*O.U.08*)**

12. Find the equation to the plane through the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ and parallel to the line $ax+by+cz+d=0=a_1x+b_1y+c_1z+d_1$.

13. (*a*) Find the equation of the plane containing the line $2x-5y+2z-6=0=2x+3y-z-5$ and parallel to the line $x=\frac{-y}{6}=\frac{z}{7}$. **(*O. U. M 14*)**

(*b*) Show that the distance of the point (3, 8, 2) from the line $\frac{x-1}{2}=\frac{y-3}{4}=\frac{z-2}{3}$ measured parallel to the plane $3x+2y-2z+5=0$ is 7.

(*c*) Find the distance of the point $(-2,3,-4)$ from the line $\frac{x+2}{3}=\frac{2y+3}{4}=\frac{3z+4}{5}$ measured parallel to the plane $4x+12y-3z=0$. **(*A. U. M 13*)**

14. Find the equation of the line through $(1,2,3)$ and parallel to the planes $2x+3y+4z=11$ and $3x+4y+5z=12$.

15. Find the equation to a pair of parallel planes each plane containing one of the two lines $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$

ANSWERS

4. First equation represents a plane through the point (x_1, y_1, z_1) and having l_1, m_1, n_1 as d.rs. to its normal.

   Second equation represents the line normal to the plane at the point (x_1, y_1, z_1).

5. $-x-3y+11z=35$. 6. $9x-2y-5z+4=0$

7. $4y-3z+1=0$, $2x-z+1=0$, $3x-2y+1=0$. 8. $x\cos\theta+y\sin\theta=0$

9. (*i*) $4x-y-2z=6$ (*ii*) $-33x-4y+9z+70=0$

11. (*i*) $9x+3y-5z+29=0$ (*ii*) $x+y+z=0$

12. $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l & m & n \\ bc_1-b_1c & ca_1-c_1a & ab_1-a_1b \end{vmatrix}=0$ 13. $6x+y-16=0$ 14. $\dfrac{x-1}{1}=\dfrac{y-2}{-2}=\dfrac{z-3}{1}$

15. $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$ $\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$

4. 7. COPLANARITY OF LINES

Theorem. L_1, L_2 *are lines whose equations are* $\dfrac{x-x_1}{l_1}=\dfrac{y-y_1}{m_1}=\dfrac{z-z_1}{n_1}$ .... **(1)**

$\dfrac{x-x_2}{l_2}=\dfrac{y-y_2}{m_2}=\dfrac{z-z_2}{n_2}$ ....**(2).** L_1, L_2 *are coplanar* $\Rightarrow \begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$

(O. U. 08, A11, K.U.) (S. V. U., N. U. 88, A. U. M 13)

Proof : First Method.

Equation to the plane containing line (1) is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ .... (3)

with the condition that $al_1+bm_1+cn_1=0$ .... (4) where not all *a, b, c* being zero.

The line (2) lies on the plane (3) if

(*i*) the point (x_2, y_2, z_2) lies on (3) $\Rightarrow a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1)=0$ ..... (5)

and (*ii*) the line (2) is perpendicular to the normal to the plane (3)

$\Rightarrow al_2+bm_2+cn_2=0$ ... (6)

The given lines (1) and (2) are coplanar if the three linear homogeneous equations (4), (5), (6) in *a, b, c* are consistent.

Eliminating a, b, c from the equations (4),(5),(6) we get $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$

.... (7)

This is the condition for the lines (1) and (2) to be coplanar.

Note 1. If the above condition is satisfied then the equation to the plane containing (1) and (2) is obtained by eliminating *a, b, c* from equations (3), (4), (6)

$$i.e., \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Second Method. Proof. Let $A=(x_1,y_1,z_1)$ and $B=(x_2,y_2,z_2)$

$\therefore$ $A \in L_1$ and $B \in L_2$. $\therefore$ $\overline{BA}$ is coplanar with L_1 and L_2

Let $\bar{n}_1=(l_1,m_1,n_1)$ and $\bar{n}_2=(l_2,m_2,n_2)$.

Since l_1,m_1,n_1 are d.rs. of L_1 and l_2,m_2,n_2 are d.rs. of L_2, we have $\bar{n}_1 \parallel L_1$ and $\bar{n}_2 \parallel L_2$.

L_1, L_2 are coplanar.

$\Leftrightarrow [\overline{BA}, \bar{n}_1, \bar{n}_2]=0 \quad \Leftrightarrow [(x_1-x_2, y_1-y_2, z_1-z_2),\ (l_1,m_1,n_1),\ (l_2,m_2,n_2)]=0$

$$\Leftrightarrow \begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \ .... \ \text{I.}$$

$\therefore$ For L_1, L_2 to be coplanar, I is the required condition.

Note 2. Condition for the lines $\bar{r}=\bar{a}+t\bar{b}$ and $\bar{r}=\bar{c}+s\bar{d}$ to be concurrent.

The line $\bar{r}=\bar{a}+t\bar{b}$ passes through the point $\bar{a}$ and is parallel to $\bar{b}$ and the line $\bar{r}=\bar{c}+s\bar{d}$ passes through the point $\bar{c}$ and is parallel to $\bar{d}$.

Lines $\bar{r}=\bar{a}+t\bar{b}$ and $\bar{r}=\bar{c}+s\bar{d}$ are coplanar $\Leftrightarrow \bar{a}-\bar{c}, \bar{b}, \bar{d}$ are coplanar.

$\Leftrightarrow [\bar{a}-\bar{c}, \bar{b}, \bar{d}]=0 \Leftrightarrow [\bar{a}\ \bar{b}\ \bar{d}]=[\bar{c}\ \bar{b}\ \bar{d}]$.

3. If the lines $\bar{r}=\bar{a}+t\,(\bar{b}+\bar{c})$, t being real and $\bar{r}=\bar{b}+s\,(\bar{c}+\bar{a})$, s being real, intersect, then they are coplanar.

For their point of intersection $\bar{r}, \bar{a}+t\,(\bar{b}+\bar{c})=\bar{b}+s\,(\bar{c}+\bar{a})$

$\Leftrightarrow t\bar{b}=\bar{b},\ t\bar{c}=s\bar{c}, \bar{a}=s\bar{a} \quad \Leftrightarrow s=t=1$.

$\therefore$ $\bar{a}+\bar{b}+\bar{c}$ is the point of intersection of the lines.

4. 8. Theorem. ***Equation to the plane containing the line*** L_1 ***with equations*** $\dfrac{x-x_1}{l_1}=\dfrac{y-y_1}{m_1}=\dfrac{z-z_1}{n_1}$ ***and parallel to the line*** L_2 ***with equations*** $\dfrac{x-x_2}{l_2}=\dfrac{y-y_2}{m_2}=\dfrac{z-z_2}{n_2}$ ***is*** $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$ **(S. V. U.)**

First Method.

Proof. The lines are $\dfrac{x-x_1}{l_1}=\dfrac{y-y_1}{m_1}=\dfrac{z-z_1}{n_1}$ .... (1) $\dfrac{x-x_2}{l_2}=\dfrac{y-y_2}{m_2}=\dfrac{z-z_2}{n_2}$ .... (2)

the equation to the plane containing the line (1) is

$a\,(x-x_1)+b\,(y-y_1)+c\,(z-z_1)=0$ .... (3)

with the condition $al_1 + bm_1 + cn_1 = 0$ .... (4) where not all a, b, c are zero.

The plane (3) will be parallel to the line (2), if $al_2 + bm_2 + cn_2 = 0$ .... (5)

We obtain the equation to the required plane by eliminating a, b, c from (3), (4), (5)

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Second Method. Let π be the plane containing L_1 and parallel to L_2

Let $A = \bar{a} = (x_1, y_1, z_1)$. $\therefore A \in L_1$ *i.e.,* $A \in \pi$

Let $\bar{P} = \bar{r} = (x, y, z) \in \pi$.

Let $\bar{n}_1 = (l_1, m_1, n_1)$. $\therefore \bar{n}_1 \parallel L_1$. Let $\bar{n}_2 = (l_2, m_2, n_2)$. $\therefore \bar{n}_2 \parallel L_2$

$A \in \pi,\ P \in \pi \Rightarrow P = A$ or $P \neq A \Rightarrow \bar{r} - \bar{a} = 0$ or $\bar{r} - \bar{a} \neq 0$.

If $\bar{r} - \bar{a} = 0$, then $[\bar{r} - \bar{a}, \bar{n}_1, \bar{n}_2] = 0$ .... (I)

If $\bar{r} - \bar{a} \neq 0$, then $\bar{r} - \bar{a}, \bar{n}_1, \bar{n}_2$ are coplanar. *i.e.,* $[\bar{r} - \bar{a}, \bar{n}_1, \bar{n}_2] = 0$ .... (II)

$P \in \pi,\ L_1 \subset \pi,\ L_1 \parallel \pi$ (using I, II)

$\Leftrightarrow [\bar{r} - \bar{a}, \bar{n}_1, \bar{n}_2] \quad \Leftrightarrow [(x - x_1, y - y_1, z - z_1)\,(l_1, m_1, n_1)\,(l_2, m_2, n_2)] = 0$

$$\Leftrightarrow \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$\therefore$ Equation to the plane π containing L_1 and parallel to L_2 is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

Note. Equation to the plane containing the line $\dfrac{x-x_2}{l_2} = \dfrac{y-y_2}{m_2} = \dfrac{z-z_2}{n_2}$ and parallel to the line $\dfrac{x-x_1}{l_1} = \dfrac{y-y_1}{m_1} = \dfrac{z-z_1}{n_1}$ is $\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

4. 9. Theorem. $\mathbf{L_1, L_2}$ ***are lines whose equations are***

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \quad .... (1) \qquad \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \quad .... (2)$$

Then equation to the plane containing L_1, L_2 ***is*** $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

Proof. First Method. The equations to the lines L_1, L_2 are

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \quad .... (1) \qquad \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \quad .... (2)$$

the equation to the plane containing the line L_1 is

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ .... (3)

with the condition $al_1 + bm_1 + cn_1 = 0$ .... (4)

if L_2 lies in (3) then $al_2 + bm_2 + cn_2 = 0$ .... (5)

Eliminating *a, b, c* from (3), (4), (5) we get $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

this is the equation to the plane containing the lines L_1 and L_2

Second Method.

Proof. Let π be the plane containing L_1 and L_2

Let $A = \bar{a} = (x_1, y_1, z_1)$. $\quad \therefore A \in L_1$ *i.e.,* $A \in \pi$. $\quad$ Let $\bar{P} = \bar{r} = (x, y, z) \in \pi$.

Let $\bar{n}_1 = (l_1, m_1, n_1)$. $\quad \therefore \bar{n}_1 \parallel L_1$. Let $\bar{n}_2 = (l_2, m_2, n_2)$. $\therefore \bar{n}_2 \parallel L_2$

$A \in \pi,\ P \in \pi \Rightarrow P = A$ or $P \neq A \Rightarrow \bar{r} - \bar{a} = 0$ or $\bar{r} - \bar{a} \neq 0$.

If $\bar{r} - \bar{a} = 0$, then $[\bar{r} - \bar{a}, \bar{n}_1, \bar{n}_2] = 0$ .... (I)

If $\bar{r} - \bar{a} \neq 0$, then $\bar{r} - \bar{a}, \bar{n}_1, \bar{n}_2$ are coplanar. *i.e.,* $[\bar{r} - \bar{a}, \bar{n}_1, \bar{n}_2] = 0$ .... (II)

$P \in \pi,\ L_1 \subset \pi,\ L_1 \parallel \pi$ (using I, II)

$\Leftrightarrow [\bar{r} - \bar{a}, \bar{n}_1, \bar{n}_2] \quad \Leftrightarrow [(x - x_1, y - y_1, z - z_1)\,(l_1, m_1, n_1)\,(l_2, m_2, n_2)] = 0$

$\Leftrightarrow \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

$\therefore$ Equation to the plane π containing L_1, L_2 is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

Note. By taking the point (x_2, y_2, z_2) in the plane π containing the lines L_1 and L_2 we can find the equation to the plane π as $\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

For lines (1) and (2) to be coplanar we have $\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$ ..... (5)

(5) shows that the point (x_2, y_2, z_2) lies on the plane (3) and the point (x_1, y_1, z_1) lies on the plane (4). These two equations *i.e.* (3) and (4) are then identical.

Thus the plane containing two coplanar lines is the one which passes through one line and is parallel to the other line or, through one line and any point on the other line.

4. 10. Theorem. ***If the lines*** $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$ .... **(1)**

$a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2 = 0$ ..... **(2)**

are coplanar, then $\dfrac{a_1\alpha + b_1\beta + c_1\gamma + d_1}{a_1l + b_1m + c_1n} = \dfrac{a_2\alpha + b_2\beta + c_2\gamma + d_2}{a_2l + b_2m + c_2n}$

Proof. Any plane through the line (2) is

$\lambda_1(a_1 + b_1 y + c_1 z + d_1) + \lambda_2(a_2 y + c_2 z + d_2) = 0$, and λ_1, λ_2 being any scalars such that $(\lambda_1, \lambda_2) \neq (0,0)$. *i.e.,* $(\lambda_1 a_1 + \lambda_2 a_2)\, x + (\lambda_1 b_1 + \lambda_2 b_2)\, y + (\lambda_1 c_1 + \lambda_2 c_2) z + (\lambda_1 d_1 + \lambda_2 d_2) = 0$

If the line (1) were to lie in this plane, then

$$\lambda_1(a_1\alpha + b_1\beta + c_1\gamma + d_1) + \lambda_2(a_2\alpha + b_2\beta + c_2\gamma + d_2) = 0 \quad \text{.... (3)}$$

$$(\lambda_1 a_1 + \lambda_2 a_2)\, l + (\lambda_1 b_1 + \lambda_2 b_2)\, m + (\lambda_1 c_1 + \lambda_2 c_2)\, n = 0$$

i.e., $\lambda_1(a_1 l + b_1 m + c_1 n) + \lambda_2(a_2 l + b_2 m + c_2 n) = 0$ .... (4)

From (3) and (4), since $(\lambda_1, \lambda_2) \neq (0,0)$, we have

$$\frac{a_1\alpha + b_1\beta + c_1\gamma + d_1}{a_1 l + b_1 m + c_1 n} = \frac{a_2\alpha + b_2\beta + c_2\gamma + d_2}{a_2 l + b_2 m + c_2 n}$$

4. 11. THEOREM. ***Condition for the lines in unsymmetrical form***

$$a_1 x + b_1 y + c_1 z + d_1 = 0,\ a_2 x + b_2 y + c_2 z + d_2 = 0 \quad \text{.... (1)}$$

$$a_3 x + b_3 y + c_3 z + d_3 = 0,\ a_4 x + b_4 y + c_4 z + d_4 = 0 \quad \text{.... (2)}$$

to be coplanar is that $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0$ ***(A. U. A11)***

Proof. Equation to a plane containing the line (1) can be taken as

$$\lambda_1(a_1 x + b_1 y + c_1 z + d_1) + \lambda_2(a_2 x + b_2 y + c_2 z + d_2) = 0,$$

λ_1, λ_2 being any scalars such that $(\lambda_1, \lambda_2) \neq (0,0)$.

i.e., $(\lambda_1 a_1 + \lambda_2 a_2)\, x + (\lambda_1 b_1 + \lambda_2 b_2)\, y + (\lambda_1 c_1 + \lambda_2 c_2)\, z + (\lambda_1 d_1 + \lambda_2 d_2) = 0$ .... (3)

Similarly equation to a plane containing the line (2) can be taken as

$$(\lambda_3 a_3 + \lambda_4 a_4)\, x + (\lambda_3 b_3 + \lambda_4 b_4)\, y + (\lambda_3 c_3 + \lambda_4 c_4)\, z + (\lambda_3 d_3 + \lambda_4 d_4) = 0 \quad \text{.... (4)}$$

λ_3, λ_4 being any scalars such that $(\lambda_3, \lambda_4) \neq (0,0)$.

If (1) and (2) are coplanar, then $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ can be so chosen to make the equations (3), (4) represent the same plane.

For $\lambda \neq 0$, $\quad \lambda_1 a_1 + \lambda_2 a_2 = \lambda\, (\lambda_3 a_3 + \lambda_4 a_4) \Rightarrow \lambda_1 a_1 + \lambda_2 a_2 + (-\lambda\lambda_3)\, a_3 + (-\lambda\lambda_4)\, a_4 = 0$,

$$\lambda_1 b_1 + \lambda_2 b_2 = \lambda\, (\lambda_3 b_3 + \lambda_4 b_4) \Rightarrow \lambda_1 b_1 + \lambda_2 b_2 + (-\lambda\lambda_3)\, b_3 + (-\lambda\lambda_4)\, b_4 = 0,$$

$$\lambda_1 c_1 + \lambda_2 c_2 = \lambda\, (\lambda_3 c_3 + \lambda_4 c_4) \Rightarrow \lambda_1 c_1 + \lambda_2 c_2 + (-\lambda\lambda_3)\, c_3 + (-\lambda\lambda_4)\, c_4 = 0$$

$$\lambda_1 d_1 + \lambda_2 d_2 = \lambda\, (\lambda_3 d_3 + \lambda_4 d_4) \Rightarrow \lambda_1 d_1 + \lambda_2 d_2 + (-\lambda\lambda_3)\, d_3 + (-\lambda\lambda_4)\, d_4 = 0$$

If these equations are to have a non-zero solution *i.e.*

$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \neq (0,0,0,0)$, then $\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = 0$ *i.e.*, $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0$ .... (5)

Note. If (5) is satisfied, the point of intersection of the lines (1) and (2) is obtained by solving any three of the four equations of (1) and (2) as simultaneous equations.

4. 12. NUMBER OF ARBITRARY CONSTANTS OR PARAMETERS IN THE EQUATION OF A LINE

Consider the following equations of a line L in symmetrical form $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$

Let $m \neq 0, n \neq 0$.

The equations to L can be expressed as $x = \left(\frac{l}{m}\right)y + \left(\frac{mx_1 - ly_1}{m}\right), x = \left(\frac{1}{n}\right)z + \left(\frac{nx_1 - lz_1}{n}\right)$

i.e., $x = ay + b, x = cz + d$ where a, b, c, d are arbitrary constants.

Here the plane $x = ay + b$ is parallel to z - axis and the plane $x = cz + d$ is parallel to y-axis. Thus the equations to the line L always be expressed in terms of two first degree equations with not more than four arbitrary constants.

Now to determine a line we consider various sets of given conditions by which we will be able to evaluate the arbitrary constants in the equations to the line. For instance, when the line

(*i*) passes through a given point and has a given direction (d.rs.)

(*ii*) passes through two given points.

(*iii*) passes through a given point and is parallel to two given planes.

(*iv*) passes through two given points and perpendicular to two given planes.

(*v*) passes through a given point and intersects two given lines.

(*vi*) intersects two given lines and has a given diretion.

(*vii*) passes through a given point and is intersecting a given line at right angles.

(*viii*) is intersecting two given lines at right angles.

4.13. LINE COPLANAR WITH TWO LINES

Consider two non-coplanar lines $u_1 = 0 = v_1$ and $u_2 = 0 = v_2$.

For $(\lambda_1, \mu_1) \neq (0,0)$ and $(\lambda_2, \mu_2) \neq (0,0)$, the line $\lambda_1 u_1 + \mu_1 v_1 = 0 = \lambda_2 u_2 + \mu_2 v_2$ lies in the plane $\lambda_1 u_1 + \mu_1 v_1 = 0$ which again contains the line $u_1 = 0 = v_1$.

The two lines $\lambda_1 u_1 + \mu_1 v_1 = 0 = \lambda_2 u_2 + \mu_2 v_2$; $u_1 = 0 = v_1$ are therefore, coplanar.

Similarly the two lines $\lambda_1 u_1 + \mu_1 v_1 = 0 = \lambda_2 u_2 + \mu_2 v_2$; $u_2 = 0 = v_2$ are coplanar.

Thus, $\lambda_1 u_1 + \mu_1 v_1 = 0 = \lambda_2 u_2 + \mu_2 v_2$ is the line coplanar with both the lines

$u_1 = 0 = v_1; u_2 = 0 = v_2$

SOLVED PROBLEMS

Ex. 1. *Prove that the lines* $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$; $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ *are coplanar. Also find their point of intersecton and the plane containing the lines.*

(A. N. U. M15, A12, S. V. U. M15, A. U. M 11, V.S.P.V M18, K.U M18, A.U M18)

Sol. Given lines are $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ .... (1); $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ .... (2)

$\therefore$ (1) passes through the point (1, 2, 3) and (2) passes through the point $(2,3,4)$.

D.rs. of (1) are 2, 3, 4 and d.rs. of (2), 3, 4, 5. For (1) and (2) to be coplanar.,

$$\begin{vmatrix} 1-2 & 2-3 & 3-4 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ 3 & 4 & 5 \end{vmatrix} = 0 \mid \ R_2 - R_3$$

$\therefore$ Lines (1) and (2) are coplanar and (1) and (2) are not parallel.

$\therefore$ (1) and (2) intersect. Two planes through (1) are

$3(x-1)=2(y-2)$ *i.e.* $3x-2y=-1$ .... (3) $4(y-2)=3(z-3)$ *i.e.*, $4y-3z=-1$ .... (4)

A plane through (2) is $4(x-2)=3(y-3)$ *i.e.* $4x-3y=-1$ ..... (5)

Solving (3), (4) and (5), $x=y=z=-1$

The plane containing the lines (1) and (2) is $\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}=0$

i.e., $(-1)(x-1)-(-2)(y-2)-1(z-3)=0$ *i.e.*, $x-2y+z=0$

OR

Let $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=p$, $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}=q$.

If the given lines intersect, then we can take

$(2p+1, 3p+2, 4p+3)=(3q+2, 4q+3, 5q+4)$

$\Rightarrow 2p+1=3q+2$ *i.e.* $2p-3q-1=0$; $3p+2=4q+3$ *i.e.*, $3p-4q-1=0$

$4p-5q-1=0$ *i.e.*, $4p-5p-1=0$

Solving $p=q=-1$ and the equation $4p-5q-1=0$ is satisfied by $p=q=-1$

$\therefore$ Given lines intersect and their point of intersection is

$(-2+1, -3+2, -4+3)$ *i.e.*, $(-1,-1,-1)$

The plane containing the given lines is $\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}=0$

Ex. 2. *Show that the lines* $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$, $\frac{x}{a\alpha}=\frac{y}{b\beta}=\frac{z}{c\gamma}$, $\frac{x}{\alpha}=\frac{y}{\beta}=\frac{z}{\gamma}$ *are coplanar if* $\frac{l}{\alpha}(b-c)+\frac{m}{\beta}(c-a)+\frac{n}{\gamma}(a-b)=0$

Sol. D. rs. of the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ are l,m,n. $\therefore$ A vector along the line is (l,m,n).

Similarly vectors along the lines $\frac{x}{a\alpha}=\frac{y}{b\beta}=\frac{z}{c\gamma}$, $\frac{x}{\alpha}=\frac{y}{\beta}=\frac{z}{\gamma}$

are respectively $(a\alpha, b\beta, c\gamma), (\alpha,\beta,\gamma)$. The three given lines are coplanar

$\Rightarrow$ the vectors $(l,m,n),(a\alpha,b\beta,c\gamma),(\alpha,\beta,\gamma)$ are coplanar

$\Rightarrow [(l,m,n),(a\alpha,b\beta,c\gamma),(\alpha,\beta,\gamma)]=0$

$$\Rightarrow \begin{vmatrix} l & m & n \\ a\alpha & b\beta & c\gamma \\ \alpha & \beta & \gamma \end{vmatrix}=0 \qquad \Rightarrow l\beta\gamma\,(b-c)+m\gamma\alpha\,(c-a)+n\alpha\beta\,(a-b)=0$$

$$\Rightarrow \frac{l}{\alpha}(b-c)+\frac{m}{\beta}(c-a)+\frac{n}{\gamma}(a-b)=0$$

Ex. 3. *Prove that the lines* $\frac{x+1}{1}=\frac{y+1}{2}=\frac{z+1}{3}$ *and* $x+2y+3z-8=0=2x+3y+4z-11$ *are intersecting and find the point of their intersection. Find also the equation to the plane containing them.* **(S.V.U. M06, N. U. S 98)**

Sol. Given lines are $\frac{x+1}{1}=\frac{y+1}{2}=\frac{z+1}{3}$ .... (1)

$x+2y+3z-8=0=2x+3y+4z-11$ .... (2)

Any point P on (1) is $(r-1, 2r-1, 3r-1)$

If P lies on the first plane containing the line (2),

$r-1+4r-2+9r-3-8=0$ *i.e.,* $14r=14$ *i.e.,* $r=1$.

$\therefore$ $P=(0,1,2)$. Clearly P also lies on the plane $2x+3y+4z-11=0$

$\therefore$ (1) and (2) intersect at the point $P\ (0,1,2)$.

A plane through the line (2) is $x+2y+3z-8+k\,(2x+3y+4z-11)=0$

i.e., $(1+2k)\,x+(2+3k)\,y+(3+4k)\,z-(8+11k)=0$.

If this plane is parallel to (1), $1\,(1+2k)+2\,(2+3k)+3\,(3+4k)=0$

i.e., $20k=-14$ *i.e.,* $k=-(7/10)$.

$\therefore$ Equation to the plane containing line (2) and parallel to (1) is

$$x+2y+3z-8-\frac{7}{10}(2x+3y+4z-11)=0$$

i.e. $-4x-y+2z-3=0$ *i.e.* $4x+y-2z+3=0$.

This plane clearly contains the point $(-1,-1,-1)$ (a point on (1)).

$\therefore$ $4x+y-2z+3=0$ is the plane containing the lines (1) and (2).

Note. 1. If the point $(-1,-1,-1)$ on the base (1) does not lie on the plane $4x+y-2z+3=0$ then $4x+y-2z+3=0$ is the plane containing the line (2) and parallel to line (1).

2. To show that the lines (1) and (2) are coplanar, find the point of intersection of (1) and (2) or find the plan containing the line (2) and the line (1).

Ex. 4. *Prove that* $\frac{x+4}{3}=\frac{y+6}{5}=\frac{z-1}{-2}$ *and* $3x-2y+z+5=0=2x+3y+4z-4$ *are coplanar. Find the point of intersection* **(K. U. M13, S.K.U M18)**

Sol. Given lines are $\frac{x+4}{3}=\frac{y+6}{5}=\frac{z-1}{-2}=t$ (say) ... (1)

and $2x+3y+4z-4=0=3x-2y+z+5$ ... (2)

Any point P on (1) is $P=(3t-4, 5t-6, -2t+1)$

If P lies on the 1st plane of (2), we get

$$3(3t-4)-2(5t-6)+(-2t+1)+5=0$$

$\Rightarrow 9t-12-10t+12-2t+1+5=0 \Rightarrow -3t+6=0 \Rightarrow t=2$. $\therefore$ $P=(2,4,-3)$

Clearly P lies the second plane of the line (2)

$\therefore$ The lines (1) & (2) intersect the point $(2,4,-3)$.

Hence the two lines are coplanar.

Ex. 5. *Show that the lines* $x+2y+3z-4=0=2x+3y+4z-5$ $2x-3y+3z-5=0=3x-2y+4z-6$ *are coplanar. Find the plane containing the lines.*

Sol. Given lines are $x+2y+3z-4=0,\ 2x+3y+4z-5=0$ ..... (1)

and $2x-3y+3z-5=0, 3x-2y+4z-6=0$ .... (2)

$$\text{Now} \begin{vmatrix} 1 & 2 & 3 & -4 \\ 2 & 3 & 4 & -5 \\ 2 & -3 & 3 & -5 \\ 3 & -2 & 4 & -6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & -4 \\ 0 & -1 & -2 & 3 \\ 0 & -7 & -3 & 3 \\ 0 & -8 & -5 & 6 \end{vmatrix} \begin{matrix} R_2-2R_1 \\ R_3-2R_1 \\ R_4-3R_1 \end{matrix}$$

$$= \begin{vmatrix} -1 & -2 & 3 \\ -7 & -3 & 3 \\ -8 & -5 & 6 \end{vmatrix} = \begin{vmatrix} -1 & -2 & 3 \\ 0 & 11 & -18 \\ 0 & 11 & -18 \end{vmatrix} \begin{matrix} R_2-7R_1=0 \\ R_3-8R_1 \end{matrix}$$

$\therefore$ Lines (1) and (2) are coplanar.

A plane through (1) is $(x+2y+3z-4)+\lambda\,(2x+3y+4z-5)=0$

i.e., $(1+2\lambda)\,x+(2+3\lambda)\,y+(3+4\lambda)\,z-(4+5\lambda)=0$ .... (3)

A plane through (2) is $(2x-3y+3z-5)+\mu\,(3x-2y+4z-6)=0$

i.e. $(2+3\mu)\,x+(-3-2\mu)y+(3+4\mu)\,z-(5+6\mu)=0$ .... (4)

If (3) and (4) represent the same plane, then

$1+2\lambda=k\,(2+3\mu)$ *i.e.,* $2\lambda-2k-3\mu k=-1$ .... (5)

$2+3\lambda=k(-3-2\mu)$ *i.e.,* $3\lambda+3k+2\mu k=-2$ .... (6)

$3+4\lambda=k(3+4\mu)$ *i.e.,* $4\lambda-3k-4\mu k=-3$ .... (7)

$4+5\lambda=k(5+6\mu)$ *i.e.,* $5\lambda-5k-6\mu k=-4$ .... (8)

$2\times(6)+(7): 10\lambda+3k=-7$ *i.e.,* $(10)\,\lambda+3k+7=0$ .... (9)

$3\times(6)+(8): 14\lambda+4k=-10$ *i.e.,* $(7)\,\lambda+2k+5=0$ .... (10)

From (9) and (10), $\frac{\lambda}{1}=\frac{k}{-1}=\frac{1}{-1}$. $\therefore\ \lambda=-1, k=1$

$\therefore$ From (6), $-3+3+2\mu=-2$ *i.e.,* $\mu=-1$. Clearly λ,μ,k saitsfy (5).

$\therefore$ Equation to the plane containing the lines (1) and (2) is

$(1-2)\,x+(2-3)\,y+(3-4)\,z-(4-5)=0$ from (3) *i.e.* $x+y+z-1=0$.

Ex. 6. *A, A'; B, B'; C, C' are points on the axes. Show that the lines of intersection of the planes $A'BC, AB'C'$; $B'CA, BC'A$; $C'AB, CA'B'$ are coplanar.*

Sol. Let $\mathrm{A}=(a,0,0), \mathrm{B}=(0,b,0), \mathrm{C}=(0,0,c)$, $\mathrm{A}'=(a',0,0), \mathrm{B}'=(0,b',0), \mathrm{C}'=(0,0,c')$.

$\therefore$ Equation of the plane $\mathrm{A'BC}$ is $\frac{x}{a'}+\frac{y}{b}+\frac{z}{c}=1$ .... (1)

and equation of the plane $\frac{x}{a}+\frac{y}{b'}+\frac{z}{c'}=1$ .... (2)

A plane containing the line of intersection of the planes (1) and (2) is

$$\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1\right)+1\left(\frac{x}{a'}+\frac{y}{b'}+\frac{z}{c'}-1\right)=0 \text{ i.e., } \left(\frac{1}{a'}+\frac{1}{a}\right)x+\left(\frac{1}{b'}+\frac{1}{b}\right)y+\left(\frac{1}{c'}+\frac{1}{c}\right)z=2 \quad .... (3)$$

By symmetry of the equation to the plane (3), the other two lines of intersection also lie in (3)

$\therefore$ The three given lines are coplanar and the plane containing them is (3).

OR : Line of intersection of the planes $\overleftrightarrow{\mathrm{A'BC}}, \overleftrightarrow{\mathrm{AB'C'}}$ is

$\frac{x}{a^1}+\frac{y}{b}+\frac{z}{c}-1=0,\ \frac{x}{a}+\frac{y}{b^1}+\frac{c_1}{c^1}-1=0.$

Similarly the line of intersection of the planes $\overleftrightarrow{\mathrm{B'CA}}.\overleftrightarrow{\mathrm{BC''A''}}$ is

$\frac{x}{a}+\frac{y}{b^1}+\frac{z}{c}-1=0,\ \frac{x}{a^1}+\frac{y}{b}+\frac{z}{c^1}-1=0.$

$$\text{Clearly} \begin{vmatrix} 1/a^1 & 1/b & (1/c)-1 \\ 1/a & 1/b^1 & (1/c^1)-1 \\ 1/a & 1/b^1 & (1/c)-1 \\ 1/a^1 & 1/b & (1/c^1)-1 \end{vmatrix}=0 \qquad (\because\ \mathrm{R}_1+\mathrm{R}_2=\mathrm{R}_3+\mathrm{R}_4)$$

$\therefore$ Lines $\overleftrightarrow{\mathrm{A'BC}}, \overleftrightarrow{\mathrm{AB'C'}}$, and $\overleftrightarrow{\mathrm{B'CA}}$, $\overleftrightarrow{\mathrm{BC'A'}}$ are coplanar. Similarly we can prove the other pairs of lines to be coplanar. Hence the result.

Ex. 7. *Find the equations of the line through the origin and intersecting each of the lines* $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ *and* $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$

Sol. Given lines are $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ ..... (1) $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ .... (2)

Let the equation to the plane containing (1) be

$a\,(x-x_1)+b\,(y-y_1)+c\,(z-z_1)=0$ .... (3) $\therefore\ al_1+bm_1+cn_1=0$ .... (4)

If the plane (3) passes through the origin, then

$-ax_1 - by_1 - cz_1 = 0$ *i.e.* $ax_1 + by_1 + cz_1 = 0$ .... (5)

Solving (4) and (5), $\dfrac{a}{m_1z_1 - n_1y_1} = \dfrac{b}{n_1x_1 - l_1z_1} = \dfrac{c}{l_1y_1 - m_1x_1}$.

$\therefore$ Equation to the plane containing the line (1) and passing through the origin is

$ax + by + cz = ax_1 + by_1 + cz_1$ *i.e.* $(m_1z_1 - n_1y_1)\,x + (n_1x_1 - l_1z_1)\,y + (l_1y_1 - m_1x_1)\,z = 0$ ..... (6)

[using (5)].

Similarly, equation to the plane containing the line (2) and passing through the origin is

$(m_2z_2 - n_2y_2)\,x + (n_2x_2 - l_2z_2)\,y + (l_2y_2 - m_2x_2)\,z = 0$ ..... (7)

$\therefore$ (6) and (7) represent the required line.

Ex. 8. *Find the equations of the line through the point* $(1,1,1)$ *and intersecting the lines* $2x - y - z - 2 = 0 = x + y + z - 1;\ x - y - z - 3 = 0 = 2x + 4y - z - 4$ ***(S. V. U., A.U M18)***

Sol. Given lines are $2x - y - z - 2 = 0, x + y + z - 1 = 0$ .... (1)

$x - y - z - 3 = 0, 2x + 4y - z - 4 = 0$ .... (2)

Let the equation to the plane containing (1) and passing through $(1,1,1)$ be

$(2x - y - z - 2) + \lambda\,(x + y + z - 1) = 0$. $\therefore$ $(2 - 1 - 1 - 2) + \lambda\,(1 + 1 + 1 - 1) = 0$ *i.e.* $\lambda = 1$.

$\therefore$ required plane is $x - 1 = 0$.

Let the equation to the plane containing (2) and passing through $(1,1,1)$ be

$(x - y - z - 3) + \mu\,(2x + 4y - z - 4) = 0$. $\therefore$ $(1 - 1 - 1 - 3) + \mu\,(2 + 4 - 1 - 4) = 0$ *i.e.* $\mu = 4$

$\therefore$ Required plane is $9x + 15y - 5z - 19 = 0$.

$\therefore$ Equations of the required line are

$x - 1 = 0,\ x + 15y - 5z - 19 = 0$ *i.e.* $x - 1 = 0, 15\,(y - 1) = 5\,(z - 1)$ *i.e.* $x - 1 = 0, \dfrac{y-1}{1} = \dfrac{z+1}{3}$

Ex. 9 . *Find the equations of the straight line passing through the point* $(1,0,-1)$ *and intersecting the lines* $4x - y - 13 = 0 = 3y - 4z - 1;\ \ y - 2z + 2 = 0 = x - 5$

Sol. Equations of given lines are

$4x - y - 13 = 0, 3y - 4z - 1 = 0$ .... (1) and $y - 2z + 2 = 0, x - 5 = 0$ .... (2)

equations of planes passing through (1), (2) are

$(4x - y - 13) + \lambda_1(3y - 4z - 1) = 0$ .... (3) $(y - 2z + 2) + \lambda_2(x - 5) = 0$ .... (4)

If the planes (3), (4) passes through $(1,0,-1)$ substitute the points in (3), (4) we have $\lambda_1 = 3, \lambda_2 = 1$, then equations of the planes passing through $(1,0,-1)$ and containing (1), (2) are given by $x + 2y - 3z - 4 = 0$ and $x + y - 2z - 3 = 0$ .... (5)

converting these equations into symmetric form we get

$\dfrac{x-0}{1} = \dfrac{y+1}{1} = \dfrac{z+2}{1}$ (or) $x = y + 1 = z + 2$

Ex. 10. *Find the equations of the line with d.cs. proportional to* $7, 4, -1$ *which intersects the lines* $x-1=-9+3y=3z+6, \frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-5}{4}$.

Sol. Given lines are $\frac{x-1}{3}=\frac{y-3}{1}=\frac{z+2}{1}=p$ .... (1) $\frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-5}{4}=q$ ..... (2)

A point P on (1) is $(3p+1,\ p+3,\ p-2)$ and a point Q on (2) is $(-3q-3, 2q+3, 4q+5)$

D.rs. of $\overleftrightarrow{PQ}$, are $(3p+3q+4,\ +p-2q,\ p-4q-7)$

If $\overleftrightarrow{PQ}$ is the required line, then

$$\left.\begin{array}{l} 3p+3q+4=7 \text{ i.e. } p+q=1 \\ +p-2q=4 \text{ i.e. } +p-2q=4 \end{array}\right\} \therefore\ p=2, q=-1 \text{ and these values satisfy } p-4q=6$$

$p-4q-7=-1$ i.e. $p-4q=6$. $\therefore$ P $=(7,5,0)$ and Q $=(0,1,1)$

$\therefore$ Equations to the required line are $\frac{x-7}{7}=\frac{y-5}{4}=\frac{z}{-1}$ or $\frac{x}{7}=\frac{y-1}{4}=\frac{z-1}{-1}$

Ex. 11. *Find the equations of the line intersecting the lines* $2x+y-1=0=x-2y+3z$; $3x-y+z+2=0=4x+5y-2z-3$ *and is parallel to the line* $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$.

Sol. Let the equations to the required line be

$$2x+y-1+\lambda(x-2y+3z)=0, (3x-y+z+2)+\mu(4x+5y-2z-3)=0$$

i.e. $(2+\lambda)x+(1-2\lambda)y+3\lambda z-1=0, (3+4\mu)x+(-1+5\mu)y+(1-2\mu)z+2-3\mu=0$

Since the required line is parallel to $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$, we have

$1(2+\lambda)+2(1-2\lambda)+3(3\lambda)=0, 1(3+4\mu)+2(-1+5\mu)+3(1-2\mu)=0$

i.e. $\lambda=-2/3, \mu=-1/2$

$\therefore$ Equations of the required line are $4x+7y-6z=3, 2x-7y+4z=-7$

EXERCISE 4 (*c*)

1. (*i*) Show that the lines $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}, \frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{+7}$ intersect. Also find their point of intersection and the plane containing the lines. ***(K. U. 11, M 14, N.U.06)***

(*ii*) Prove that the lines $\frac{x-a+d}{\alpha-\delta}=\frac{y-a}{\alpha}=\frac{z-a-d}{\alpha+\delta}, \frac{x-b+c}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-b-c}{\beta+\gamma}$ are coplanar and find the equation of the plne in which they lie. ***(S. V. U. A 93)***

2. Show that the lines $x=ay+b=cz+d, x=py+q=rz+s$ are coplanar if $(r-c)(aq-bp)=(p-a)(cs-dr)$.

3. Find the equation of the plane through the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ and perpendicular to the plane containing the lines $\frac{x}{m}=\frac{y}{n}=\frac{z}{l}$ and $\frac{x}{n}=\frac{y}{l}=\frac{z}{m}$ $(lm+mn+nl\neq 0)$.

4. Find the equations of the line intersecting the lines $\frac{x-5}{1}=\frac{y}{1}=\frac{z-5}{1}, x+5=y=\frac{z+5}{2}$ and parallel to the line $\frac{x-5}{2}=\frac{y-5}{1}=\frac{z-10}{3}$. ***(O. U. 07, N. U. M. 98)***

5. Prove that the lines $\frac{x-l_1}{l_2}=\frac{y-m_1}{m_2}=\frac{z-n_1}{n_2}$ and $\frac{x-l_2}{l_1}=\frac{y-m_2}{m_1}=\frac{z-n_2}{n_1}$ intersect and find the point of intersection.

6. Prove that the three lines through O with d.rs. $1,-1,1;2,-3,0;1,0,3$ are coplanar.

7. Prove that the lines $x+2y-5z+9=0=3x-y+2z-5;\ 4x-5y+z+3=0=$ $2x+3y-z-3$ are coplanar. Also find their point of intersection. ***(O. U. 08)***

8. Show that the distance of the point (–2, 3, –4) from the line $(x+2)/3=(2y+3)/4$ $=(3z+4)/5$ measured parallel to the plane $4x+12y-3z+1=0$ is 17/2. ***(K.U)***

9. Find the equations of the line through the point $(2,-3,1)$ parallel to the plane $2x+y-z=6$ so as to meet the line $\frac{x-2}{2}=\frac{y}{-3}=\frac{z-2}{-1}$. Find also the point of intersection.

10. Show that the lines $x+5y-2z=6, 6x-4y+5z=2;\frac{x-6}{2}=\frac{y-2}{1}=\frac{z-4}{-1}$ intersect nd find the plane containing the lines.

11. Find the equation to the line through the origin and intersecting the lines $2x-3y+4z+1=0=3x+2y+4z-5, 2x-4y+z+6=0=3x-4y+z-3$. ***(O. U. 07)***

12. Find the equations of the line through the point $(2,-1,1)$ and intersecting the lines $2x+y-4=0=y+2z;\ x+3z-4=0=2x+5z-8$. ***(A. U. A12)***

13. A line with d.rs. $2,7,-5$ intersects each of the lines $x-5=3(7-y)=3(z+2)$ and $\frac{x+3}{-3}=\frac{y-3}{2}=\frac{6-z}{-4}$. Find the points of intersection.

14. A line with d.rs. 2, 1, 2 meets each of the lines $x=y+a=z;\ x+a=2y=2z$. Find each point of intersection.

15. Find the equations of the line intersecting each of the lines $2x+y-1=0=x-2y+3z$; $3x-y+z+2=0=4x+5y-2z-3$ and is parallel to the line $6x=3y=2z-8$.

16. Find the equations to the line parallel to $\frac{x}{4}=\frac{y}{1}=\frac{z}{1}$ and intersecting the lines $5x-6=4y+3=z;\ 2x-4=3y+5=z$. ***(S. V. U. A 93)***

17. If the lines $x=mz+a, y=nz+b$ and $x=m_1z+a_1, y=n_1z+b_1$ intersect, show that $(a_1-a)(n_1-n)=(b_1-b)(m_1-m)$. ***(S. V. U. O 97)***

18. Find the equations of the line through (2, 3, 4), perpendicular to the $x-$axis and intersecting the line $x=y=z$.

19. Find the equations of the line through (2, 2, 2) and intersecting the lines $\frac{x-2}{2}=\frac{y+2}{3}=\frac{z-4}{4};\ x=2y=3z$ in the unsymmetrical form.

20. Find the equations of the line intersecting the lines $2(x+w)=2y=z+w,\ x-w=y=z-w$ and parallel to the line $3(x-w)=6(y-w)=2(z-2w)$.

ANSWERS

1. $(5,-7,6)\,;\,11x-6y-5z=67$ **3.** $\sum(m-n)\,x=0$. **4.** $\dfrac{x-5}{2}=\dfrac{y}{1}=\dfrac{z-5}{3}$

5. $(l_1+l_2,\ m_1+m_2,\ n_1+n_2)$ **7.** $\left(\dfrac{1}{2},\dfrac{3}{2},\dfrac{5}{2}\right)$ **8.** $x+2y+3z=2$

9. $\dfrac{x-2}{1}=\dfrac{y+3}{-3}=\dfrac{z-1}{-1}\,;\,(0,3,3)$ **10.** $x-y+z=0$ **11.** $13x-13y+24z=0=8x-12y+3z$

12. $x+y+z=2,\ x+2z=4$ **13.** $(2,8,-3),(0,1,2)$ **14.** $(3a,2a,3a),(a,a,a)$

15. $4x+7y-6z-3=0=2x-7y+4z+7$ **16.** $15x-76y+16z-75=0,\ 4x-21y+5z-43=0$

17. $\dfrac{x-2}{1}=\dfrac{y+3}{-3}=\dfrac{z-1}{-1}$ **18.** $x=2,\ 2y-z=2$ **19.** $11x-2y-4z-10=0=x-4y+3z$.

20. $x+y-z=0=2x-y-z-w$.

4.14. SHORTEST DISTANCE BETWEEN TWO SKEW LINES

Skew lines. Any two non-parallel and non-intersecting lines are called Skew lines. Skew lines are non- coplanar. ***(N. U. 07)***

Let L_1, L_2 be two skew lines. We know that there exists one and only one line L intersecting L_1, L_2 such that $L \perp L_1$ and $L \perp L_2$. Let L intersect L_1 at M and L_2 at N so that MN is the line segment on L and in between L_1, L_2. Also MN is the shortest distance (S.D) between L_1, L_2 and $\overleftrightarrow{MN}\,(=L)$ is the line of S.D. Let A, B are any two points on L_1, L_2.

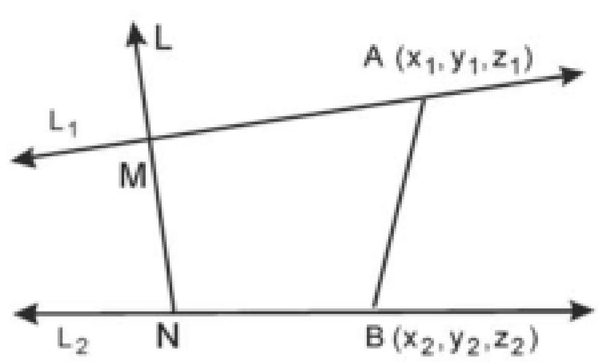


Fig. 48

$$\overline{MN}=\big|\,|\overline{AB}|\cos(\overline{MN},\overline{AB})\big|=\overline{AB}\,|\cos(\overline{MN},\overline{AB})|\le\overline{AB}$$

Theorem. *The S.D. between the skew lines* $\dfrac{x-x_1}{l_1}=\dfrac{y-y_1}{m_1}=\dfrac{z-z_1}{n_1}$ ...(1)

and $\dfrac{x-x_2}{l_2}=\dfrac{y-y_2}{m_2}=\dfrac{z-z_2}{n_2}$ ...(2) *is* $\left|\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}\right| \div \sqrt{\Sigma(m_1n_2-m_2n_1)^2}$

First Method.

Proof. Let L_1, L_2 be the two given lines (1) and (2) respectively.

A point on L_1 is $A(x_1, y_1, z_1)$. A point on L_2 is $B(x_2, y_2, z_2)$.

Let MN be the S. D. between the lines L_1 and L_2.

Let (l,m,n) be the D.cs. of MN. MN is perpendicular to both L_1 and L_2.

$\Rightarrow ll_1 + mm_1 + nn_1 = 0$ .... (3) $ll_2 + mm_2 + nn_2 = 0$ .... (4)

From (3) and (4) we get $\dfrac{l}{m_1n_2 - m_2n_1} = \dfrac{m}{n_1l_2 - n_2l_1} = \dfrac{n}{l_1m_2 - l_2m_1} = \dfrac{1}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}$

Since MN is perpendicular to both (1) and (2) the magnitude of the S. D. is the projection on the line of S. D. of the line joining A and B.

$\therefore$ MN = SD between L_1 and L_2.

= | Projection of AB on MN | $= |(x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n|$

$$= \frac{|(x_2 - x_1)(m_1n_2 - m_2n_1) + (y_2 - y_1)(n_1l_2 - n_2l_1) + (z_2 - z_1)(l_1m_2 - l_2m_1)}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}$$

$$= \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div \sqrt{\Sigma(m_1n_2 - m_2n_1)^2}$$

Equations to the line of SD

The SD is coplanar with L_1

$\therefore$ The equation to the plane determined by L_1 and MN is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0$

The SD is also coplanar with L_2. .... (5)

$\therefore$ The equation to the plane determined by L_2 and MN is $\begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0$

$\therefore$ Equations to the line of SD between (1) and (2) are (5) and (6). .... (6)

Second Method.

Proof. Let L_1, L_2 be the two given lines (1) and (2) respectively.

A point on L_1 is (x_1, y_1, z_1). Let $A = \bar{a} = (x_1, y_1, z_1)$.

A point on L_2 is (x_2, y_2, z_2). Let $B = \bar{b}(x_2, y_2, z_2)$.

Since d.rs. of L_1 are l_1, m_1, n_1, a vector along L_1 is $\bar{c} = (l_1, m_1, n_1)$.

Since d.rs. of L_2 are l_2, m_2, n_2, a vector along L_2 is $\bar{d} = (l_2, m_2, n_2)$.

Let MN be the S.D. between L_1 and L_2.

L_1, L_2 are non-coplanar $\Rightarrow \bar{c}, \bar{d}$ are non-coplanar $\Rightarrow \bar{c} \times \bar{d} \neq 0$

$\Rightarrow \left| \bar{c} \times \bar{d} \right| \neq 0$ and $\bar{c} \times \bar{d} \parallel \overleftrightarrow{MN} \Rightarrow \dfrac{\bar{c} \times \bar{d}}{|\bar{c} \times \bar{d}|}$ is a unit vector along $\overleftrightarrow{MN}$.

Now $\overline{AB} = \bar{b} - \bar{a} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$ and

$\bar{c} \times \bar{d} = (m_1n_2 - m_2n_1, n_1l_2 - n_2l_1, l_1m_2 - l_2m_1)$

$\therefore$ $\overleftrightarrow{MN}$ = S.D. between L_1 and L_2 = | Proj. of $\overline{AB}$ along $\overleftrightarrow{MN}$ |

$= |$ Proj. of $(\bar{b}-\bar{a})$ along $\overleftrightarrow{MN}\,|$

$$= \frac{|(\bar{b}-\bar{a}).(\bar{c}\times\bar{d})|}{|\bar{c}\times\bar{d}|} = \frac{|[(\bar{b}-\bar{a})\,\bar{c}\,\bar{d}]|}{|\bar{c}\times\bar{d}|}$$

$$= \frac{|[(x_2-x_1,\, y_2-y_1,\, z_2-z_1)\,(l_1, m_1, n_1)\,(l_2, m_2, n_2)]|}{|(l_1, m_1, n_1)\times(l_2, m_2, n_2)|}$$

$$= \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div \sqrt{\Sigma\,(m_1n_2-m_2n_1)^2}$$

Equations to the line of S.D.

$\bar{r}$ is any point in the plane determined by $L_1, \overleftrightarrow{MN}$.

$\Leftrightarrow$ $\bar{r}-\bar{a}\ (\neq 0), \bar{c}, \bar{c}\times\bar{d}$ are coplanar or $\bar{r}-\bar{a}\ (=0), \bar{c}, \bar{c}\times\bar{d}$ are three vectors

$\Leftrightarrow$ $[\bar{r}-\bar{a}, \bar{c}, \bar{c}\times\bar{d}]=0$

$\therefore$ Equation to the plane determined by $L_1, \overleftrightarrow{MN}$ is $[\bar{r}-\bar{a}, \bar{c}, \bar{c}\times\bar{d}]=0$

i.e. $$\begin{vmatrix} x-x_1 & y-y_1 & z-z_2 \\ l_1 & m_1 & n_1 \\ m_1n_2-m_2n_1 & n_1l_2-n_2l_1 & l_1m_2-l_2m_1 \end{vmatrix}=0 \qquad ...(3)$$

Similarly, equation to the plane determined by $L_2, \overleftrightarrow{MN}$ is $[\bar{r}-\bar{b}, \bar{d}, \bar{c}\times\bar{d}]=0$

i.e. $$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ m_1n_2-m_2n_1 & n_1l_2-n_2l_1 & l_1m_2-l_2m_1 \end{vmatrix}=0 \qquad ...(4)$$

$\therefore$ Equation to the line of S.D. between (1) and (2) are (3) and (4).

Note 1. S.D. between lines L_1, L_2 is zero $\Leftrightarrow$ L_1, L_2 are intersecting

$$\Leftrightarrow L_1, L_2 \text{ are coplanar } \Leftrightarrow \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

2. If $\bar{r}=\bar{a}+t\,\bar{c}$ and $\bar{r}=\bar{b}+s\,\bar{d}$ be two skew lines, then the S.D. between them is

$$\frac{|[\bar{b}-\bar{a}, \bar{c}\;\bar{d}]|}{|\bar{c}\times\bar{d}|}.$$

Equation to the line of S.D. between the skew lines is

$$[\bar{r}-\bar{a}, \bar{c}, \bar{c}\times\bar{d}]=0,\ [\bar{r}-\bar{b}, \bar{d}, \bar{c}\times\bar{d}]=0.$$

4.15. EQUATIONS OF TWO SKEW LINES IN A SIMPLIFIED FORM (N.U.91)

(Fig. 49). L_1, L_2 are two skew lines and $\overleftrightarrow{MN}$ be the line of S.D. between them. Let O be the mid point of MN. Let $\overleftrightarrow{OC} \parallel L_1$ and $\overleftrightarrow{OD} \parallel L_2$. In the plane $\overleftrightarrow{COD}$, let $\overleftrightarrow{OX}, \overleftrightarrow{OY}$ be the bisectors of angles between $\overleftrightarrow{OC}, \overleftrightarrow{OD}$ so that $\overleftrightarrow{OX}$ is the bisector of $(\overrightarrow{OC}, \overrightarrow{OD})$.

$\therefore\ \overrightarrow{OX} \perp \overrightarrow{OY}$ and $\overleftrightarrow{XOY} = \overleftrightarrow{COD}$

$\overrightarrow{OM} \perp L_1,\ \overrightarrow{OC} \parallel L_1 \Rightarrow \overrightarrow{OM} \perp \overrightarrow{OC}$

and $\overrightarrow{OM} \perp L_2, \overrightarrow{OD} \parallel L_2 \Rightarrow \overrightarrow{OM} \perp \overrightarrow{OD}$.

$\therefore\ \overrightarrow{OM} \perp \overleftrightarrow{COD} \Rightarrow \overrightarrow{OM} \perp \overleftrightarrow{XOY}$.

Let $Z \in \overrightarrow{OM}$. Then $\overrightarrow{OM} = \overrightarrow{OZ}$.

$\therefore\ \overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ are mutually perpendicular rays.

$\therefore$ We can take $\overleftrightarrow{OX}, \overleftrightarrow{OY}, \overleftrightarrow{OZ}$ as rectangular coordinate axes.

Let $MN = 2c$. $\therefore\ OM = ON = c$.

$\therefore\ M = (0, 0, c), N = (0, 0, -c)$.

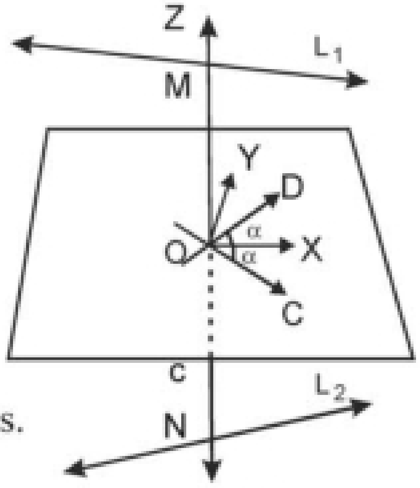


Fig. 49

Let $(\overrightarrow{OC}, \overrightarrow{OD}) = 2\alpha$. $\therefore$ D.rs. of $\overrightarrow{OC}, \overrightarrow{OD}$ *i.e.,* D.rs. of L_1, L_2 are

$$\left(\cos\alpha, \cos\left(\frac{\pi}{2}+\alpha\right), \cos\frac{\pi}{2}\right); \left(\cos\alpha, \cos\left(\frac{\pi}{2}-\alpha\right), \cos\frac{\pi}{2}\right)$$

i.e., $(\cos\alpha, -\sin\alpha, 0), (\cos\alpha, \sin\alpha, 0)$

$\therefore$ Equations to L_1, L_2 are $\dfrac{x-0}{\cos\alpha} = \dfrac{y-0}{-\sin\alpha}, z - c = 0$, $\dfrac{x-0}{\cos\alpha} = \dfrac{y-0}{\sin\alpha}, z + c = 0$.

i.e., $y = -x\tan\alpha, z = c$; $y = x\tan\alpha, z = -c$.

Note. Any point on L_1 is $(r_1, -r_1\tan\alpha, c)$, r_1 is any parameter. Any point on L_2 is $(r_2, +r_2\tan\alpha, -c)$, r_2 is any parameter.

4.16. LENGTH OF THE PERPENDICULAR FROM A POINT TO A LINE. (N. U.88)

Theorem. *If L is the line* $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$ *and* $P = (x_1, y_1, z_1)$, *then length of the perpendicular from P to L* $= \dfrac{1}{\sqrt{l^2+m^2+n^2}}[\Sigma\{m(z_1-\gamma)-n(y_1-\beta)\}^2]^{1/2}$

First Method.

Proof. Let M be the projection of P in L.

Let N be the point (α, β, γ) on L.

$P = (x_1, y_1, z_1)$, $NP = \sqrt{\Sigma(x_1-\alpha)^2}$.

ΔPMN is a right-angled triangle.

D.cs. of NM are $\left(\dfrac{l}{\sqrt{\Sigma l^2}}, \dfrac{m}{\sqrt{\Sigma l^2}}, \dfrac{n}{\sqrt{\Sigma l^2}}\right)$

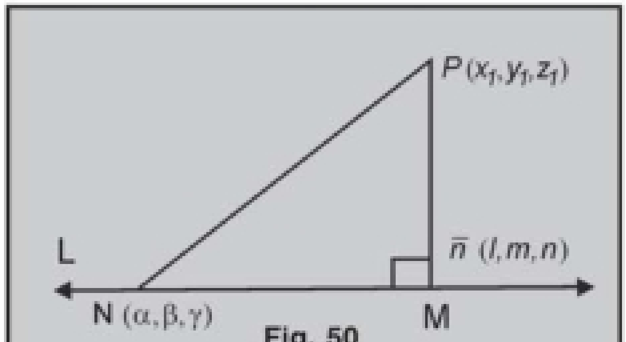


Fig. 50

NM = Projection of PN on the given line

$$= \frac{l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)}{\sqrt{\Sigma l^2}}$$

$$PM^2 = PN^2 - NM^2 = \Sigma(x_1-\alpha)^2 - \left[\frac{\Sigma l(x_1-\alpha)}{\sqrt{\Sigma l^2}}\right]^2$$

$$= \frac{(\Sigma l^2)(\Sigma \overline{x_1-\alpha}^2)-\Sigma l(x_1-\alpha)^2}{\sqrt{\Sigma l^2}} = \frac{[\Sigma(m(z_1-\alpha)-n(y_1-\beta)^2]^{1/2}}{\sqrt{\Sigma l^2}} \text{ (using Art 2.41)}$$

Second Method.

Proof. (Fig. 50). Let M be the projection of P in L. Let N be the point (α, β, γ) on L. Since $P=(x_1, y_1, z_1)$, $\overline{NP}=(x_1-\alpha, y_1-\beta, z_1-\gamma)$. ΔPMN is a right-angled triangle.

Let $\overline{n}$ be a unit vector along $\overleftrightarrow{NM}$ so that $\overline{n} = \frac{1}{\sqrt{l^2+m^2+n^2}}(l, m, n)$.

$$PM = NP.\sin\angle PNM = |\overline{n}||\overline{NP}|\sin(\overline{NP}, \overline{n}) = |\overline{n}\times\overline{NP}|$$

$$= \left|\frac{1}{\sqrt{l^2+m^2+n^2}}(l, m, n)\times(x_1-\alpha, y_1-\beta, z_1-\gamma)\right|$$

$$= \frac{1}{\sqrt{l^2+m^2+n^2}}|(m\overline{z_1-\gamma}-n\overline{y_1-\beta}, n\overline{x_1-\alpha}-l\overline{z_1-\gamma}, m\overline{x_1-\alpha}-l\overline{y_1-\beta})|$$

$$= \frac{1}{\sqrt{l^2+m^2+n^2}}|(m\overline{z_1-\gamma}-n\overline{y_1-\beta}, n\overline{x_1-\alpha}-l\overline{z_1-\gamma}, m\overline{x_1-\alpha}-l\overline{y_1-\beta})|$$

$$= \frac{1}{\sqrt{l^2+m^2+n^2}}\left[\Sigma\{m(z_1-\gamma)-n(y_1-\beta)\}^2\right]^{1/2}$$

$\therefore$ Length of the perpendicular from P to L

$$= \frac{1}{\sqrt{l^2+m^2+n^2}}\left[\Sigma\{m(z_1-\gamma)-n(y_1-\beta)\}^2\right]^{1/2}.$$

Note. Length of the perpendicular from P to L is $|\overline{n}\times\overline{NP}| = \frac{|\overline{NM}\times\overline{NP}|}{|\overline{NM}|}$

SOLVED PROBLEMS

Ex. 1. *Find the S.D. between the lines* $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}, \frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$. *Find also the equations and the points in which the S.D. meets the given lines.*

(S. K. U. M15, A11, O.U. 07,01 M, A12, S.V.U. M15, O 90, A11, A. N. U. M14)

Sol. Let L_1, L_2 be the lines

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1} \quad ...(1); \quad \frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4} \quad ...(2) \text{ respectively.}$$

First Method : Let $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{+1}=c$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}=d$.

Let $C \in L_1$ and $D \in L_2$ so that CD = S.D. between L_1 and L_2.

Let $C = (3c+3, -c+8, c+3)$ and $D = (-3d-3, 2d-7, 4d+6)$.

$\therefore\ \overline{CD} = (-3d-3c-6, 2d+c-15, 4d-c+3)$

$L_1 \perp \overline{CD} \Rightarrow 3(-3d-3c-6) - 1(2d+c-15) + 1(4d-c+3) = 0 \Rightarrow 11c + 7d = 0$ ...(3)

$L_2 \perp \overline{CD} \Rightarrow -3(-3d-3c-6) + 2(2d+c-15) + 4(4d-c+3) = 0 \Rightarrow 29d + 7c = 0$ ...(4)

$\therefore$ From (3) and (4), $c = d = 0$. $\quad\therefore\ C = (3, 8, 3)$ and $D = (-3, -7, 6)$.

Also $\overline{CD} = (-6, -15, 3)$. $\quad\therefore$ S.D. $= |\overline{CD}| = |(-6, -15, 3)| = \sqrt{36+225+9} = 3\sqrt{30}$.

Since $\overline{CD} = (-6, -15, 3)$, d.rs. of CD are $2, 5, -1$.

Since $L_1, \overleftrightarrow{CD}$ are coplanar and $L_2, \overleftrightarrow{CD}$ are coplanar, equations to the line of S.D. $(\overleftrightarrow{CD})$

are $\begin{vmatrix} x-3 & y-8 & z-3 \\ 3 & -1 & 1 \\ 2 & 5 & -1 \end{vmatrix} = 0 = \begin{vmatrix} x+3 & y+7 & z-6 \\ -3 & 2 & 4 \\ 2 & 5 & -1 \end{vmatrix}$

i.e., $4x - 5y - 17z + 79 = 0 = 22x - 5y + 19z - 83$.

OR : Equation to $\overleftrightarrow{CD}$ is $\dfrac{x-3}{2} = \dfrac{y-8}{5} = \dfrac{z-3}{-1}$.

Second Method : Let $A = (3, 8, 3)$ and $A \in L_1$. Let $B = (-3, -7, 6)$ and $B \in L_2$.

Since d.rs. of L_1 are $3, -1, 1$, a vector along L_1 is $(3, -1, 1)$.

Since d.rs. of L_2 are $-3, 2, 4$, a vector along L_2 is $(-3, 2, 4)$.

Let CD be the S.D. between L_1, L_2 so that $C \in L_1$ and $D \in L_2$.

$CD \perp L_1, CD \perp L_2 \Rightarrow$ a vector along CD is $(3, -1, 1) \times (-3, 2, 4)$

$\Rightarrow \overleftrightarrow{CD} \parallel (-6, -15, 3) \Rightarrow \overleftrightarrow{CD} \parallel (2, 5, -1)$

$\therefore$ S.D. = CD = | Projection of $\overleftrightarrow{AB}$ along $\overleftrightarrow{CD}$

$$= \left| \overleftrightarrow{AB} \cdot \frac{\overleftrightarrow{CD}}{|\overleftrightarrow{CD}|} \right| = \left| (-6, -15, 3) \cdot \frac{(2, 5, -1)}{\sqrt{4+25+1}} \right| = \left| \frac{-12-75-3}{\sqrt{30}} \right| = 3\sqrt{30}$$

Since $L_1, \overleftrightarrow{CD}$ are coplanar and $L_2, \overleftrightarrow{CD}$ are coplanar, equations to the line of S.D. $(\overleftrightarrow{CD})$

are $\begin{vmatrix} x-3 & y-8 & z-3 \\ 3 & -1 & 1 \\ 2 & 5 & -1 \end{vmatrix} = 0 = \begin{vmatrix} x+3 & y+7 & z-6 \\ -3 & 2 & 4 \\ 2 & 5 & -1 \end{vmatrix}$

i.e., $-4(x-3) + 5(y-8) + 17(z-3) = 0 - 22(x+3) + 5(y+7) - 19(z-6)$

i.e., $4x - 5y - 17z + 79 = 0$ ...(3) $\quad 22x - 5y + 19z - 83 = 0$ ...(4)

Since $C \in L_1 \cap \overleftrightarrow{CD}$, by solcing the equations

$x + 3y - 27 = 0, y + z - 11 = 0$ [from (2)], (3) and (4), we get $D = (-3, -7, 6)$.

Third Method : Equation to the plane containing L_1 and parallel to L_2 is

$\begin{vmatrix} x-3 & y-8 & z-3 \\ 3 & -1 & 1 \\ -2 & 2 & 4 \end{vmatrix} = 0$ *i.e.*, $2x + 5y - z - 43 = 0$ ...(3)

S.D. between L_1 and L_2 = Distance of any point on L_2 from (3)

= Distance of $(-3,-7,6)$ from $2x+5y-z-43=0 = \left|\dfrac{-6-35-6-43}{\sqrt{4+25+1}}\right| = 3\sqrt{30}$.

S.D. is a line common to the planes

(*i*) containing L_1 and perpendicular to $2x+5y-z-43=0$

(*ii*) containing L_2 and perpendicular to

$$\begin{vmatrix} x+3 & y+7 & z-6 \\ -3 & 2 & 4 \\ 3 & -1 & 1 \end{vmatrix} = 0 \quad i.e., \quad 2x+5y-z+47=0$$

$\therefore$ Equations to the line of S.D. are $\begin{vmatrix} x-3 & y-8 & z-3 \\ 3 & -1 & 1 \\ 2 & 5 & -1 \end{vmatrix} = 0 = \begin{vmatrix} x+3 & y+7 & z-6 \\ -3 & 2 & 4 \\ 2 & 5 & -1 \end{vmatrix}$

i.e., $4x-5y-17z+79=0=22x-5y+19z-83$.

The points of intersection of S.D. with L_1, L_2 can be found out as in the first method.

Ex. 2. *Find the distance between the straight lines*

$\dfrac{x-3}{1}=\dfrac{y-5}{-2}=\dfrac{z-7}{1}; \dfrac{x+1}{7}=\dfrac{y+1}{-6}=\dfrac{z+1}{1}$ ***(S.V. U. 08, S. K. U. 01 M)***

Sol. Equations of given straight lines are $\dfrac{x-3}{1}=\dfrac{y-5}{-2}=\dfrac{z-7}{1}; \dfrac{x+1}{7}=\dfrac{y+1}{-6}=\dfrac{z+1}{1}$

Let $\dfrac{x-3}{1}=\dfrac{y-5}{-2}=\dfrac{z-7}{1}=r_1$ and $\dfrac{x+1}{7}=\dfrac{y+1}{-6}=\dfrac{z+1}{1}=r_2$

Any point on (1) is $P(r_1+3, -2r_1+5, r_1+7)$

Any point on (2) is $Q(7r_2-1, -6r_2-1, r_2-1)$

Dr's of PQ are $(r_1+3)-(7r_2-1), (-2r_1+5)-(-6r_2-1), (r_1+7)-(r_2-1)$

i.e., $(r_1-7r_2+4, -2r_1+6r_2+6, r_1-r_2+8)$

Let PQ be the shortest distance between (1), (2) then PQ is perpendicular to (1), (2)

$(r_1-7r_2+4).1+(-2r_1+6r_2+6)(-2)+(r_1-r_2+8).1=0$

$\Rightarrow r_1+4r_1+r_1-7r_2-12r_2-r_2+4-12+8=0 \Rightarrow 6r_1-20r_2=0 \Rightarrow 3r_1-10r_2=0$ ...(3)

and $(r_1-7r_2+4)(7)+(-2r_1+6r_2+6)(-6)+(r_1-r_2+8).1=0$

$\Rightarrow 7r_1+12r_1+r_1-49r_2-36r_2-r_2+28-36+8=0 \Rightarrow 20r_1-86r_2=0$

$\Rightarrow 10r_1-43r_2=0$ ...(4)

on solving (3), (4) $r_1=r_2=0$

Co-ordinates of D : (3, 5, 7) and Co-ordinates of Q : $(-1,-1,-1)$

Shortest distnce between (1), (2) is

$PQ=\sqrt{(3+1)^2+(5+1)^2+(7+1)^2} = \sqrt{16+36+64} = \sqrt{116} = 2\sqrt{29}$ units.

Ex. 3. *Find the length and equations of the line of S.D. between the lines* $\dfrac{x}{1}=\dfrac{y}{2}=\dfrac{z}{1}$ *and* $x+y+2z-3=0=2x+3y+3z-4$. ***(A. N. U. M15, A. U. A11, A.K.N.U M18, S.U.M M18)***

Sol. Given lines are $\frac{x}{1}=\frac{y}{2}=\frac{z}{1}$ ...(1) and $x+y+2z-3=0$ ...(2)

$2x+3y+3z-4=0$ ...(3)

A plane through the second line and parallel to (1) is

$x+y+2z-3+\lambda\,(2x+3y+3z-4)=0$ ...(4)

i.e., $(1+2\lambda)x+(1+3\lambda)y+(2+3\lambda)z-3-4\lambda=0$.

$\therefore\ 1\,(1+2\lambda)+2\,(1+3\lambda)+1\,(2+3\lambda)=0 \Rightarrow 11\lambda=-5 \Rightarrow \lambda=-\frac{5}{11}$

$\therefore$ From (4), equation to the plane through the second line and parallel to (1) is

$x-4y+7z-13=0$ ...(5) A point on (1) is (0, 0, 0).

$\therefore$ S.D. between the lines = Distance of (0, 0, 0) from (5) $=\left|\frac{-13}{\sqrt{1+16+49}}\right|=\frac{13}{\sqrt{(66)}}$

Equation to the plane through the first line and perpendicular to (5) is

$$\begin{vmatrix} x & y & z \\ 1 & 2 & 1 \\ 1 & -4 & 7 \end{vmatrix}=0 \quad \textit{i.e.,}\ 3x-y-z=0 \qquad ...(6)$$

Let a plane through the second line be $x+y+2z-3+\lambda_1\,(2x+3y+3z-4)=0$

i.e., $(1+2\lambda_1)x+(1+3\lambda_1)y+(2+3\lambda_1)z-3-4\lambda_1=0$

If this plane is perpendicular to (5),

$1\,(1+2\lambda_1)-4\,(1+3\lambda_1)+7\,(2+3\lambda_1)=0 \Rightarrow 11\lambda_1+11=0 \Rightarrow \lambda=-1$.

$\therefore$ Equation to the plane through the second line and perpendicular to (5) is

$x+2y+z-1=0$ ...(7) $\therefore$ Equations to the line of S.D. are (6) and (7).

Ex. 4. *Find the S.D. and the equations of the line of S.D. between the lines* $3x-9y+5z=0=x+y-z$ *and* $6x+8y+3z-10=0=x+2y+z-3$.

(K. U. 12, S. K. U. M 13, K.U M18, A.U. M18)

Sol. Given lines are $3x-9y+5z=0=x+y-z$ ...(1)

and $6x+8y+3z-10=0=x+2y+z-3$ ...(2)

A plane through (1) is $3x-9y+5z+\lambda\,(x+y-z)=0$

i.e., $(3+\lambda)x+(-9+\lambda)y+(5-\lambda)=0$ ...(3)

A plane through (2) is $(6x+8y+3z-10)+\mu\,(x+2y+z-3)=0$

i.e., $(6+\mu)x+(8+2\mu)y+(3+\mu)z-(10+3\mu)=0$ ...(4)

If (3) and (4) are parallel, then

$3+\lambda=k\,(6+\mu)$ *i.e.,* $\lambda-6k-k\mu=-3$ ...(5)

$-9+\lambda=k\,(8+2\mu)$ *i.e.,* $\lambda-8k-2k\mu=9$ ...(6)

$5-\lambda=k\,(3+\mu)$ *i.e.,* $+\lambda+3k+k\mu=5$ ...(7)

From (5) and (6), $-\lambda+4k=15$ Solving, $k=\frac{32}{5}, \lambda=\frac{53}{5}$.

From (5) and (7), $2\lambda-3k=2$ $\therefore$ From (5), $\mu=-\frac{31}{8}$

$\therefore$ Equations to the planes through (1) and (2) and parallel to each other from (3) and (4) are $17x+2y-7z=0$ ...(8) $17x+2y-7z-13=0$ ...(9) A point on (8) is (0, 0, 0).

$\therefore$ S.D. between (1) and (2) = Distance of (0, 0, 0) from (9) $=\left|\dfrac{-13}{\sqrt{289+4+49}}\right|=\dfrac{13}{3\sqrt{38}}$

If (3) is perpendicular to (8), then $17(3+\lambda)+2(-9+\lambda)-7(5-\lambda)=0$

$\Rightarrow \lambda=1/13$. Similarly, $\mu=-97/14$.

$\therefore$ Equation to the plane through (1) and perpendicular to (8) is

$3x-9y+5z+\dfrac{1}{13}(x+y-z)=0$ *i.e.,* $10x-29y+16z=0$ ...(10)

Also equation to the plane through (2) and perpendicular to (8) is

$13x+82y+55z-151=0$ ...(11)

$\therefore$ Equations to the lines of S.D. are (10) and (11).

Ex. 5. *Show that the equation to the plane containing the line* $\dfrac{y}{b}+\dfrac{z}{c}=1, x=0$ *and parallel to the line* $\dfrac{x}{a}-\dfrac{z}{c}=1, y=0$ *is* $\dfrac{x}{a}-\dfrac{y}{b}-\dfrac{z}{c}+1=0$ *and if 2d is the S.D., prove that* $\dfrac{1}{d^2}=\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}$. ***(N.U.S 98, S.V.U., O.U.2001 Oct.)***

Sol. Given lines are $\dfrac{y}{b}+\dfrac{z}{c}-1=0, x=0$ ...(1)

and $\dfrac{x}{a}-\dfrac{z}{c}=1, y=0$ ...(2)

The line (2) can be written as $\dfrac{x-a}{a}=\dfrac{z}{c}, y=0$.

A plane through the line (1) is $\dfrac{y}{b}+\dfrac{z}{c}-1+\lambda x=0$.

If this is parallel to (2), then

$\lambda . a+\dfrac{1}{b}.0+\dfrac{1}{c}.c=0 \Rightarrow \lambda=-\dfrac{1}{a}$.

$\therefore$ Equation to the plane containing the line (1) and parallel to (2) is

$\dfrac{y}{b}+\dfrac{z}{c}-1-\dfrac{1}{a}x=0$ *i.e.,* $\dfrac{x}{a}-\dfrac{y}{b}-\dfrac{z}{c}+1=0$. A point on the line (2) is $(a, 0, 0)$.

Since $2d$ is the S.D. between (1) and (2), therefore

$2d$ = distance of $(a, 0, 0)$ from the plane $\dfrac{x}{a}-\dfrac{y}{b}-\dfrac{z}{c}+1=0$

$$\Rightarrow 2d=\left|\frac{\frac{a}{a}-\frac{0}{b}-\frac{0}{c}+1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right| \Rightarrow \frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

Ex. 6. *a, b, c are the lengths of the edges of a rectangular parallelopiped. Prove that the S.D. between the diagonals and the edges not meeting them are*

$$\frac{bc}{\sqrt{b^2+c^2}}, \frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}}$$

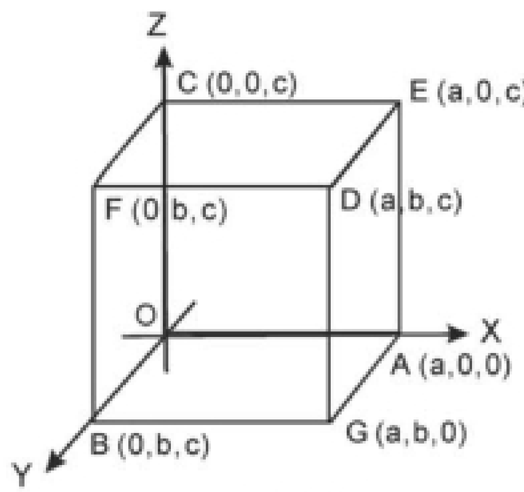


Fig. 51

Sol. (Fig. 51). Let (OBGA ; CFDE) be the rectangular parallelopiped with edges a, b, c.

Let $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ be the axes.

$\therefore$ A = $(a, 0, 0)$, B = $(0, b, 0)$, C = $(0, 0, c)$

Equation to $\overleftrightarrow{GC}$ is $\frac{x-0}{a} = \frac{y-0}{b} = \frac{z-c}{-c}$ ...(1)

Let a plane through $\overleftrightarrow{OB}$ and parallel to $\overleftrightarrow{CG}$ be $x + \lambda z = 0$.

$1 . a + a . b + \lambda(-c) = 0 \rightarrow \lambda = a / c$

$\therefore$ Equation to the plane through the edge OB and parallel to the diagonal CG is $x + \frac{a}{c} z = 0$ *i.e.*, $cx + az = 0$.

$\therefore$ S.D. between OB and CG is the distance of a point $(0, 0, c)$ from the plane $cx + az = 0$ is $\frac{c . 0 + a . c}{\sqrt{c^2+a^2}} = \frac{c\,a}{\sqrt{c^2+a^2}}$. Similarly other distances can be found.

Ex. 7. *Find the length of the perpendicular from the point (1, 2, 3) to the line through the point (6, 7, 7) whose d.rs. are $3, 2, -2$.* ***(A. N. U. M 14)***

(or) Find the perpendicular distance of the point $(1,2,3)$ from the line $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$ ***(K. U. M 13)***

Sol. Let L be the line through (6, 7, 7) with d.rs. $3, 2, -2$.

$\therefore$ Equation to L is $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$ $(= r$, say)

Let P = (1, 2, 3) and N = (6, 7, 7). $\therefore$ $\overline{NP} = (5, 5, 4)$

Let $\bar{n}$ be a unit vector along L.

Since d.rs. of L are $3, 2, -2$, we have $\bar{n} = \left(\frac{3}{\sqrt{(17)}}, \frac{2}{\sqrt{(17)}}, \frac{-2}{\sqrt{(17)}}\right)$.

$\therefore$ Length of the perpendicular from P to L.

$$= \left| \overline{NP} \times \bar{n} \right| = \left| (5, 5, 4) \times \frac{3, 2, -2}{\sqrt{(17)}} \right| = \frac{1}{\sqrt{(17)}} |-18, 22, -5| = \frac{\sqrt{324+484+25}}{\sqrt{(17)}} = \sqrt{\frac{833}{17}} = 7$$

OR : Let $Q \in L$ and $Q = (3r+6, 2r+7, -2r+7)$

If $PQ \perp L$ then $3(3r+6-1)+2(2r+7-2)-2(-2r+7-3)=0$

since $\overline{PQ} = (3r+6-1, 2r+7-2, -2r+7-3)$

$\therefore\ 17r = -17$ *i.e.,* $r=-1$ $\therefore\ Q=(3,5,9)$ $\therefore\ PQ=\sqrt{4+9+36}=7$.

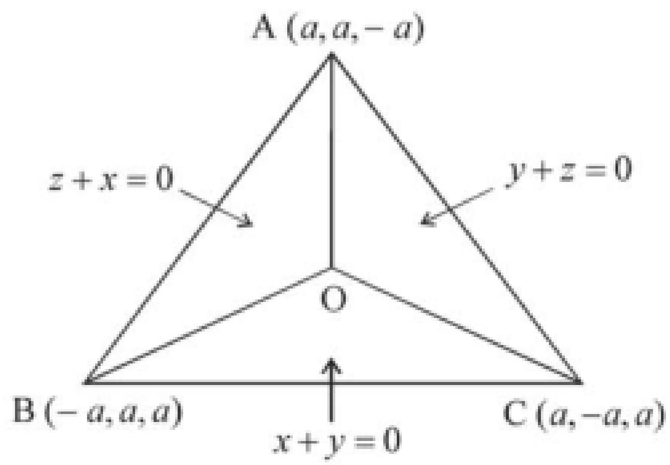
Ex. 8. *Show that the shortest distance between any two opposite edges of the tetrahedron formed by the planes* $y+z=0, z+x=0, x+y=0$ *and* $x+y+z=a$ *is* $\frac{2|a|}{\sqrt{6}}$ *and that three lines of shortest distance intersect the point* $x=y=z=-a$ **(*A. U. M 13*)**

Sol. Let planes representing the three faces of the tetrahedron $y+z=0, z+x=0$, $x+y=0$ intersect at the origin. If you consider the corners as A, B, C then tetrahedron would be OABC and the equation of the face ABC is $x+y+z=a$.

We have A is the point of intersection of the planes $z+x=0$ and $y+z=0, x+y+z=0$

$\therefore\ A=(a,a,-a)$

Similarly, $B=(-a,a,a); C=(a,-a,a)$



Equations of $\overline{OA}$ are, $\frac{x}{1}=\frac{y}{1}=\frac{z}{-1}$

Equations of $\overline{BC}$ are, $\frac{x-a}{1}=\frac{y+a}{-1}=\frac{z-a}{0}$

$\therefore$ Equation of the plane through $\overline{OA}$ and parallel to $\overline{BC}$ is $\begin{vmatrix} x & y & z \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{vmatrix} = 0 \Rightarrow -x-y-2z=0 \Rightarrow x+y+2z=0$

The S. D. between $\overline{OA}$ and $\overline{BC}$ is the perpendicular distance from any point $\overline{BC}$ to the plane $x+y+2z=0$.

Since $(a,-a,a)$ is a point on $\overline{BC}$, the S. D. is $\left|\frac{a-a+2a}{\sqrt{6}}\right| = \frac{2|a|}{\sqrt{6}}$

Let $x+y+z=0$ be the plane $\pi=0$

Then the equation of the plane through $\overline{OA}$ and perpendicular to π is $\begin{vmatrix} x & y & z \\ 1 & 1 & -1 \\ 1 & 1 & 2 \end{vmatrix} = 0$

$\Rightarrow 3x-3y=0 \Rightarrow x-y=0$

Equation of the plane through $\overline{BC}$ and perpendicular to π is

$$\begin{vmatrix} x-a & y+a & z-a \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 0 \Rightarrow x+y-z+a=0$$

$\therefore\ x-y=0, x+y-z+a=0$ are the equations of the line of S.D. between $\overline{OA}$ and $\overline{BC}$. Since the point $(-a,-a,-a)$ lies on both the planes, $(-a,-a,-a)$ lies on this line of S. D. We can observe by symmetry that the S. D. between other points of opposoite edges is $\frac{2|a|}{6}$ and lines of S. D. pass through $(-a,-a,-a)$.

EXERCISE 4 (*d*)

1. Find the length and equations to the line of S.D. between the lines

(*a*) $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-1}{2}, \frac{x-4}{4}=\frac{y-3}{5}=\frac{z-2}{3}$

(S. K. U. 2001 Oct. O. U. M 98, N. U. A12, S.K.V M18)

(*b*) $\frac{x}{2}=\frac{y}{-3}=\frac{z}{1}$ and $\frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}$

(N.U.06, O. U. M 97, N. U. 89, 91, S. K. U. 2002 A, S.V.U.)

2. (*i*) Find the length and equations to the line of S.D. between the lines

$\frac{x-3}{-1}=\frac{y-4}{2}=\frac{z+2}{1}, \frac{x-1}{1}=\frac{y+7}{3}=\frac{z+2}{2}$. ***(K. U. 08, M 14, S. V. U. M 13, V.S.P.U M18)***

(*ii*) Find the length and equations to the line of S.D. between the lines

$\frac{x-10}{1}=\frac{y-9}{3}=\frac{z+2}{-2}$, $2(x+1)=y-12=4(z-5)$. Find the points where the line of S.D. intersects the given lines. ***(K. U. 2001 M)***

3. If the position vectors of A, B, C, D are respectively $-\bar{i}+2\bar{j}-3\bar{k}, -16\bar{i}+6\bar{j}+4k$, $\bar{i}-\bar{j}+3\bar{k}$ and $4\bar{i}+9\bar{j}+7\bar{k}$, find the S.D. between the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. ***(N. U. 89)***

4. Find the length and equations to the line of S.D. between the lines $\frac{x}{4}=\frac{y+1}{3}=\frac{z-2}{2}$; $5x-2y-3z+6=0=x-3y+2z-3$.***(K.U. M15, O.U. O97, A11, N.U.M. 98, A.U. M14, M18)***

5. Find the S.D. between the lines $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}; \frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$.***(S. V. U. 01 S)***

Hence show that the lines are coplanar.

6. Find the point on the line through the points $(-6, 1, -10), (-3, 7, -13)$ which is nearest to the line. $3x+2y-15z-144=0=3x-y-3z-42$

7. Find the length and equations to the line of S.D. between the lines

(*i*) $2x-3y+4z=0=x-y+z$; $x+y+2z-3=0=2x+3y+3z-4$

(*ii*) $-5x-y-z=0=x-2y+z+3$; $7x-4y-2z=0=x-y+z-3$.

8. Find the S.D. between the Z-axis and the line

$a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2$ ***(S. V. U. 98)***

9. Show that the S.D. between the lines $x+k=2y=-12z$; $x=y+2k=6z-6k$ is $2k$ where $k>0$. ***(O.U. 08, K.U. Model Paper)***

10. Prove that the S.D. between a diagonal and an edge not meeting it in a cube of side a is $a/\sqrt{2}$.

11. Show that the S.D. between any two opposite edges of the tetrahedron with faces $y+z=0, z+x=0, x+y=0, x+y+z=a$ is $2a/\sqrt{6}$.

12. Find the perpendicular distance of the point $(-2, 1, 5)$ from the line $\dfrac{x-2}{-2}=\dfrac{y-3}{3}=\dfrac{z-5}{-6}$.

(S.V.U.M06)

13. Find the perpendicular distance of the point $(2, 4, -1)$ from the line through the point $(-5, -3, 6)$ whose d.rs. are $1, 4, -9$.

14. Find the foot of the perpendicular from the origin to the line $2x+3y+4z+5=0=x+2y+3z+4$. Hence find the distance of the origin from the line.

(S. V. U. 2001 May, S. V. U. A 93)

15. Find the perpendicular distance of the point $(-1, 3, 9)$ from $\dfrac{x-13}{5}=\dfrac{y+8}{8}=\dfrac{z-31}{1}$.

(N. U. 89)

ANSWERS

1. (*a*) $1\sqrt{6}$; $2x-7y+11z+6=0=2x-10y+14z-6$.

(*b*) $\dfrac{1}{\sqrt{3}}$; $4x+y-5z=0=7x+y-8z-31$

2. (*i*) $\sqrt{35}; \dfrac{x-4}{1}=\dfrac{y-2}{3}=\dfrac{-z-3}{5}$ (*ii*) $5\sqrt{6}; \dfrac{x-8}{11}=\dfrac{y-3}{-5}=\dfrac{z-2}{-2}$; $(8, 3, 2), (-3, 8, 4)$

4. $\sqrt{6/39}$; $7x-2y-11z+20=0=13x-13y+24$ **5.** 0 **6.** $(-7, -1, -9)$

7. (*i*) $\dfrac{13}{\sqrt{66}}$; $3x-y-z=0, x+2y+z=1$, (*ii*) $\dfrac{13\sqrt{3}}{15}$; $17x+20y-19z=39, 8x+5y-31z+67=0$

8. $|d_1c_2-d_2c_1|/\sqrt{\{(a_1c_2-a_2c_1)^2+(b_1c_2-b_2c_1)^2\}}$ **12.** $\dfrac{\sqrt{61}}{7}$. **13.** 7.

14. $\left(\dfrac{2}{3}, \dfrac{-1}{3}, \dfrac{-4}{3}\right), \sqrt{\dfrac{7}{3}}$ **15.** $\dfrac{1}{3\sqrt{10}}(268.47)$

4. 17. AREA OF A TRIANGLE

Theorem. ***Are of the tringle with vertices (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is $\frac{1}{2}[\Sigma\{(y_2-y_1)(z_3-z_1)-(y_3-y_1)(z_2-z_1)\}^2]^{1/2}$ sq. units.***

Proof. Let $A=(x_1, y_1, z_1), B=(x_2, y_2, z_2), C=(x_3, y_3, z_3)$.

$\therefore$ Area of Δ ABC $=\dfrac{1}{2}|\overline{AB}\times\overline{AC}|$

$=\dfrac{1}{2}|(x_2-x_1, y_2-y_1, z_2-z_1)\times(x_3-x_1, y_3-y_1, z_3-z_1)$

$=\dfrac{1}{2}[\Sigma\{(y_2-y_1)(z_3-z_1)-(y_3-y_1)(z_2-z_1)\}^2]^{1/2}$ square units.

Note. If $A=(x_1, y_1, z_1), B=(x_2, y_2, z_2), C=(x_3, y_3, z_3), D=(x_4, y_4, z_4)$ are the vertices of a parallelogram ABCD, then its area $=|\overline{AB}\times\overline{AD}|$

$|(x_2-x_1, y_2-y_1, z_2-z_1)\times(x_4-x_1, y_4-y_1, z_4-z_1)|$ sq. units.

4. 18. DIHEDRAL ANGLE

Half planes of a plane (Fig. 52)

Definition. *Let E be a plane and L be a line in it. Then L is said to divide E into two half planes* H_1, H_2 *(say).* H_1, H_2 *are also called sides of L.*

Dihedral angle (Fig. 53). Let π_1, π_2 be two planes intersecting in a line L. Let H_1 be a half plane of π_1 and H_2 be a half plane of π_2.

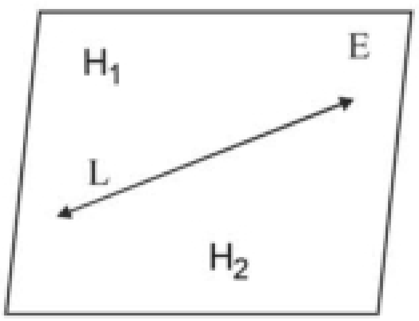


Fig. 52

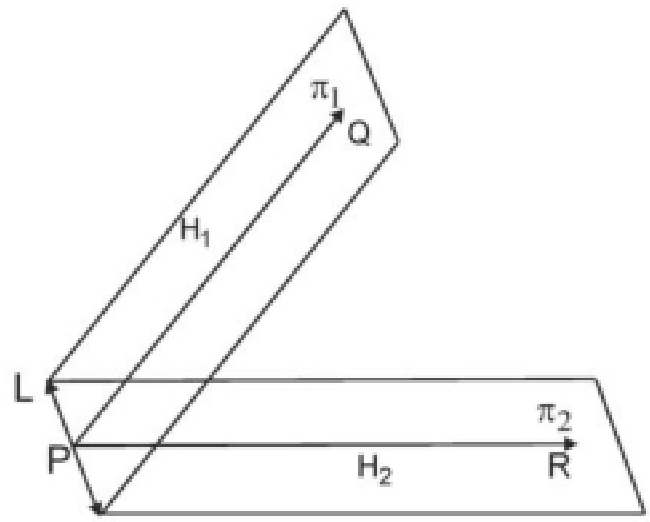


Fig. 53

Then $H_1 \cup H_2 \cup L$ is called dihedral angle. $H_1 \cup L, H_2 \cup L$ are called its sides and L is called its edge.

Let P be any point on L. Let a plane π_3 through P and perpendicular to L intersect π_1 and π_2 in an $\angle QPR$. It is denoted by $(\overrightarrow{PQ}, \overrightarrow{PR})$ or (H_1, H_2).

For all P on L, the measure of the angle is constant. This measure of the angle $(\overrightarrow{PQ}, \overrightarrow{PR})$ is called the measure of the dihedral angle $H_1 \cup H_2 \cup L$.

If $(H_1, H_2) = 90^\circ$, we say that $H_1 \perp H_2$.

Further there will be four dihedral angles between π_1, π_2. If one is θ, others are $180^\circ - \theta, \theta, 180^\circ + \theta$. But we take the angles between π_1, π_2 as $\theta, 180^\circ - \theta$. If $(\pi_1, \pi_2) = 0$ or 180°, we say that $\pi_1 \parallel \pi_2$.

We can observe that the angles between two intersecting planes are equal to the angles between their normals.

4.19. ORTHOGONAL PROJECTION ON A PLANE

Please refer to Art 1.3 - Points 37, 38.

Projection of a triangle in a plane (Fig.54)

Let ABC be a triangle not in the plane σ and also none of the sides is parallel to σ. For convenience we can take A in σ. In σ, let B_1 be the projection of B so that AB_1 be the projection of AB and C_1 be the projection of C so that $B_1 C_1$ be the projection of BC. Clearly in σ, AC_1 is the projection of AC. Now the triangle $A_1 B_1 C_1$ is called the projection of the triangle ABC in σ.

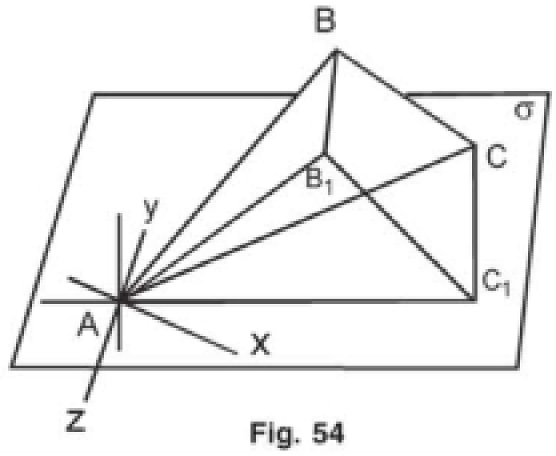


Fig. 54

Theorem. *ABC is a triangle and* σ *is a plane such that* $(\overleftrightarrow{ABC}, \sigma) = \theta$. *If* Δ *is the area of* Δ ABC *and* Δ_1 *is the area of its projection in* σ, *then* $\Delta_1 = \Delta \cos\theta$.

4. 20. VOLUME OF THE TETRAHEDRON

Theorem. *If* $A = (x_1, y_1, z_1), B = (x_2, y_2, z_2), C = (x_3, y_3, z_3), D = (x_4, y_4, z_4)$ *are the vertices of a tetrahedron ABCD, then its volume* $= \dfrac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$

First Proof. Let V be the volume of the tetrahedron ABCD with vertices $A = (x_1, y_1, z_1), B = (x_2, y_2, z_2), C = (x_3, y_3, z_3), D = (x_4, y_4, z_4)$.

Let p be the perpendicular distance of D from the plane ABC.

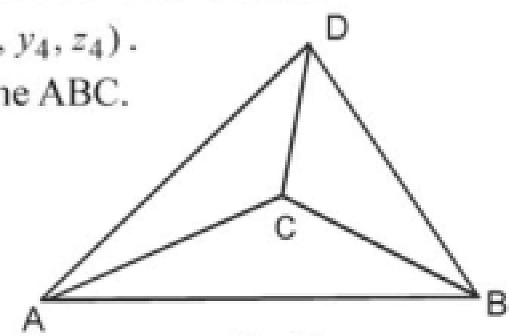


Fig. 55

Equation to the plane ABC is $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$ .... (1)

Let $\Delta_x, \Delta_y, \Delta_z$ be the orthogonal projections of Δ on the coordinate plans

$2\Delta_x = \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}$ = coeffecient of x in (1), $2\Delta_y = \begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}$ = coeffecient of y in (1)

$2\Delta_z = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ = coeffecient of z in (1)

we also have $p = \dfrac{1}{2\sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2}} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = \dfrac{1}{2\Delta} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$

$$\therefore\ V = \left(\frac{1}{3}p\right)(\Delta) = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

Second Proof. (Fig. 55)

Volume of the tetrahedron $= \frac{1}{6}|[\overline{AB}, \overline{AC}, \overline{AD}]|$

$$= \frac{1}{6}|[(x_2 - x_1, y_2 - y_1, z_2 - z_1), (x_3 - x_1, y_3 - y_1, z_3 - z_1), (x_4 - x_1, y_4 - y_1, z_4 - z_1)$$

$$= \frac{1}{6}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 & 0 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 & 0 \end{vmatrix}$$

$$= \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \begin{matrix} R_2 + R_1 \\ R_3 + R_1 \\ R_4 + R_1 \end{matrix} \text{ cubic units} \qquad \begin{bmatrix} \text{taking positive value of} \\ \text{the determinant in each case} \end{bmatrix}$$

Note. If

$[(x_2 - x_1, y_2 - y_1, z_2 - z_1), (x_3 - x_1, y_3 - y_1, z_3 - z_1), (x_4 - x_1, y_4 - y_1, z_4 - z_1)] = 0$,

then A, B, C, D are coplanar.

SOLVED PROBLEMS

Ex. 1. *The areas of projections of* Δ ABC *in the coordinate planes YZ, ZX, XY are respectively* A_x, A_y, A_z. *If A is the area of* Δ ABC, show that $A^2 = A_x^{\ 2} + A_y^{\ 2} + A_z^{\ 2}$.

Sol. Let $(\overleftrightarrow{ABC}, \overleftrightarrow{YOZ}) = \theta$ and L be normal to $\overleftrightarrow{ABC}$ and d.cs. (l, m, n).

$\therefore\ \theta = (L, \overrightarrow{OX}) \Rightarrow \cos\theta = l$.

$\therefore$ Area of projection of Δ ABC in YZ plane $= A_x = A\cos\theta = A_l$

Similarly, $A_y = A_m$ and $A_z = A_n$. $\quad \therefore\ A_x^{\ 2} + A_y^{\ 2} + A_z^{\ 2} = A^2\ (l^2 + m^2 + n^2) = A^2$

Note. The above result may be remembered as a formula.

Ex. 2. *In* $\Delta OAB, O = (0, 0, 0)$, $A = (x_1, y_1, z_1)$ *and* $B = (x_2, y_2, z_2)$. *If* ΔOA_1B_1 *is the projection of* ΔOAB *in the XY plane, show that area of* ΔOA_1B_1

$$= \textit{positive value of} \begin{vmatrix} x_1 & x_2 & 1 \\ y_1 & y_2 & 1 \\ 0 & 0 & 1 \end{vmatrix} \textit{ square units.}$$

Sol. A_1 is the projection of A and B_1 is the projection of B in the XY plane.

$\therefore\ A_1 = (x_1, y_1, 0)$ and $B_1 = (x_2, y_2, 0)$. $\quad \therefore\ \overline{OA_1} = (x_1, y_1, 0)$ and $\overline{OB_1} = (x_2, y_2, 0)$.

$\therefore$ Area of $\Delta OA_1B = \frac{1}{2}|\overline{OA_1} \times \overline{OB_1}| = \frac{1}{2}|(x_1, y_1, 0) \times (x_2, y_2, 0)|$

$= \frac{1}{2}|(0, 0, x_1y_2 - x_2y_1)| = \frac{1}{2}\sqrt{[(x_1y_2 - x_2y_1)^2]} = \frac{1}{2}\{$ positive value of $(x_1y_2 - x_2y_1)\}$

$$=\frac{1}{2}\left(\text{positive value of }\begin{vmatrix} x_1 & x_2 & 1 \\ y_1 & y_2 & 1 \\ 0 & 0 & 1 \end{vmatrix}\right) \text{ square units.}$$

Note. If ΔOA_2B_2, ΔOA_3B_3 are the projections of ΔOAB in YZ, ZX planes respectively then, area of

$$\Delta OA_2B_2 = \frac{1}{2}\left(\text{positive value of }\begin{vmatrix} 0 & 0 & 1 \\ y_1 & y_2 & 1 \\ z_1 & z_2 & 1 \end{vmatrix}\right) \text{ square units}$$

and area of $$\Delta OA_3B_3 = \frac{1}{2}\left(\text{positive value of }\begin{vmatrix} x_1 & y_1 & 1 \\ 0 & 0 & 1 \\ z_1 & z_2 & 1 \end{vmatrix}\right) \text{ square units.}$$

$$\therefore (\text{ Area of } \Delta OAB)^2 = \frac{1}{4}\begin{vmatrix} x_1 & x_2 & 1 \\ y_1 & y_2 & 1 \\ 0 & 0 & 1 \end{vmatrix}^2 + \frac{1}{4}\begin{vmatrix} 0 & 0 & 1 \\ y_1 & y_2 & 1 \\ z_1 & z_2 & 1 \end{vmatrix}^2 + \frac{1}{4}\begin{vmatrix} x_1 & x_2 & 1 \\ 0 & 0 & 1 \\ z_1 & z_2 & 1 \end{vmatrix}$$

Note. The above result may be remembered as a formula.

Ex.3. *OABC is a tetrahedron.* $\overline{OA} = \bar{a}, \overline{OB} = \bar{b}, \overline{OC} = \bar{c}$. $OA = a, OB = b, OC = c$.

$(\overrightarrow{OA}, \overrightarrow{OB}) = \alpha, (\overrightarrow{OB}, \overrightarrow{OC}) = \beta, (\overrightarrow{OC}, \overrightarrow{OA}) = \gamma$. *Show that the volume of the tetrahedron.*

$$=\frac{1}{6}abc\begin{vmatrix} 1 & \cos\alpha & \cos\gamma \\ \cos\alpha & 1 & \cos\beta \\ \cos\gamma & \cos\beta & 1 \end{vmatrix}^{1/2} \quad \textit{cubic units.}$$

First Method. Let O be the origin, and let $OA = a, OB = b, OC = c$ be the three coterminals edges, and let the angles between them be α, β, γ

i.e., $\alpha = (\overrightarrow{OA}, \overrightarrow{OB}), \beta = (\overrightarrow{OB}, \overrightarrow{OC})$ and $\gamma = (\overrightarrow{OC}, \overrightarrow{OA})$

Let the D.cs. of OA, OB, OC be (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) respectively.

Then the coordinates of A, B, C are $(l_1a,\ m_1a,\ n_1a)$, $(l_2b,\ m_2b,\ n_2b)$ and $(l_3c,\ m_3c,\ n_3c)$

$$\therefore \text{ Volume of the tetrahedron OABC } = \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ l_1a & m_1a & n_1a & 1 \\ l_2b & m_2b & n_2b & 1 \\ l_3c & m_3c & n_3c & 1 \end{vmatrix} = -\frac{1}{6}abc\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$\text{(or)}\quad V = \pm\frac{1}{6}abc\begin{vmatrix} \Sigma l_1^2 & \Sigma l_1l_2 & \Sigma l_1l_3 \\ \Sigma l_1l_2 & \Sigma l_2^2 & \Sigma l_2l_3 \\ \Sigma l_1l_3 & \Sigma l_2l_3 & \Sigma l_3^2 \end{vmatrix}^{1/2} = \pm\frac{1}{6}abc\begin{vmatrix} 1 & \cos\alpha & \cos\gamma \\ \cos\alpha & 1 & \cos\beta \\ \cos\alpha & \cos\beta & 1 \end{vmatrix}^{1/2}$$

Second Method.

Sol. If $\bar{a}, \bar{b}, \bar{c}$ are three non-coplanar vectors, we know that

$$[\bar{a}, \bar{b}, \bar{c}]^2 = [\bar{a}, \bar{b}, \bar{c}][\bar{a}, \bar{b}, \bar{c}] = \begin{vmatrix} \bar{a}.\bar{a} & \bar{a}.\bar{b} & \bar{a}.\bar{c} \\ \bar{b}.\bar{a} & \bar{b}.\bar{b} & \bar{b}.\bar{c} \\ \bar{c}.\bar{a} & \bar{c}.\bar{b} & \bar{c}.\bar{c} \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & ab\cos\alpha & ac\cos\gamma \\ ba\cos\alpha & b^2 & bc\cos\beta \\ ca\cos\gamma & cb\cos\beta & c^2 \end{vmatrix} = a^2b^2c^2 \begin{vmatrix} 1 & \cos\alpha & \cos\gamma \\ \cos\alpha & 1 & \cos\beta \\ \cos\gamma & \cos\beta & 1 \end{vmatrix}$$

$\therefore$ Volume of the tetrahedron OABC $= \frac{1}{6}|[abc]|$

$$= \frac{1}{6}abc \begin{vmatrix} 1 & \cos\alpha & \cos\gamma \\ \cos\alpha & 1 & \cos\beta \\ \cos\gamma & \cos\beta & 1 \end{vmatrix}^{\frac{1}{2}} \text{ cubic units.}$$

Ex. 4. *In a tetrahedron OABC,* $OA = a, BC = b, (\overleftrightarrow{OA}, \overleftrightarrow{BC}) = \theta$ *and d is the S.D. between* $\overleftrightarrow{OA}, \overleftrightarrow{BC}$. *If V is the volume of the tetrahedron, show that* $V = \frac{1}{6}abd\sin\theta$ *cubic units.*

First Method.

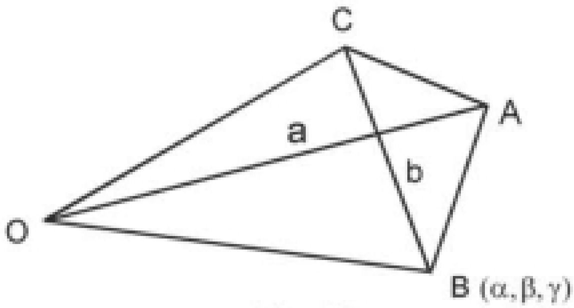


Fig. 56

Sol. OABC is a tetra hedron. $\mathrm{OA} = a, \mathrm{BC} = b$ are the non-intersecting edges.

Let (l_1, m_1, n_1) be the D.cs. of OA and let (l_2, m_2, n_2) be the D.cs. of BC.

$\Rightarrow$ the coordinates of A are $(l_1 a, m_1 a, n_1 a)$

Let $\mathrm{B} = (\alpha, \beta, \gamma) \Rightarrow \mathrm{C} = (\alpha + l_2 b, \beta + m_2 b, \gamma + n_2 b)$

$$\therefore \text{Volume of the tetrahedron OABC } = \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ al_1 & am_1 & an_1 & 1 \\ \alpha & \beta & \gamma & 1 \\ \alpha + bl_2 & \beta + bm_2 & \gamma + bn_2 & 1 \end{vmatrix}$$

$$= -\frac{1}{6}\begin{vmatrix} l_1 a & m_1 a & n_1 a \\ \alpha & \beta & \gamma \\ \alpha + bl_2 & \beta + bm_2 & \alpha + bn_2 \end{vmatrix} = \frac{-a}{6}\begin{vmatrix} l_1 & m_1 & n_1 \\ \alpha & \beta & \gamma \\ bl_2 & bm_2 & bn_2 \end{vmatrix} = \mathrm{R}_3 - \mathrm{R}_2$$

$$= -\frac{1}{6}ab\begin{vmatrix} l_1 & m_1 & n_1 \\ \alpha & \beta & \gamma \\ l_2 & m_2 & n_2 \end{vmatrix} \quad \text{... (A). But } d = \text{S.D.} = \begin{vmatrix} \alpha & \beta & \gamma \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div \sqrt{\Sigma(m_1 n_2 - m_2 n_1)^2}$$

$$= \begin{vmatrix} \alpha & \beta & \gamma \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div \sin\theta \qquad (\because \ (m_1n_2 - m_2n_1)^2 = \sin^2\theta)$$

$$= -\begin{vmatrix} l_1 & m_1 & n_1 \\ \alpha & \beta & \gamma \\ l_2 & m_2 & n_2 \end{vmatrix} \div \sin\theta \Rightarrow \begin{vmatrix} l_1 & m_1 & n_1 \\ \alpha & \beta & \gamma \\ l_2 & m_2 & n_2 \end{vmatrix} = -d\sin\theta$$

substituting this value in (A) we get vol. of the tetrahedron = $\frac{1}{6} abd\sin\theta$ cubic units.

Second Method.

Sol. Let O be the origin and $A = \bar{a}, B = \bar{b}, C = \bar{c}$.

Since $\overleftrightarrow{OA}, \overleftrightarrow{OB}$ are non-coplanar, therefore $d =$ S.D. between $\overleftrightarrow{OA}, \overleftrightarrow{BC}$

$$\Rightarrow d = \left| AC \,.\, \frac{OA \times BC}{|OA \times BC|} \right| = \left| |\bar{c} - \bar{a}| \frac{\bar{a} \times (\bar{c} - \bar{b})}{|OA||BC|\sin\theta} \right|$$

$$\Rightarrow d = \frac{|(\bar{c} - \bar{a}).(\bar{a} \times \bar{c} - \bar{a} \times \bar{b})|}{ab\sin\theta} = \frac{|[\bar{c}\bar{a}\bar{c}] - [\bar{c}\bar{a}\bar{b}] - [\bar{a}\bar{a}\bar{c}] + [\bar{a}\bar{a}\bar{b}]|}{ab\sin\theta} = \frac{|[\bar{a}\bar{b}\bar{c}]|}{ab\sin\theta}$$

$\Rightarrow |[\bar{a}\ \bar{b}\ \bar{c}]| = abd\sin\theta$. But $V = \frac{1}{6}|[\bar{a}, \bar{b}, \bar{c}]|$. $\therefore$ $V = \frac{1}{6} abd\sin\theta$ cubic units.

Ex. 5. *Planes through OX, OY include an angle* α. *Show that their lines of intersection lies on the cone* $z^2(x^2 + y^2 + z^2) = x^2y^2\tan^2\alpha$. ***(N. U. 91)***

Sol. Equation of $\overleftrightarrow{OX}$ are $y = 0, z = 0$. Equations of $\overleftrightarrow{OY}$ are $x = 0, z = 0$.

Equations of a plane through $\overleftrightarrow{OX}$ is $y + kz = 0$ ...(1)

and equation of a plane through $\overleftrightarrow{OY}$ is $x + lz = 0$ ...(2)

where k, l are any two real numbers.

D.rs. of a normal line to (1) plane $= (0, 1, k)$.

D.rs. of a normal line to (2) plane $= (1, 0, l)$.

Equations of line of intersection of the planes (1) and (2) are

$y = -kz, x = -kz$. *i.e.,* $\frac{-y}{z} = k, \frac{-x}{z} = l$ ...(3)

$\alpha =$ Angle between the planes (1) and (2) = Angle between normal lines.

$$\therefore \quad \cos\alpha = \frac{(0, 1, k).(1, 0, l)}{\sqrt{1^2 + k^2}\sqrt{1^2 + l^2}} \Rightarrow (1 + k^2)(1 + l^2) = k^2l^2\sec^2\alpha$$

$\Rightarrow 1 + k^2 + l^2 = k^2l^2\tan^2\alpha$ ...(4)

Eliminating k, l in (3) and (4) the locus of the line (3) is $1 + \frac{y^2}{z^2} + \frac{x^2}{z^2} = \frac{x^2y^2}{z^4}\tan^2\alpha$

i.e., $z^2(x^2 + y^2 + z^2) = x^2y^2 + \tan^2\alpha$

EXERCISE 4 (e)

1. Find the area of the triangle with vertices $(1, 1, -1), (2, -1, -1), (-1, 2, 3)$.
(*A. U. A10, M 14*)
2. Find the areas of the projections of the triangle with vertices (0, 0, 0), (1, 2, 3), $(2, -1, 4)$ in the coordinate planes XY, YZ, ZX. Hence find the area of the triangle.
3. Find the volume of the tetrahedron with vertices
(*i*) $(0, 0, 0), (-1, 1, 1), (1, 1, -1), (1, -1, 1)$ **(*O. U. 07, A. U. M 13*)**
(*ii*) $(1, 2, 1), (3, 2, 5), (2, -1, 0), (-1, 0, 1)$ (*iii*) $(0,1,2),(3,0,1),(4,3,6),(2,3,2)$ **(*A. U. 08*)**
4. Show that the points (*i*) $(6, -4, 4), (1, 2, -5), (-1, -2, -3)$ $(0, 0, -4)$ are coplanar.
(*ii*) $(0,4,3),(-1,-5,-3),(-2,-2,1),(1,1,-1)$ **(*A. U. 08*)**
5. A, B, C are respectively points on $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ such that $OA = a, OB = b, OC = c$. Show that the area of ΔABC is $\frac{1}{2}\sqrt{(b^2c^2 + c^2a^2 + a^2b^2)}$ square units.
6. If P = (2, 2, 1) and if the plane through P and perpendicular to $\overleftrightarrow{OP}$ intersects the coordinate axes in A, B, C, show that the area of ΔABC is $\frac{243}{8}$ square units.
7. O,A,B are the vertices of a triangle such that O = (0, 0, 0),A =(2, 6, 3) , B = (5, 12, 0). Find the area of ΔOAB and hence find the length of the altitude OC of ΔOAB.
8. $A = (a, 0, 0), B = (0, b, 0), C = (0, 0, c)$. If PABC is a tetrahedron with constant volume $\frac{kabc}{6}$ cubic units, show that the locus of P is a pair of planes each parallel to $\overleftrightarrow{ABC}$.
9. A variable plane forms a tetrahedron with the coordinate planes of constant volume $64k^3$ cubic units. Show that the locus of the centroid of the tetrahedron is $x^2y^2z^2 = 36k^6$.
10. Show that the volume of the tetrahedron of which a pair of opposite edges is formed by lengths r_1, r_2 on the lines whose equations are $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ where $l_1^2 + m_1^2 + n_1^2 = l_2^2 + m_2^2 + n_2^2 = 1$ is the positive value of $\frac{1}{6}r_1r_2\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$

ANSWERS

1. $\frac{1}{2}\sqrt{73}$ sq. units. **2.** $\frac{5}{2}$ sq. units, $\frac{11}{2}$ sq. units, 1 sq. unit, $\frac{\sqrt{150}}{2}$ sq. units

3. (*i*) 2/3 cubic units (*ii*) 6 cubic units **7.** $\frac{1}{2}\sqrt{1557}$ sq. units, $\sqrt{\frac{173}{6}}$

4. 21. INTERSECTION OF THREE PLANES - TRIANGULAR PRISM

Suppose we are given three distinct planes (1), (2), (3) such that no two of them are parallel. Then by taking them in pairs we get three lines L_1, L_2, L_3 of intersection with the following possibilities.

(*i*) (Fig. 57). The three lines of intersection have only one point in common *i.e.*,the three planes of intersection have one and only one point in common.

This case arises when the line of intersection of any two planes is perpendicular to the third.

(*ii*) (Fig. 58). The three lines of intersection are coincident *i.e.* the three planes of intersection have a line in common.

This case arises when the line of intersection of any two planes lies in the third plane.

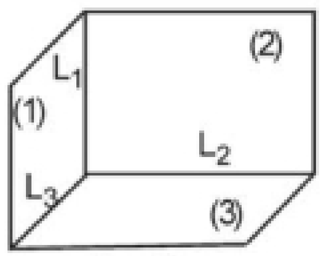


Fig. 57

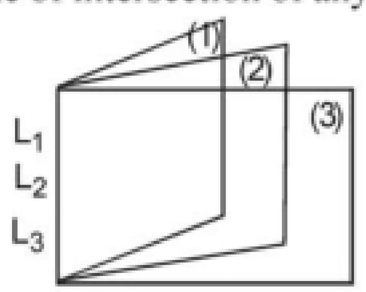


Fig. 58

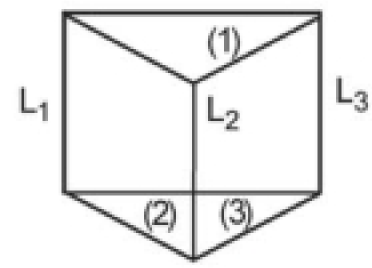


Fig. 59

(*iii*) (Fig. 59) **The three lines of intersection are parallel *i.e.*,the three planes of intersection form a triangular prism.**

Three planes of intersection are said to form a triangular prism if the three lines of intersection of the planes taken in pairs are distinct and parallel.

This case arises when the line of intersection of any two planes is parallel to the third plane without lying in it.

We now establish the conditions for the three distinct planes *(i)* to have a point in common *(ii)* to have a line in common and *(iii)* to form a triangular prism.

Let the three distinct planes so that no two of them are parallel be

$a_1x+b_1y+c_1z+d_1=0$ ...(1) $a_2x+b_2y+c_2z+d_2=0$ ...(2) $a_3x+b_3y+c_3z+d_3=0$ ...(3)

We know that the equation to the line of intersection of (1) and (2) (Art. 4.4) is

$$\frac{x-\dfrac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}}{b_1c_2-b_2c_1}=\frac{y-\dfrac{a_2d_1-a_1d_2}{a_1b_2-a_2b_1}}{c_1a_2-c_2a_1}=\frac{z-0}{a_1b_2-a_2b_1} \quad ...(4)$$

(*i*) The planes (1), (2), (3) have a point in common if the line (4) intersects the plane (3). For this to happen, the line (4) must not be parallel to the plane (3) and the condition for which is

$$a_3\,(b_1c_2-b_2c_1)+b_3\,(c_1a_2-c_2a_1)+c_3\,(a_1b_2-a_2b_1)\neq 0$$

i.e., $a_1\,(b_2c_3-b_3c_2)+b_1\,(a_2c_3-a_3c_2)+c_1\,(a_2b_3-a_3b_2)\neq 0$

i.e., $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\neq 0$ *i.e.*, $\Delta\neq 0$ where $\Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(*ii*) The planes (1), (2), (3) have a line in common if and only if the line (4) lies in the plane (3). For this to happen

(A) the line (4) is parallel to the plane (3) and

(B) the point $\left(\dfrac{b_1d_2-b_2d_1}{a_1b_2-a_2b_2}, \dfrac{a_2d_1-a_2d_2}{a_2b_2-a_2b_2}, 0\right)$ lies in the plane (3).

The condition for (A) is $a_3\,(b_1c_2-b_2c_1)+b_3\,(c_1a_2-c_2a_1)+c_3\,(a_1b_2-a_2b_1)=0$ *i.e.*, $\Delta=0$.

and the condition for (B) is $a_3\left(\dfrac{b_1d_2-b_2d_1}{a_2b_2-a_2b_1}\right)+b_3\left(\dfrac{a_2d_1-a_2d_2}{a_1b_2-a_2b_1}\right)+c_3\,(0)+d_3=0$

i.e., $a_3\,(b_1d_2-b_2d_1)+b_3\,(a_2d_1-a_2d_2)+d_3\,(a_1b_2-a_2b_1)=0$

i.e., $a_1\,(b_2d_3-b_3d_2)+b_1\,(a_2d_3-a_3d_2)+d_1\,(a_2b_3-a_3b_2)=0$

i.e., $\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}=0$ *i.e.*, $\Delta_3=0$ where $\Delta_3=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$.

(*iii*) The planes (1), (2), (3) form a triangular prism if the line (4) is parallel to the plane (3) without lying in the plane (3). For this to happen the condition is

$a_3\,(b_1c_2-b_2c_1)+b_3\,(c_1a_2-c_2a_1)+c_3\,(a_1b_2-a_2b_1)=0$

i.e., $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$ and $a_3\left(\dfrac{b_1d_2-b_2d_1}{a_2b_2-a_2b_1}\right)+b_3\left(\dfrac{a_2d_1-a_1d_2}{a_1b_2-a_2b_1}\right)+c_3\,(0)+d_3\neq 0$

i.e., $a_1\,(b_2d_3-b_3d_2)-b_1\,(a_2d_3-a_3d_2)+d_1\,(a_2b_3-a_3b_2)\neq 0$ *i.e.*, $\Delta_3\neq 0$.

Working Rule. Suppose the equations to three distinct planes are given .

Step I. Check the parallelism between planes. If all the three planes are praralle, there is nothing to proceed further.

If any two of the three planes are parallel, then find the common lines of intersection of these two planes with the third plane (Art. 4.4). Observe that the two lines are parallel.

Step II. If no two of the three planes are parallel, then find Δ. If $\Delta\neq 0$, then the three planes intersect in a unique point. This point can be obtained by solving the equations as simultaneous equations in x, y, z.

Step III. If $\Delta=0$, then find any one of $\Delta_1, \Delta_2, \Delta_3$ where

$$\Delta_1=\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}, \Delta_2=\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}, \Delta_3=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

To Remember : Coefficient matrix of the equation is

$$\left(\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} \begin{array}{l} \text{First column is deleted to form } \Delta_1. \\ \text{Second column is deleted to form } \Delta_2. \\ \text{Third column is deleted to form } \Delta_3. \\ \text{Fourth column is deleted to form } \Delta_1. \end{array}\right)$$

If any one of $\Delta_1, \Delta_2, \Delta_3$ say, $\Delta_1 \neq 0$, then the three planes form a prism.

If $\Delta_1 = 0$, then the three planes have a line of intersection in common.

SOLVED PROBLEMS

Ex. 1. *Examine the nature of intersection of the plans*

$x+2y+3z+1=0$ ...(1) $x-y-z-2=0$ ...(2) $x+2y+3z+4=0$ ...(3)

Sol. Clearly planes (1) and (3) are parallel.

D.rs. of the line of intersection of (1) and (2) are given by

$$\frac{l}{-2+3}=\frac{m}{3+1}=\frac{n}{-1-2} \qquad i.e., \quad \frac{l}{1}=\frac{m}{4}=\frac{n}{-3}$$

A point on the line is obtained by putting $z=0$ in (1) and (2).

$\therefore\ x+2y+1=0, x=y-2=0$. Solving, $x=1, y=-1$.

$\therefore$ Equation to the line of intersection (1) and (2) is $\dfrac{x-1}{1}=\dfrac{y+1}{4}=\dfrac{z-0}{-3}$ ...(4)

Similarly, the line of intersection of (2) and (3) is $\dfrac{x-0}{1}=\dfrac{y+2}{4}=\dfrac{z-0}{-3}$ ...(5)

Clearly, lines (4) and (5) are parallel.

Ex. 2. *Examine the nature of intersection of the plaes*

(*i*) $2x+3y-z-2=0$ ...(1) $3x+3y+z-4=0$ ...(2) $x-y+2z-5=0$ ...(3)

(*ii*) $x+y+z+6=0$ ...(1) $x+2y+2z+6=0$ ...(2) $x+3y+3z+6=0$ ...(3)

(*iii*) $x-2y+z-3=0$ ...(1) $x+y-2z-3=0$ ...(2) $x-z-1=0$ ...(3)

Sol. (*i*) No two of the given planes (1), (2), (3) are parallel.

Coefficient matrix of the given equations is $\begin{bmatrix} 2 & 3 & -1 & -2 \\ 3 & 3 & 1 & -4 \\ 1 & 1 & 2 & -5 \end{bmatrix}$

$$\therefore\ \Delta = \begin{vmatrix} 2 & 3 & -1 \\ 3 & 3 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 2(7)-3(5)-1(-6)=5 \neq 0$$

$\therefore$ The planes (1), (2), (3) intersect in a unique point.

$$\left.\begin{array}{l} (1)-2\times(3): 5y-5z+8=0 \text{ ...(4)} \\ (2)-3\times(3): 6y-5z+11=0 \text{...(5)} \end{array}\right\} \begin{array}{l} \therefore y=-3, z=-7/5 \\ \therefore \quad x=\dfrac{24}{5} \end{array}$$

$\therefore\ \left(\dfrac{24}{5}, -3, -\dfrac{7}{5}\right)$ is the unique point of intersection of the planes.

(*ii*) No two of the given planes (1), (2), (3) are parallel.

Coefficient matrix of the given equation is $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 2 & 6 \\ 1 & 3 & 3 & 6 \end{bmatrix}$

$$\therefore \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 3 \end{vmatrix} = 0, \Delta_1 = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 2 & 6 \\ 3 & 3 & 6 \end{vmatrix} = 0, \ \Delta_2 = 0, \Delta_3 = 0.$$

$\therefore$ The planes intersect in a line. If l, m, n are the d.rs. of the common line, then $\frac{l}{0} = \frac{m}{-1} = \frac{n}{1}$ and a point on the common line is $(-6, 0, 0)$.

$\therefore$ Equation to the common is $x + 6 = 0, \frac{y}{-1} = \frac{z}{1}$.

(*iii*) No two of the given planes (1), (2), (3) are parallel.

Coefficient matrix of the given equations is $\begin{bmatrix} 1 & -2 & 1 & -3 \\ 1 & 1 & -2 & -3 \\ 1 & 0 & -1 & -1 \end{bmatrix}$

$$\therefore \ \Delta = \begin{vmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & -1 \end{vmatrix} = 1(-1) + 2(1) + 1(-1) = 0 \text{ and}$$

$$\Delta_1 = \begin{vmatrix} -2 & 0 & -3 \\ 1 & -2 & -3 \\ 0 & -1 & -1 \end{vmatrix} = -2(-1) - 1(-4) = 6 \neq 0$$

$\therefore$ The planes (1), (2), (3) form a triangular prism.

Ex. 3. *Find the area and the lengths of the edges of a normal section of the prism* $2x + y + z - 3 = 0$ *...(1),* $x - y + 2z - 4 = 0$ *...(2),* $x + z - 2 = 0$ *...(3)*

Sol. Consider a plane section of the prism through the origin which is normal to the three planes (1), (2), (3).

Let this plane be $lx + my + nz = 0$ ..(4)

From (1), (4) : $\left.\begin{array}{l} 2l + m + n = 0 \\ l - m + 2n = 0 \\ l - 0.m + n = 0 \end{array}\right\}$ $\therefore \ \frac{l}{3} = \frac{m}{-3} = \frac{n}{-3}$ *i.e.,* $\frac{l}{1} = \frac{m}{-1} = \frac{n}{-1}$

Clearly, the proportional values of l, m, n satisfy $l + n = 0$.

$\therefore$ Equation to the plane normal to (1), (2), (3) is $x - y - z = 0$ ...(5)

Let A, B, C be the points of intersection of the planes (1), (2), (5) ; (1), (3), (5) and (2), (3), (5) respectively.

$\therefore$ Solving, $A = \left(1, -\frac{1}{3}, \frac{4}{3}\right)$, $B = (1, 0, 1)$, $C = \left(\frac{2}{3}, -\frac{2}{3}, -\frac{4}{3}\right)$.

Since the normal plane section of the prism is the Δ ABC,

$AB = \sqrt{2/3}, BC = \sqrt{6/3}, CA = \sqrt{2/3}$. Also area of $\Delta \text{ ABC} = \frac{1}{2} |\ \overline{AB} \times \overline{AC}\ |$

$$= \frac{1}{2}\left|\left(0, \frac{1}{3}, -\frac{1}{3}\right) \times \left(-\frac{1}{3}, -\frac{1}{3}, 0\right)\right| = \frac{1}{2}\left|\left(-\frac{1}{9}, \frac{1}{9}, \frac{1}{9}\right)\right| = \frac{1}{2}\sqrt{\frac{1}{81} + \frac{1}{81} + \frac{1}{81}} = \frac{\sqrt{3}}{18} \text{ sq. units.}$$

Note. If $\bar{a}, \bar{b}, \bar{c}$ are three non-coplanar and non-null vectors, then the planes $r \cdot \bar{a} = 1$, $r \cdot \bar{b} = 1, r \cdot \bar{c} = 1$ intersect in the point $\dfrac{\bar{b} \times \bar{c} + \bar{c} \times \bar{a} + \bar{a} \times \bar{b}}{[\bar{a}, \bar{b}, \bar{c}]}$

Ex. 4. *The plane* $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ *meets the axes* $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ *in A, B, C. Prove that the planes through the axes and the internal bisectors of the angles A, B, C pass through the line* $\dfrac{x}{a\sqrt{(b^2+c^2)}} = \dfrac{y}{b\sqrt{(c^2+a^2)}} = \dfrac{z}{c\sqrt{(a^2+b^2)}}$.

Sol. The plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ ...(1)

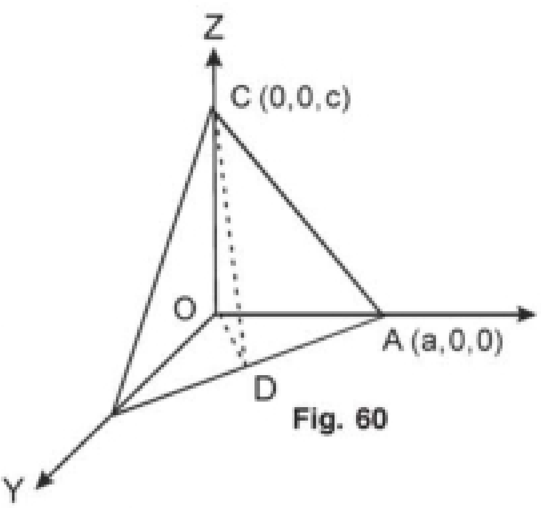


Fig. 60

intersects the axes in A, B, C (Fig.59).

$\therefore$ A = $(a, 0, 0)$, B = $(0, b, 0)$, C = $(0, 0, c)$

$\therefore$ BC = $\sqrt{(b^2+c^2)}$ and AC = $\sqrt{(a^2+c^2)}$.

Let CD be the internal bisector of angle C of Δ ABC, then BD : DC

$= \sqrt{(b^2+c^2)} : \sqrt{(a^2+c^2)}$

$$\therefore \text{ D} = \left(\frac{a\sqrt{(b^2+c^2)}}{\sqrt{(b^2+c^2)} + \sqrt{(a^2+c^2)}}, \frac{b\sqrt{(a^2+c^2)}}{\sqrt{(b^2+c^2)} + \sqrt{(a^2+c^2)}}, 0\right)$$

Since equations to the Z-axis are $x = 0, y = 0$,

a plane through the Z-axis is $x - \lambda y = 0$.

If the plane passes through the line of bisector CD,

$$\text{then } \frac{a\sqrt{(b^2+c^2)}}{\sqrt{(b^2+c^2)} + \sqrt{(a^2+c^2)}} - \lambda \frac{b\sqrt{(c^2+a^2)}}{\sqrt{(b^2+c^2)} + \sqrt{(c^2+a^2)}} = 0$$

i.e., $\lambda = a\sqrt{(b^2+c^2)} \,/\, b\sqrt{(c^2+a^2)}$

$\therefore$ Equation to the plane throuh the Z-axis and bisecting the angle C of Δ ABC is

$$x - \frac{a\sqrt{(b^2+c^2)}}{b\sqrt{(c^2+a^2)}} = y = 0 \quad i.e., \quad \frac{x}{a\sqrt{(b^2+c^2)}} = \frac{y}{b\sqrt{(c^2+a^2)}}.$$

Similarly, we can find the other bisecting planes and their line of intersection is

$$\frac{x}{a\sqrt{(b^2+c^2)}} = \frac{y}{b\sqrt{(c^2+a^2)}} = \frac{z}{c\sqrt{(a^2+b^2)}}$$

EXERCISE 4 (f)

1. Examine the nature of intersection of the set of planes
   (i) $2x-y+z-4=0, 5x+7y+2z=0, 3x+4y-2z+3=0$
   (ii) $x+2y+3z-6=0,\ 3x+4y+5z-2=0,\ 5x+4y+3z+18=0$
   (iii) $4x-5y-2z-2=0,\ 5x-4y+2z+2=0,\ 2x+2y+8z-1=0$. **(A.U. M 14)**
2. For what values of λ do the planes $x-y+z+1=0,\ \lambda x+3y+2z-3=0,\ 3x+\lambda y-z-2=0$
   (i) intersect at a point (ii) form a triangular prism.
3. Prove that the planes $x+ay+(b+c)z+d=0,\ x+by+(c+a)z+d=0,$
   $x+cy+(a+b)z+d=0$ where $a \neq b \neq c$ intersect in a unique line.
4. Show that the planes $2x+3y+4z-6=0,\ 3x+4y+5z-2=0,\ x+2y+3z-2=0$ form a prism. Hence find the area of its normal section.
5. Show that the planes $bx-ay=n,\ cy-bz=l,\ az-cx=m$: $abc \neq 0$ will intersect in a line if $al+bm+cn=0$ and the d.rs. of the line, then, are a, b, c.
6. Prove that the planes $x=cy+bz, y=az+cx, z=bx+ay$ pass through a unique line if $a^2+b^2+c^2+2abc=1$ and the line of intersection then is $\dfrac{x}{\sqrt{(1-a^2)}}=\dfrac{y}{\sqrt{(1-b)^2}}=\dfrac{z}{\sqrt{(1-c^2)}}$
7. (i) Show that the volume of a tetrahedron bounded by the planes $lx+my=0, my+nz=0,$ $nz+lx=0, lx+my+nz=p$ is $\dfrac{2p^3}{3lmn}$ cub. units.
   (ii) Show that the volume of a tetrahedron formed by the planes
   $y+z=0, z+x=0, x+y=0$ and $x+y+z=1$ is 2/3 c. units. **(O.U. 08)**
8. If the three planes through P and the three given lines $y=1, z=-1; z=1, x=-1;$ $x=1,\ y=-1$ ll pass through one line, then show that P lies on $yz+zx+xy+1=0$.

ANSWERS

1. (i) $(1,-1,1)$ (ii) $\dfrac{x+10}{1}=\dfrac{y-8}{-2}=\dfrac{z}{1}$ (iii) Prism

2. (i) $\lambda=4,-3$ (ii) $\lambda=4$ **4.** $\dfrac{8\sqrt{6}}{3}$ sq. units.

5

Change of Axes

5.1. Relative to a frame of reference we have defined the coordinates for any point in space. By changing the frame of reference also we can determine the coordinates of a point in space so that the coordinates of the point change with the change in the frame of reference. This different sets of coordinates could be got in various ways. For instance, we could shift the origin to a new position with coordinate axes similarly directed or change the directions of axes keeping the origin in its original position or shift the origin to a new position and change the direction of axes simultaneously.

By establishing relations between the coordinates of a point relatively to two suitable different frames of reference, the equations to surfaces can be brought to simplified forms.

5. 2. TRANSLATION OF AXES (CHANGE OF ORIGIN)

Definition. *If the origin is changed without changing the direction of axes we get a transformation called the translation of axes.*

Theorem. Let OXYZ be a frame. Let $O_1 = (f, g, h)$.

$\therefore$ $OO_1 = (f, g, h)$. Through O_1, take lines $\overleftrightarrow{X'X}, \overleftrightarrow{Y'Y}, \overleftrightarrow{Z'Z}$ parallel to the axes. Let O_1XYZ be another frame of reference.

(Fig. 61). Let P be a point so that

$P = (x, y, z)$ w.r.t. OXYZ as frame of reference

and

$P = (X, Y, Z)$ w.r.t. O_1XYZ as frame of reference.

$\therefore$ $\overline{OP} = (x, y, z)$, $\overline{O_1P} = (X, Y, Z)$

Now $\therefore$ $\overline{OP} = \overline{OO_1} + \overline{O_1P}$

$\Rightarrow (x, y, z) = (f, g, h) + (X, Y, Z)$

$\Rightarrow (x, y, z) = (X + f, Y + g, Z + h)$

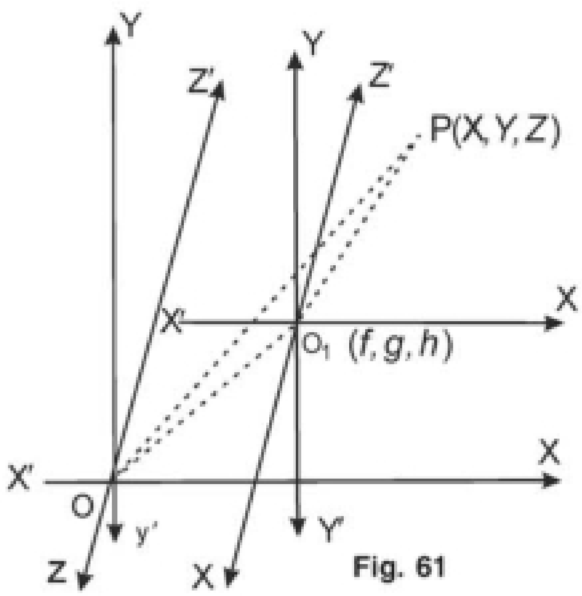


Fig. 61

$\Rightarrow x = X + \bar{f}, y = Y + \bar{g}, z = Z + \bar{h}$ **or** $\overline{X} = \bar{x} - \bar{f}, \overline{Y} = \bar{y} - \bar{g}, \overline{Z} = \bar{z} - \bar{h}$

These equations are called *transformation equations* with translation of axes.

5. 3. ROTATION OF AXES

Defintion. *If the direction of the axes is changed without changing the origin we get a transformation called the rotation of axes.*

***Theorem.** Oxyz is a frame of reference. Through* $\mathbf{O}, \overleftrightarrow{\mathbf{X'X}}, \overleftrightarrow{\mathbf{Y'Y}}, \overleftrightarrow{\mathbf{Z'Z}}$ *are three mutually perpendicular lines so that the d.cs. of* $\overleftrightarrow{\mathbf{OX}}, \overleftrightarrow{\mathbf{OY}}, \overleftrightarrow{\mathbf{OZ}}$ *are respectively* l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3. *P is a point so that* $\mathbf{P} = (x, y, z)$ *w.r.t. Oxyz as the frame of reference and* $\mathbf{P} = (\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ **w.r.t. OXYZ** *as the frame of reference. Then we have*

$x = l_1\text{X} + l_2\text{Y} + l_3\text{Z}$	$\text{X} = l_1 x + m_1 y + n_1 z$
$y = m_1\text{X} + m_2\text{Y} + m_3\text{Z}$	$\text{Y} = l_2 x + m_2 y + n_2 z$
$z = n_1\text{X} + n_2\text{Y} + n_3\text{Z}$	$\text{Z} = l_3 x + m_3 y + n_3 z$

Proof. Let $\bar{i}, \bar{j}, \bar{k}$ be the unit vectors along $\overrightarrow{\text{Ox}}, \overrightarrow{\text{Oy}}, \overrightarrow{\text{Oz}}$ and $\bar{i}_1, \bar{j}_1, \bar{k}_1$ be the unit vectors along $\overleftrightarrow{\text{OX}}, \overleftrightarrow{\text{OY}}, \overleftrightarrow{\text{OZ}}$ respectively.

$\therefore\ \bar{i} = (1, 0, 0),\ \bar{j} = (0, 1, 0),\ \bar{k} = (0, 01)$ and

$$\bar{i}_1 = (l_1, m_1, n_1),\ \bar{j}_1 = (l_2, m_2, n_2),\ \bar{k}_1 = (l_3, m_3, n_3)$$

$\therefore\ \bar{i} \cdot \bar{i}_1 = (1, 0, 0) \cdot (l_1, m_1, n_1) = l_1, \bar{i} \cdot \bar{j}_1 = l_2,\ \bar{i} \cdot \bar{k}_1 = l_3$;

$\bar{j} \cdot \bar{i}_1 = (0, 1, 0) \cdot (l_1, m_1, n_1) = m_1, \bar{j} \cdot \bar{j}_1 = m_2,\ \bar{j} \cdot \bar{k}_1 = m_3$;

$\bar{k} \cdot \bar{i}_1 = (0, 0, 1) \cdot (l_1, m_1, n_1) = n_1, \bar{k} \cdot \bar{j}_1 = n_2,\ \bar{k} \cdot \bar{k}_1 = n_3$.

$\overline{\text{OP}} = x\bar{i} + y\bar{j} + z\bar{k}$ w.r.t. Oxyz and $\overline{\text{OP}} = \text{X}\bar{i}_1 + \text{Y}\bar{j}_1 + \text{Z}\bar{k}_1$ w.r.t. OXYZ

$\therefore\ x\bar{i} + y\bar{j} + z\bar{k} = \text{X}\bar{i}_1 + \text{Y}\bar{j}_1 + \text{Z}\bar{k}_1$ ...(1)

$\Rightarrow\ \bar{i} \cdot (x\bar{i} + y\bar{j} + z\bar{k}) = \bar{i} \cdot (\text{X}\bar{i}_1 + \text{Y}\bar{j}_1 + \text{Z}\bar{k}_1) \ \Rightarrow\ x = l_1\text{X} + l_2\text{Y} + l_3\text{Z}$ ...(1)

Again from I, $\bar{j} \cdot (x\bar{i} + y\bar{j} + z\bar{k}) = \bar{j} \cdot (\text{X}\bar{i}_1 + \text{Y}\bar{j}_1 + \text{Z}\bar{k}_1) \ \Rightarrow\ y = m_1\text{X} + m_2\text{Y} + m_3\text{Z}$ ...(2)

Also from I, $\bar{k} \cdot (x\bar{i} + y\bar{j} + z\bar{k}) = \bar{k} \cdot (\text{X}\bar{i}_1 + \text{Y}\bar{j}_1 + \text{Z}\bar{k}_1) \Rightarrow\ z = n_1\text{X} + n_2\text{Y} + n_3\text{Z}$ ...(3)

Further from (I), $i_1 \cdot (\text{X}\bar{i}_1 + \text{Y}\bar{j}_1 + \text{Z}\bar{k}_1) = \bar{i} \cdot (x\bar{i} + y\bar{j} + z\bar{k}) \Rightarrow\ \text{X} = l_1 x + m_1 y + n_1 z$ ...(4)

Similarly $\text{Y} = l_2 x + m_2 y + n_2 z$ ...(5) $\text{Z} = l_3 x + m_3 y + n_3 z$ ...(6)

The equations (1), (2), (3) ; (4), (5), (6) are called transformation equations with rotation of axes.

Note. ***(i)*** With OXYZ as frame of reference, d.cs. of $\overrightarrow{\text{O}x}, \overrightarrow{\text{O}y}, \overrightarrow{\text{O}z}$ are l_1, l_2, l_3 ; m_1, m_2, m_3 ; n_1, n_2, n_3

(ii) The following table helps us to remember various results of the above theorem.

	x	y	z
X	l_1	m_1	n_1
Y	l_2	m_2	n_2
Z	l_3	m_3	n_3

(iii) Relations between the d.cs. $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ of $\overrightarrow{\text{O}x}, \overrightarrow{\text{O}y}, \overrightarrow{\text{O}z}$ w.r.t. Oxyz and the d.cs. l_1, l_2, l_3 ; m_1, m_2, m_3 ; n_1, n_2, n_3 of $\overrightarrow{\text{O}x}, \overrightarrow{\text{O}y}, \overrightarrow{\text{O}z}$ w.r.t. OXYZ :

$l_1^2+m_1^2+n_1^2=1,\ \ l_2^2+m_2^2+n_2^2=1,\ \ l_3^2+m_3^2+n_3^2=1$.

Since $OX \perp OY, OY \perp OZ, OZ \perp OX$, we have

$l_1l_2+m_1m_2+n_1n_2=0$ ...(I) $\qquad l_2l_3+m_2m_3+n_2n_3=0$ ...(II)

$l_3l_1+m_3m_1+n_3n_1=0$ ...(III)

Also $l_1^2+l_2^2+l_3^2=1,\ \ m_1^2+m_2^2+m_3^2=1,\ \ n_1^2+n_2^2+n_3^2=1$.

Since $Ox \perp Oy, Oy \perp Oz, Oz \perp Ox$, we have

$l_1m_1+l_2m_2+l_3m_3=0$ ...(IV) $\qquad m_1n_1+m_2n_2+m_3n_3=0$ ...(V)

$n_1l_1+n_2l_2+n_3l_3=0$ ...(VI)

From I and III, $\dfrac{l_1}{m_2n_3-m_3n_2}=\dfrac{m_1}{n_2l_3-n_3l_2}=\dfrac{n_1}{l_2m_3-l_3m_2}$

$$=\frac{\sqrt{l_1^2+m_1^2+n_1^2}}{\sqrt{\Sigma(m_2n_3-m_3n_2)^2}}=\frac{1}{\sin 90^\circ}=1 \qquad (\because (\overrightarrow{OY},\overrightarrow{OZ})=90^\circ)$$

$\Rightarrow l_1=m_2n_3-m_3n_2,\ m_1=n_2l_3-n_3l_2,\ n_1=l_2m_3-l_3m_2$

Similarly from I and II, we can have l_2, m_2, n_2;

II and III, we can have l_3, m_3, n_3; IV and VI, we can have l_1, l_2, l_3;

IV and V we can have m_1, m_2, m_3; V and VI we can have n_1, n_2, n_3.

(*iv*) $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = l_1(m_2n_3-m_3n_2)-m_1(l_2n_3-l_3n_2)+n_1(l_2m_3-l_3m_2)$

$=l_1(l_1)-m_1(-m_1)+n_1(n_1)=l_1^2+m_1^2+n_1^2=1$

Also $\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}=1$

(*v*) If $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ are d.cs. of three mutually perpendicular lines through a point P, then there always exist three mutually perpendicular lines through P with d.cs. $l_1, l_2, l_3; m_1, m_2, m_3; n_1, n_2, n_3$.

(*vi*) From (1), (2), (3), $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \Rightarrow P = AQ$

where $P=\begin{bmatrix} x \\ y \\ z \end{bmatrix},\ \ A=\begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix},\ \ Q=\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$.

From (4), (5), (6), $\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\Rightarrow Q = A'P$ $\Rightarrow Q = A'AQ$ $\Rightarrow A'A = 1$

$$\Rightarrow \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} l_1^2 + m_1^2 + n_1^2 & l_1l_2 + msub1m_2 + n_1n_2 & l_1l_3 + m_1m_3 + n_1n_3 \\ l_1l_2 + m_1m_2 + n_1n_2 & l_2^2m_2^2 + n_2^2 & l_2l_3 + m_2m_3 + n_2n_3 \\ l_1l_3 + m_1m_3 + n_1n_3 & l_2l_3m_2m_3 + n_2n_3 & l_3^2 + m_3^2 + n_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow l_1^2 + m_1^2 + n_1^2 = 1,\ l_2^2 + m_2^2 + n_2^2 = 1, l_3^2 + m_3^2 + n_3^2 = 1$

$l_1l_2 + m_1m_2 + n_1n_2 = 0, l_2l_3 + m_2m_3 + n_2n_3 = 0, l_3l_1 + m_3m_1 + n_3n_1 = 0$

Similarly $AA' = I \quad \Rightarrow l_1^2 + l_2^2 + l_3^2 = 1, m_1^2 + m_2^2 + m_3^2 = 1, n_1^2 + n_2^2 + n_3^2 = 1$

$l_1m_1 + l_2m_2 + l_3m_3 = 0, m_1n_1 + m_2n_2 + m_3n_3 = 0, n_1l_1 + n_2l_2 + n_3l_3 = 0$

Also $|AA'| = I \Rightarrow |A||A'| = |I| \Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1$

$$\Rightarrow \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \pm 1 \text{ or } \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = \pm 1$$

$|A| = -1$ is due to the face that if l, m, n are d.cs. of $\overrightarrow{OX}$ then $l, m, n; -l, -m, -n$ are d.cs. of $\overleftrightarrow{OX}$.

5. 4. CHANGE OF ORIGIN ALONG WITH THE CHANGE OF DIRECTION OF AXES

Theorem. $O_1 = (f, g, h)$ $Oxyz$ *as the frame of reference.* O_1XYZ *is another frame of reference. The d.cs. of* $\overrightarrow{O_1X}, \overrightarrow{O_1Y}, \overrightarrow{O_1Z}$ *are* $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ *w.r.t.* $Oxyz$ *as the frame of reference.* $P = (x, y, z)$ *w.r.t.* $Oxyz$ *and* $P = (X, Y, Z)$ *w.r.t.* O_1XYZ. *The transformation equations are*

$$x = f + l_1X + l_2Y + l_3Z, X = l_1(x - f) + m_1(y - g) + n_1(z - h)$$
$$y = g + m_1X + m_2Y + m_3Z, Y = l_2(x - f) + m_2(y - g) + n_2(z - h)$$
$$z = h + n_1X + n_2Y + n_3Z, Z = l_3(x - f) + m_3(y - g) + n_3(z - h)$$

Proof. By translation of axes to O_1, $P = (x - f, y - g, z - h)$

But $P = (X, Y, Z)$ w.r.t. O_1XYZ.

$X = l_1(x - f) + m_1(y - g) + n_1(z - h)$; $Y = l_2(x - f) + m_2(y - g) + n_2(z - h)$

$Z = l_3(x - f) + m_3(y - g) + n_3(z - h)$

Also $x - f = l_1X + l_2Y + l_3Z$, $y - g = m_1X + m_2Y + m_3Z$, $z - h = n_1X + n_2Y + n_3Z$

5. 5. THE DEGREE OF A POLYNOMIAL EQUATION $\phi(x, y, z) = 0$ TO ANY SURFACE IS INVARIANT BY ANY TRANSFORMATION.

For, $\phi(x, y, z) = \phi(X + f, Y + g, Z + h)$,

$\phi(x, y, z) = \phi(l_1X + l_2Y + l_3Z, m_1X + m_2Y + m_3Z, n_1X + n_2Y + n_3Z)$

$\phi(x, y, z) = \phi(f + l_1X + l_2Y + l_3Z, g + m_1X + m_2Y + m_3Z, h + n_1X + n_2Y + n_3Z)$

5. 6. INVARIANTS

Theorem. *By the transformation - Rotation of axes - if a polynomial* $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$ *is transformed into* $\mathbf{AX^2+BY^2+CZ^2+2FYZ+2GZX+2HXY}$, *then (i)* $\mathbf{a+b+c=A+B+C}$,

(ii) $ab+bc+ca-f^2-g^2-h^2=\mathbf{AB+BC+CA-F^2-G^2-H^2}$

(iii) $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} \mathbf{A} & \mathbf{H} & \mathbf{G} \\ \mathbf{H} & \mathbf{G} & \mathbf{F} \\ \mathbf{G} & \mathbf{F} & \mathbf{C} \end{vmatrix}$.

Proof. Let $Oxyz$, OXYZ be two frames of reference by the transformation - Rotation of axes.

If $P=(x, y, z)$ w.r.t. $Oxyz$ frame and $P=(X, Y, Z)$ w.r.t. OXYZ frame, then

$x^2+y^2+z^2=OP^2=X^2+Y^2+Z^2$

Thus $x^2+y^2+z^2$ becomes $X^2+Y^2+Z^2$ by the transformation.

Also it is given that $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$ is transformed into

$AX^2+BY^2+CZ^2+2FYZ+2GZX+2HXY$.

Thus if λ is any real number, the polynomial

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy-\lambda(x^2+y^2+z^2)$

$=(a-\lambda)x^2+(b-\lambda)y^2+(c-\lambda)z^2+2fyz+2gzx+2hxy$ ...(1)

becomes $AX^2+BY^2+CZ^2+2FYZ+2GZX+2HXY-\lambda(X^2+Y^2+Z^2)$

$=(A-\lambda)X^2+(B-\lambda)Y^2+(C-\lambda)Z^2+2FYZ+2GZX+2HXY$ ...(2)

But we know that the degree of the polynomial (1) is invariant by a transformation. Therefore, the degree of (2) is equal to the degree of (1). Hence, for any value of λ, the polynomial becomes a product of two linear factors, then for the same value of λ the polynomial (2) must also become a product of two linear factors. Further the linear factors of (1) become the linear factors of (2).

Now, the values of λ, for which the polynomials (1) and (2) are the products of the linear factors are respectively the roots of the cubic equations.

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0, \quad \begin{vmatrix} A-\lambda & h & g \\ h & B-\lambda & f \\ g & f & C-\lambda \end{vmatrix}=0$$

$\Rightarrow \lambda^3-(a+b+c)\lambda^2+(ab+bc+ca-f^2-g^2-h^2)\lambda+D=0$ .... (3)

$\lambda^3-(A+B+C)\lambda^2+(AB+BC+CA-F^2-G^2-H^2)\lambda+D'=0$ .... (4)

where $D=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, $D'=\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$

Since (3) and (4) have the same roots, we have

$1:1=(a+b+c):(A+B+C)=(ab+bc+ca-f^2-g^2-h^2)$

$\Rightarrow a+b+c = \mathrm{A}+\mathrm{B}+\mathrm{C},\ ab+bc+ca-f^2-g^2-h^2 = \mathrm{AB}+\mathrm{BC}+\mathrm{CA}-\mathrm{F}^2-\mathrm{G}^2-\mathrm{H}^2,$

$$\mathrm{D}=\mathrm{D}' \quad i.e., \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} \mathrm{A} & \mathrm{H} & \mathrm{G} \\ \mathrm{H} & \mathrm{B} & \mathrm{F} \\ \mathrm{G} & \mathrm{F} & \mathrm{C} \end{vmatrix}$$

Note. $a+b+c, bc+aca+ab-f^2-g^2-h^2$, in the above context are called invariants.

SOLVED PROBLEMS

Ex. 1. *Show that the lines through a point with d.cs.* $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$; $\left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)$; $\left(\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$ *form a system of three mutually perpendicular lines. Hence transform the equation* $x^2+y^2+z^2+2yz+2zx+2xy=0$ *of a surface w.r.t. the axes through the same origin and with the above d.cs.*

Sol. Let $l_1=\frac{1}{\sqrt{3}}, m_1=\frac{1}{\sqrt{3}}, n_1=\frac{1}{\sqrt{3}}$; $l_2=\frac{1}{\sqrt{2}}, m_2=0, n_2=-\frac{1}{\sqrt{2}}$;
$l_3=\frac{1}{\sqrt{6}}, m_4=-\frac{2}{\sqrt{6}}, n_3=\frac{1}{\sqrt{6}}$.

We observe that $l_1^2+m_1^2+n_1^2=1,\ l_2^2+m_2^2+n_2^2=1,\ l_3^2+m_3^2+n_3^2=1$

$l_1l_2+m_1m_2+n_1n_2=0,\ l_2l_3+m_2m_3+n_2n_3=0,\ l_1l_3+m_1m_3+n_1n_3=0$

$\therefore$ The lines with the given d.cs. form a system of three mutually perpendicular lines.

The transformation equations with rotation of axes through the same origin are

$x=l_1\mathrm{X}+l_2\mathrm{Y}+l_3\mathrm{Z}=\frac{1}{\sqrt{3}}\mathrm{X}+\frac{1}{\sqrt{2}}\mathrm{Y}+\frac{1}{\sqrt{6}}\mathrm{Z},\ y=m_1\mathrm{X}+m_2\mathrm{Y}+m_3\mathrm{Z}=\frac{1}{\sqrt{3}}\mathrm{X}-\frac{2}{\sqrt{6}}\mathrm{Z},$

$z=n_1\mathrm{X}+n_2\mathrm{Y}+n_3\mathrm{Z}=\frac{1}{\sqrt{3}}\mathrm{X}-\frac{1}{\sqrt{2}}\mathrm{Y}+\frac{1}{\sqrt{6}}\mathrm{Z}.\quad i.e., \begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}\begin{bmatrix} \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{bmatrix}$

Taking transpose of the matrices, $[x \quad y \quad z]=[\mathrm{X} \quad \mathrm{Y} \quad \mathrm{Z}]\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$

We can write $x^2+y^2+z^2+2yz+2zx+2xy$ in the form

$[x \quad y \quad z]\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ which transforms to

$$[\mathrm{X} \quad \mathrm{Y} \quad \mathrm{Z}]\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{bmatrix}$$

$$=[X\quad Y\quad Z]\begin{bmatrix}\frac{3}{\sqrt{3}} & \frac{3}{\sqrt{3}} & \frac{3}{\sqrt{3}}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\end{bmatrix}\begin{bmatrix}X\\ Y\\ Z\end{bmatrix}$$

$$=[X\quad Y\quad Z]\begin{bmatrix}3 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}X\\ Y\\ Z\end{bmatrix}=[3X\quad 0\quad 0]\begin{bmatrix}X\\ Y\\ Z\end{bmatrix}=[3X^2]$$

$\therefore$ Transformed equation is $3X^2=0$

Ex. 2. *Find the point to which the origin should be shifted, the direction of the axes remaining the same, such that the transformed equation of the surface* $3x^2+5y^2-2z^2-12x+10y+20z-50=0$ *does not contain any term of the first degree.*

Sol. $3x^2+5y^2-2z^2-12x+10y+20z-50=0$

$$\equiv 3(x^2-4x+4)+5(y^2+2y+1)-2(z^2-10z+25)-17$$

$$\equiv 3(x-2)^2+5(y+1)^2-2(z-5)^2-17.$$

Transferring the origin to $(2,-1,5)$ by the translation of axes, transformation equations are $X=x-2$ *i.e.,* $x=X+2$, $Y=y+1$ *i.e.,* $y=Y-1$, $Z=z-5$ *i.e.,* $z=Z+5$.

The equation to the surface can be transformed into $3X^2+5Y^2-2Z^2-17-0$ in which first degree terms are missing.

$\therefore$ The origin should be shifted to $(2,-1,5)$

Ex. 3. *The d.cs. of three mutually perpendicular rays* $\overrightarrow{OA},\overrightarrow{OB},\overrightarrow{OC}$ *are respectively* $l_1,m_1,n_1;\ l_2,m_2,n_2;\ l_3,m_3,n_3$. *If* $OA=OB=OC=a$, *show that the equation to the plane* $\overleftrightarrow{ABC}$ *is* $(l_1+l_2+l_3)x+(m_1+m_2+m_3)y+(n_1+n_2+n_3)z=a$.

Sol. Let OABC be the new frame of reference.

$\therefore$ Transformation equations are

$X=l_1x+m_1y+n_1z,\ Y=l_2x+m_2y+n_2z,\ Z=l_3x+m_3y+n_3z.$

Now for the plane $\overleftrightarrow{ABC}$ intercepts on the new axes OA, OB, OC *i.e., a, a, a.*

$\therefore$ Equation to the plane $\overleftrightarrow{ABC}$ w.r.t. OABC frame is $\frac{X}{a}+\frac{Y}{a}+\frac{Z}{a}=1$ *i.e.,* $X+Y+Z=a$

$\therefore$ Equation to the plane $\overleftrightarrow{ABC}$ w.r.t. O*xyz* frame is

$(l_1x+m_1y+n_1z)+(l_2x+m_2y+n_2z)+(l_3x+m_3y+n_3z)=a$

i.e. $(l_1+l_2+l_3)x+(m_1+m_2+m_3)y+(n_1+n_2+n_3)z=a$.

Ex. 4. *The equation* $p_r-l_rx=m_ry-n_r=0,\ r=1,2,3$ *represent three mutually perpendicular planes and* l_r,m_r,n_r *are d.cs. of the normals to the respective planes. Prove that if* (α,β,γ) *is a point which is at a distance d from each of the planes such that* $l_r\alpha+m_r\beta+n_r\gamma>p_r$ *then*

$$d=\frac{\alpha-(l_1p_1+l_2p_2+l_3p_3)}{l_1+l_2+l_3}=\frac{\beta-(m_1p_1+m_2p_2+m_3p_3)}{m_1+m_2+m_3}=\frac{\gamma-(n_1p_1+n_2p_2+n_3p_3)}{n_1+n_2+n_3}$$

Sol. Distance of (α,β,γ) from each of the planes is given by

$|p_r - l_r, \alpha - m_r, \beta - n_r\gamma| = l_r\alpha + m_r\beta + n_r\gamma - p_r \qquad (r = 1,2,3)$

$\therefore \quad d = l_1\alpha + m_1\beta + n_1\gamma - p_1 \quad \text{.... (1)} \quad d = l_2\alpha + m_2\beta + n_2\gamma - p_2 \quad \text{..... (2)}$

$d = l_3\alpha + m_3\beta + n_3\gamma - p_3 \quad \text{.... (3)}$

$$l_1 \times (1) + l_2 \times (2) + l_3 \times (3): \; d\,(l_1 + l_2 + l_3) = (l_1^2 + l_2^2 + l_3^2)\,\alpha + (l_1m_1 + l_2m_2 + l_3m_3)\,\beta$$
$$+ (l_1n_1 + l_2n_2 + l_3n_3)\,\gamma - (l_1p_1 + l_2p_2 + l_3p_3) = \alpha - (l_1p_1 + l_2p_2 + l_3p_3)$$

since $l_1, m_1, n_1;\ l_2, m_2, n_2;\ l_3, m_3, n_3$ are d.cs. of three mutually perpendicular lines.

$\therefore \quad d = \dfrac{\alpha - (l_1p_1 + l_2p_2 + l_3p_3)}{l_1 + l_2 + l_3}.$ Similarly other results can be found.

EXERCISE 5

1. Find the equation to the plane $x + 5y + 6z - 8 = 0$ when the translation of axes is made through the point $(-1, 2, -2)$.

2. Find the point to which the origin should be shifted, the direction of the axes remaining the same, such that the transformed equation of the surface

   $2x^2 + 3y^2 + 4z^2 - 4x - 12y - 24z + 99 = 0$ does not contain any term of the first degree.

3. With reference to $Oxyz$ frame the equation to a surface is $4x^2 + 2y^2 + 3z^2 + 4yz - 4zx = 0$. If the lines joining the origin to the points $(1,-2,2), (2,2,1), (-2,1,2)$ are taken as the axes, find the transformed equation with reference to the new frame.

4. With reference to $Oxyz$ frame the equation to a surface is

   $5x^2 - 16y^2 + 5z^2 + 8yz - 14zx + 8xy = 0$. If the lines through the origin with d.rs. $(-1,0,1)$, $(1,-4,1), (2,1,2)$ are taken as the axes, find the transformed equation w.r.t. the new frame.

5. With reference to $Oxyz$ frame $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{2};\ \dfrac{x}{2} = \dfrac{y}{-2} = \dfrac{z}{1};\ \dfrac{x}{2} = \dfrac{y}{1} = \dfrac{z}{-2}$ are three lines. With these three lines as axes find the transformed equation of $x^2 + 7y^2 + z^2 + 8yz + 16zx - 8xy = 9$.

6. $O_1 = (1,-2,4)$ w.r.t. $Oxyz$ as the frame of reference. The d.cs. $\overrightarrow{O_1X}, \overrightarrow{O_1Y}, \overrightarrow{O_1Z}$ are $\dfrac{1}{3}, \dfrac{2}{3}, \dfrac{2}{3};\ \dfrac{-14}{15}, \dfrac{2}{15}, \dfrac{1}{3};\ \dfrac{2}{15}, \dfrac{-11}{15}, \dfrac{2}{3}$ w.r.t. OXYZ as the frame of reference.

   If $2x - 3y - 2z + 5 = 0$ is the equation to a plane w.r.t. OXYZ frame, find the equation to the plane w.r.t. OXYZ frame.

7. If $Oxyz$, OXYZ are two frames of reference connected by the equations $x = \dfrac{X}{\sqrt{3}} + \dfrac{Y}{\sqrt{2}} + \dfrac{Z}{\sqrt{6}},\ y = \dfrac{X}{\sqrt{3}} - \dfrac{2Z}{\sqrt{6}},\ z = \dfrac{X}{\sqrt{3}} - \dfrac{Y}{\sqrt{2}} + \dfrac{Z}{\sqrt{6}}$, show that the equation $x + y + z = 0$ is transformed into $X = 0$. Hence show that the section of the surface $yz + zx + xy + 1 = 0$ by the plane $x + y + z = 0$ is a circle of radius $\sqrt{2}$.

8. If $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ are the d.cs. of three mutually perpendicular rays $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ show that the ray $\overrightarrow{OP}$ with d.rs. $l_1 + l_2 + l_3,\ m_1 + m_2 + m_3,\ n_1 + n_2 + n_3$ makes equal angles with $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$.

9. If $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ are d.cs. $(l_r\ m_r\ n_r \neq 0)$ of three mutually perpendicular lines, prove that

(*i*) $m_2 n_2 (m_3^2 - n_3^2) - m_3 n_3 (m_2^2 - n_2^2) = l_1 l_2 l_3.$

(*ii*) $\sum l_1 m_1 n_1 (m_2 n_3 - m_3 n_2) = l_1 l_2 l_3.$

(*iii*) $\dfrac{a}{l_1} + \dfrac{b}{m_1} + \dfrac{c}{n_1} = 0, \dfrac{a}{l_2} + \dfrac{b}{m_2} + \dfrac{c}{n_2} = 0 \Rightarrow \dfrac{a}{l_3} + \dfrac{b}{m_3} + \dfrac{c}{n_3} = 0$ and

$a : b : c = l_1\ l_2\ l_3 : m_1\ m_2\ m_3 : n_1\ n_2\ n_3$

ANSWERS

1. $x + 5y + 6z - 11 = 0$ **2.** $(1, 2, 3)$ **3.** $y^2 + 2z^2 = 0$

4. $2x^2 - 3z^2 = 0$ **5.** $x^2 + y^2 - z^2 = 1$ **6.** $8x + 36y - 33z - 15 = 0$

UNIT - III

6. **The Sphere**

Definition and equation of the sphere, Equation of the sphere through four non - coplanar points, Plane sections of sphere, Intersection of two spheres, Equation of a circle, sphere through a given circle, Intersection of a sphere and a line, Tangent planes, Plane of contact, Polar plane, Pole of a plane, Conjugate points, Conjugate planes, Angle of intersection of two spheres, Orthogonal spheres, Power of a point, Radical plane, Coaxal system of spheres, Simplified form of the equation to a coaxal system of spheres.

6

The Sphere

6. 1. Definition. The set of points in space which are at a constant distance $a (\geq 0)$ from a fixed point C is called a sphere.

In other words a sphere is the locus of the points in space which are at a constant distance $a (\geq 0)$ from a fixed point C. ***(N.U.06)***

C is called the *centre* and a is called the *radius* of the sphere. ***(N.U.06)***

If $a = 0$ the sphere is called a *point sphere*.

6. 2. EQUATION OF A SPHERE

Theorem. *Equation to the sphere with centre (x_1, y_1, z_1) and radius a is $(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = a^2$.*

Proof. Let C = centre = (x_1, y_1, z_1)

Let P = (x, y, z) be any point on the sphere.

By def. CP = radius = a .... (1) $\Leftrightarrow \sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2} = a$

$\Leftrightarrow (x-x_1)^2+(y-y_1)^2+(z-z_1)^2 = a^2$

$\Rightarrow$ Equation to the sphere is $(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = a^2$ .... (2)

having (x_1, y_1, z_1) as centre and a as radius.

Aliter. Let C = $\bar{c} = (x_1, y_1, z_1)$ be the centre of sphere Σ (Fig. 62).

a is the radius of Σ. Let $P = \bar{r} = (x, y, z) \in \Sigma$.

$P \in \Sigma$.

$\Leftrightarrow CP = |\overline{CP}| = a \Leftrightarrow |\bar{r} - \bar{c}| = a \Leftrightarrow (\bar{r} - \bar{c})^2 = a^2$ .... (1)

$\Leftrightarrow (\bar{r} - \bar{c}).(\bar{r} - \bar{c}) = a^2$

$\Leftrightarrow (x - x_1, y - y_1, z - z_1).(x - x_1, y - y_1, z - z_1) = a^2$

$\Leftrightarrow (x-x_1)^2+(y-y_1)^2+(z-z_1)^2 = a^2$ ..... (2)

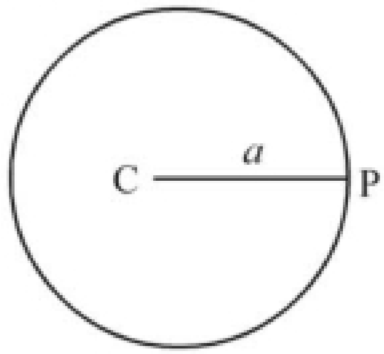


Fig. 62

$\therefore$ Equation to the sphere with centre (x_1, y_1, z_1) and radius a is

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = a^2$$

Note 1. The equation to the sphere with centre (0, 0, 0) and radius a is $x^2 + y^2 + z^2 = a^2$. ***(A. U. A12)***

2. The equation of the sphere with centre (x_1, y_1, z_1) and radius 0 is $(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = 0$.

3. Let $\bar{c} (\neq 0)$ be the centre of a sphere (Fig. 63) with non-zero radius a.

If A is a point on the sphere with position vector $\bar{c} + t\,\bar{b}$ where t is a real number, then from (1) we see that $\bar{c} - t\,\bar{b}$, say B is also a point on the sphere.

Further C is the mid point of AB. AB is called a diameter of the sphere and A, B are called the ends of the diameter AB. Since $t\,\bar{b}$ can have infinitely many values, a sphere will have infinitely many diameters.

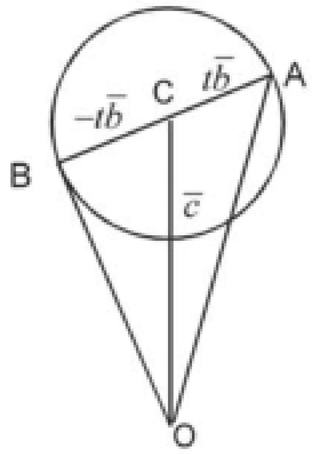


Fig. 63

4. For a sphere only one centre and one radius exist. Thus where centre and radius of a sphere are given, its equation is unique.

5. Equation (2) to the sphere Σ can be written as

$x^2 - 2x_1x + x_1^2 + y^2 - 2y_1y + y_1^2 + z^2 - 2z_1z + z_1^2 = a^2$

i.e. $x^2 + y^2 + z^2 + 2(-x_1)x + 2(-y_1)y + 2(-z_1)z + (x_1^2 + y_1^2 + z_1^2 - a^2) = 0$.

i.e., $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

where $x_1 = -u,\ y_1 = -v,\ z_1 = -w,\ d = x_1^2 + y_1^2 + z_1^2 - a^2 = u^2 + v^2 + w^2 - a^2$

i.e., $a^2 = u^2 + v^2 + w^2 - d$ *i.e.,* $u^2 + v^2 + w^2 - d \geq 0$ $(\because\ a \geq 0)$

$\therefore$ Equation to sphere is of the form

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \ldots(3)$$

where $(-u, -v, -w)$ is the centre and $\sqrt{u^2 + v^2 + w^2 - d}$ is the radius.

Equation (3) is taken as the *general equation of a sphere*. We denote :

$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

$S' \equiv x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$

If the sphere $S = 0$ passes through the origin, then its equation is of the form

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0$.

6. The equation to the point sphere is of the form

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

where $d = u^2 + v^2 + w^2$ $(\because\ a = 0)$

7. The equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

i.e., $(x+u)^2 + (y+v)^2 + (z+w)^2 = u^2 + v^2 + w^2 - d$

does not represent a sphere if $u^2 + v^2 + w^2 < d$.

In fact, there is no point which satisfies the above equation.

8. We may observe the following characteristics in the equation of a sphere :

(*i*) *It is a second degree equation in x, y, z.*

(*ii*) *The coefficients of* x^2, y^2, z^2 *are equal.*

(*iii*) *The product terms xy, yz, zx are absent.*

9. Concentric spheres. Spheres with the same centre are known as concentric spheres. If $S = 0$ is a sphere, then its concentric sphere is always $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + k = 0$ where k is an unknown constant.

Ex. 1. Equation of the sphere of radius 3 concentric with the sphere $x^2 + y^2 + z^2 - 2x - 2y - 2z = 1$ is $(x-1)^2 + (y-1)^2 + (z-1)^2 = 3^2$.
i.e. $x^2+y^2+z^2-2x-2y-2z+(-6)=0$ since (1, 1, 1) is the centre of the given sphere.

Ex. 2. Equation of the sphere concentric with the sphere $x^2 + y^2 + z^2 - 2x - 2y - 2z = 1$ and of double its surface area is $(x-1)^2 + (y-1)^2 + (z-1)^2 = 8$.

Since : Given sphere radius $= r = \sqrt{1+1+1+1} = 2$ and radius of required sphere = R. Here $4\pi R^2 = 2 \times 4\pi r^2 = 2 \times 4\pi \times 4 \Rightarrow R^2 = 8$.

6. 3. THEOREM. ***The equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$ represents a sphere if $a = b = c \neq 0, f = g = h = 0$ and $u^2 + v^2 + w^2 > ad$.***

Proof. If $a=b=c\neq 0, f=g=h=0$, then the equation becomes

$ax^2+ay^2+az^2+2ux+2vy+2wz+d=0$.

i.e., $x^2+y^2+z^2+2\left(\frac{u}{a}\right)x+2\left(\frac{v}{a}\right)y+2\left(\frac{w}{a}\right)z+\frac{d}{a}=0$,

Since $u^2+v^2+w^2>ad$, $\frac{u^2}{a^2}+\frac{v^2}{a^2}+\frac{w^2}{a^2}-\frac{d}{a}=\frac{u^2+v^2+w^2-ad}{a^2}>0$ *i.e.* $(\text{radius})^2 > 0$

$\therefore$ The given equation represents a sphere with centre $=\left(\frac{-u}{a},\frac{-v}{a},\frac{-w}{a}\right)$

and radius $=\frac{\sqrt{u^2+v^2+w^2-ad}}{a}$ if $a=b=c\neq 0, f=g=h=0$ and $u^2+v^2+w^2>ad$.

e.g. The equation $2x^2+2y^2+2z^2-3x+5y+2z+7=0$ does not represents a sphere since $\left(\frac{-3}{2}\right)^2+\left(\frac{5}{2}\right)^2+\left(\frac{2}{2}\right)^2=\frac{9+25+4}{4}=\frac{38}{4}\not> (2)(7)$. $(\because u^2+v^2+w^2 \not> ad)$

6. 4. ARBITRARY CONSTANTS OR PARAMETERS IN THE EQUATION OF A SPHERE.

Consider the equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$. It represents a sphere for a set of values of u, v, w, d such that $u^2 + v^2 + w^2 - d \geq 0$. Since there are four parameters in the equation, the equation to a sphere can be uniquely determined only if four conditions, such that each condition gives rise to one relation linear between the four parameters, are given. *In particular, we can have a unique sphere when four non-coplanar points on the sphere are given.* On the other hand if lesser number of conditions are given we can have infinitely many spheres satisfying the given conditions.

6. 5. Theorem. ***Equation to the sphere passing through four non-coplanar points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$, $D(x_4, y_4, z_4)$ is***

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0 \quad \text{... I} \qquad \textbf{(O.U. M 98)}$$

Proof. Let a sphere through A, B, C, D be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{...(1)}$$

Since (1) contains the points $A(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3), D(x_4, y_4, z_4)$ we have

$$x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0 \quad \text{...(2)}$$

$$x_2^2 + y_2^2 + z_2^2 + 2ux_2 + 2vy_2 + 2wz_2 + d = 0 \quad \text{...(3)}$$

$$x_3^2 + y_3^2 + z_3^2 + 2ux_3 + 2vy_3 + 2wz_3 + d = 0 \quad \text{...(4)}$$

$$x_4^2 + y_4^2 + z_4^2 + 2ux_4 + 2vy_4 + 2wz_4 + d = 0 \quad \text{...(5)}$$

Further A, B, C, D are non-coplanar. $\Rightarrow \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \neq 0$

Hence equations (2), (3), (4), (5) have a unique solution. For this solution i.e., for unique values of u, v, w, d there exists a unique sphere passing through the non-coplanar points. Its equation is obtained by eliminating u, v, w, d from the equations (1), (2), (3), (4), (5).

$\therefore$ Equation to the required sphere is I.

Note. 1. In numerical problems it is convenient to solve the equations (2), (3), (4), (5) for u, v, w, d and substitute the values in (1) to get the equation to the required sphere.

2. If $\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$ then A, B, C, D are coplanar and no sphere through A, B, C, D is possible

SOLVED PROBLEMS

Ex. 1. *Find t, if the radius of the sphere $x^2 + y^2 + z^2 + 6x - 8y - t = 0$ is 6.* **(A. U. A11)**

Sol. Radius $= \sqrt{9 + 16 + 0 + t} = 6$ (given). $\therefore 25 + t = 36 \Rightarrow t = 11$

Ex. 2. *Find the equation to the sphere through $O = (0, 0, 0)$ and making intercepts a, b, c on the axes.* **(O. U. A12, S. V. U. 08, A. N. U. M15, 90, 93, 04,07, S.K.U.99, K.U.01)**

Sol. Let the sphere through O intersect the axes at A, B, C.

$\therefore$ $A = (a, 0, 0), B = (0, b, 0), C = (0, 0, c)$.

Let the equation to the sphere through O, A, B, C be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{...(1)} \quad \therefore \quad d = 0$$

Also $a^2 + 2ua = 0$ i.e., $u = -\dfrac{a}{2}$. Similarly $v = -\dfrac{b}{2}; w = -\dfrac{c}{2}$.

$\therefore$ Equation to the sphere passing the origin and making intercepts a, b, c on the axes is $x^2 + y^2 + z^2 - ax - by - cz = 0$.

Its centre $= \left(\dfrac{a}{2}, \dfrac{b}{2}, \dfrac{c}{2}\right)$ and radius $= \dfrac{\sqrt{a^2 + b^2 + c^2}}{2}$

Note. 1. The above result may be taken as formula.

2. The above equation is the equation of the sphere passing through $(0,0,0),(a,0,0),(0,b,0)$ and $(0,0,c)$.

3. Find the euqation of the sphere through the points $(0,0,0),(1,0,0),(0,0,1),(0,1,0)$

(K. U. M 13)

Sol. Put $a=1, b=1, c=1$ in the above problem $x^2+y^2+z^2-x-y-z=0$.

Ex. 3. *Find the equation of the sphere circumscribing the tetrahedron formed by the planes* $\frac{y}{b}+\frac{z}{c}=0, \frac{z}{c}+\frac{x}{a}=0, \frac{x}{a}+\frac{y}{b}=0, \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. ***(O. U. 07)***

Sol. Given faces of the tetrahedron are $\frac{y}{b}+\frac{z}{c}=0$ ...(1) $\frac{z}{c}+\frac{x}{a}=0$ ...(2)

$\frac{x}{a}+\frac{y}{b}=0$ ...(3) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ...(4)

Solving (1), (2), (3) : O = (0, 0, 0). Solving (1), (2), (4) : A = $(a, b, -c)$.

Solving (1), (3), (4) : B = $(a, -b, c)$. Solving (2), (3), (4) : C = $(-a, b, c)$.

Let the sphere through O, A, B, C be

$$x^2+y^2+z^2+2ux+2vy+2wz=0 \quad ...(5)$$

$$A \in (5) \Rightarrow a^2+b^2+c^2+2ua+2vb-2wc=0 \quad ...(6)$$

$$B \in (5) \Rightarrow a^2+b^2+c^2+2ua-2vb+2wc=0 \quad ...(7)$$

$$C \in (5) \Rightarrow a^2+b^2+c^2-2ua+2vb+2wc=0 \quad ...(8)$$

$$(6)+(7) \Rightarrow 4ua=-2(a^2+b^2+c^2) \Rightarrow 2u=-\frac{a^2+b^2+c^2}{a}$$

Similarly $2v=-\frac{a^2+b^2+c^2}{b}$, $2w=-\frac{a^2+b^2+c^2}{c}$

Substituting these values in (5) we get

$$x^2+y^2+z^2-(a^2+b^2+c^2)\frac{x}{a}-(a^2+b^2+c^2)\frac{y}{b}-(a^2+b^2+c^2)\frac{z}{c}=0$$

i.e., $\frac{x^2+y^2+z^2}{a^2+b^2+c^2}-\frac{x}{a}-\frac{y}{b}-\frac{z}{c}=0$ is the equation to the required sphere.

Ex. 4. *A point is such that the sum of the squares of its distances from the six faces of a cube is a constant k (>0). Show that the point lies on a sphere.*

Sol. Let a be the edge of the cube (Fig. 64). Let Oxyz be the frame as shown. (Fig. 63). Equations to the six faces are $y=0$, $x=a$, $x=0$, $y=a$, $z=0$, $z=a$.

Let $P(x_1, y_1, z_1)$ be a point such that the sum of the squares of its distance p from the faces is k.

$$\therefore x_1^2+(x_1-a)^2+y_1^2+(y_1-a)^2+z_1^2+(z_1-a)^2=k$$

i.e., $x_1^2+y_1^2+z_1^2-ax_1-ay_1-az_1=\frac{1}{2}(k-3a^2)$

$\therefore$ $P(x_1, y_1, z_1)$ lies on the sphere.

$$x^2+y^2+z^2-a(x+y+z)+\frac{1}{2}(3a^2-k)=0$$

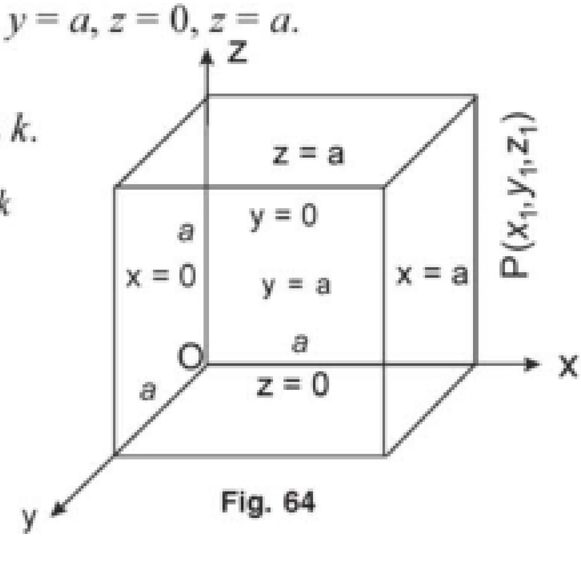


Fig. 64

Ex. 5. *A plane passes through a fixed point (a, b, c) and intersects the axes in A, B, C. Show that the centre of the sphere OABC lies on* $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$.

(N.U. 07, A.U.98, N.U.97, S.K.U.96, O.U. Oct. 01, S.V. U. 98, O. U. A 88, A.K.U M18)

Sol. let the sphere through O, A, B, C be $x^2+y^2+z^2+2ux+2vy+2wz=0$ ...(1)

Since A $\in x$ axis, from (1), $x=-2u$ $\therefore$ A = $(-2u, 0, 0)$.

similarly B = $(0, -2v, 0)$ and C = $(0, 0, -2w)$

$\therefore$ Equation to the plane $\overleftrightarrow{ABC}$ is $\frac{x}{-2u}+\frac{y}{-2v}+\frac{z}{-2w}=1$

Since this plane passes through the point (a, b, c), we have

$$\frac{a}{-2u}+\frac{b}{-2v}+\frac{b}{-2w}=1 \Rightarrow \frac{a}{-u}+\frac{b}{-v}+\frac{c}{-w}=2$$

$\therefore$ The centre $(-u, -v, -w)$ of the sphere OABC lies on $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$.

Ex. 6. *A sphere of constant radius r passes through the origin O and cuts the axes in A, B, C. Prove that the foot of the perpendicular from O to the plane* $\overleftrightarrow{ABC}$ *lies on* $(x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2})=4r^2$. **(S.V. U. O 97, S.V.M M18)**

Sol. Let the plane $\overleftrightarrow{ABC}$ be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ...(1)

$\therefore$ A = $(a, 0, 0)$, B = $(0, b, 0)$, C = $(0, 0, c)$.

$\therefore$ Equation to the sphere through O, A, B, C is

$$x^2+y^2+z^2-ax-by-cz=0 \quad \text{...(2)}$$

Since radius of (2) is r, $\frac{a^2}{4}+\frac{b^2}{4}+\frac{c^2}{4}=r^2 \Rightarrow a^2+b^2+c^2=4r^2$ ...(3)

Equation to the line through O and perpendicular to the plane (1) are

$$\frac{x-0}{1/a}=\frac{y-0}{1/b}=\frac{z-0}{1/c} \ (=\lambda \text{ say}) \quad \text{...(4)}$$

$\therefore$ A point on (4) is $(\lambda/a, \lambda/b, \lambda/c)$.

$P(x_1, y_1, z_1)$ is the foot of the perpendicular from O to $\overleftrightarrow{ABC}$.

$\Rightarrow$ $x_1=\frac{\lambda}{a}, y_1=\frac{\lambda}{b}, z_1=\frac{\lambda}{c}; \frac{x_1}{a}+\frac{y_1}{b}+\frac{z_1}{c}=1$ from (1)

$\Rightarrow$ $a=\frac{\lambda}{x_1}, b=\frac{\lambda}{y_1}, c=\frac{\lambda}{z_1}; \frac{x_1}{a}+\frac{y_1}{b}+\frac{z_1}{c}=1$

$\Rightarrow$ $\lambda^2\left(\frac{1}{x_1^2}+\frac{1}{y_1^2}+\frac{1}{z_1^2}\right)=4r^2$ from (3), $\frac{x_1^2}{\lambda}+\frac{y_1^2}{\lambda}+\frac{z_1^2}{\lambda}=1$

$\Rightarrow (x_1^2+y_1^2+z_1^2)^2(x_1^{-2}+y_1^{-2}+z_1^{-2})=4r^2$

$\therefore$ Foot of the perpendicular from O to the plane $\overleftrightarrow{ABC}$ lies on

$(x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2})=4r^2$.

Ex. 7. *A sphere of constant radius k passes through the origin and intersects the axes in A, B, C. Prove that the centroid of the ΔABC lies on the sphere $9(x^2+y^2+z^2) = 4k^2$.* **(A. N. U. A12, M 13; S. V. U. A11, M 13; O. U. 82; A.U.99)**

Sol. Let OA = a, OB = b, OC = c

$\Rightarrow$ Coordinates of A, B, C are $(a, 0, 0), (0, b, 0), (0, 0, c)$

$\Rightarrow$ The equation of the sphere OABC is $x^2 + y^2 + z^2 - ax - by - cz = 0$ ...(1)

The radius of the sphere (1) is $k = \sqrt{\frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4}} \Rightarrow a^2 + b^2 + c^2 = 4k^2$ ... (2)

Let (x_1, y_1, z_1) be the centroid of the Δ ABC.

Then $$\left.\begin{aligned} x_1 &= \frac{a+0+0}{3} \Rightarrow a = 3x_1, \\ y_1 &= \frac{0+b+0}{3} \Rightarrow b = 3y_1, \\ z_1 &= \frac{0+0+c}{3} \Rightarrow c = 3z_1 \end{aligned}\right\} \quad \text{...(3)}$$

Substituting values of a, b, c from (3) in (2) we get, $9{x_1}^2 + 9{y_1}^2 + 9{z_1}^2 = 4k^2$.

$\Rightarrow$ The locus of (x_1, y_1, z_1) is $9(x^2 + y^2 + z^2) = 4k^2$ ..... (4)

$\Rightarrow$ The centroid of the Δ ABC lies on the sphere (4).

EXERCISE 6 (*a*)

1. Find the centre and radius of the sphere

(*i*) $x^2 + y^2 + z^2 - 6x + 2y - 4z + 14 = 0$

(*ii*) $2x^2 + 2y^2 + 2z^2 - 2x + 4y + 2z + 1 = 0$ ***(V.S.P.U M18)***

(*iii*) $x^2 + y^2 + z^2 - 6x + 8y - 10z + 1 = 0$ ***(A. U. M13)***

2. Find the equation of the sphere through the points (non-coplanar)

(*i*) $(0, 0, 0), (0, 1, -1), (-1, 2, 0), (1, 2, 3)$

(*ii*) $(4, -1, 2), (0, -2, 3), (1, 5, -1), (2, 0, 1)$

(*iii*) $(0, 0, 0), (-a, b, c), (a, -b, c), (a, b, -c)$ ***(A. U. M13)***

3. (*i*) Is there a sphere passing through the points $(4, 0, 1), (10, -4, 9), (-5, 6, -11), (1, 2, 3)$.

(*ii*) Find the equation of the sphere of radius 3 concentric with the sphere $x^2 + y^2 + z^2 - 2x - 2y - 2z = 1$. ***(N. U. S. 98)***

(*iii*) Find the equation of the sphere concentric with the sphere $x^2 + y^2 + z^2 + 3x - 5y + 7z - 11 = 0$ and also passing through the centre of the sphere $x^2 + y^2 + z^2 - 2x - 3y + 5z - 7 = 0$.

4. (*i*) Find the equation of the sphere circumscribing the tetrahedron whose faces are $x = 0, y = 0, z = 0$, and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. ***(K.U.Model Paper, S. V. U 93)***

(*ii*) A sphere is circumscribing the tetrahedron with vertices $(-4, 2, 4), (3, -6, 6), (1, -2, 11), (2, 2, 1)$. Show that its equation is $x^2 + y^2 + z^2 - x + 2y - 11z = 0$.

5. Find the equation of the sphere having its centre on the line $5y + 2z = 0 = 2x - 3y$ and passing through the points $(2, -1, -1)$, $(0, -2, -4)$. ***(O. U. M 13, A.U M18)***

6. A point is such that the sum of the squares of its distances from the planes $x + y + z = 0, x - 2y + z = 0, x - z = 0$ is k^2, a constant. Show that the point lies on a sphere.

7. A point is such that the ratio of its distances from two fixed points is k^2, a constant. Show that the point lies on a sphere.

8. Find the equation of the sphere through the points $(1, -4, 3)$, $(1, -5, 2)$, $(1, -3, 0)$ and whose centre lies on the plane $x + y + z = 0$. ***(A.U.2001, O. U. O 97, K.U M18)***

9. Find the equation of the sphere through the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and having the least radius. ***(S. V. U. O 98)***

10. A plane passes through a fixed point (a,b,c) and cuts the axes at A, B, C. Show that the locus of the centre of the sphere OABC ic $(a/x)+(b/y)+(c/z)=2$ ***(A. N. U. M 13)***

11. A plane passes through a fixed point (a, b, c). Show that the foot of the perpendicular to the plane from the origin lies on the sphere $x^2 + y^2 + z^2 - ax - by - cz = 0$. ***(O.U.96, N. U. A 88)***

12. A sphere of constant radius $2k$ passes through the origin and intersects the axes in A, B, C. Prove that the locus of the centroid of the tetrahedron O ABC is the sphere $x^2 + y^2 + z^2 = k^2$. ***(O. U. A11)***

13. A variable sphere passes through the origin O and intersects the axes in A, B, C so that the volume of the tetrahedron is $4\sqrt{7}/3$ cubic units. Show that the centre of the sphere lies on $x^2y^2z^2 = 7$.

ANSWERS

1. (*i*) $(3,-1,2),0$ (*ii*) $\left(\frac{1}{2},-1,-\frac{1}{2}\right),1$

2. (*i*) $7(x^2+y^2+z^2)-15x-25y-11z=0$ (*ii*) $x^2+y^2+z^2-4x-14y-22z+25=0$
(*iii*) $x^2+y^2+z^2-(a^2+b^2+c^2)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0$

3. (*i*) No (*ii*) $x^2+y^2+z^2-2x-2y-2z=6$ (*iii*) $x^2+y^2+z^2+3x-5y+7z+(25/2)=0$

4. (*i*) $x^2+y^2+z^2-ax-by-cz=0$ (*ii*) $12(x^2+y^2+z^2)=174x+116y+87z$

5. $x^2+y^2+z^2-6x-4y+10z+12=0$ 8. $x^2+y^2+z^2-4x+7y-3z+15=0$

9. $3(x^2+y^2+z^2)-2(x+y+z)=1$

6. 6. PLANE SECTIONS OF A SPHERE

Definition. *If ξ is a sphere and π is a plane, the non-empty set of points common to the sphere ξ and the plane π is called a plane section of the sphere. Then we say that the plane π intersects the sphere ξ.*

We, therefore, have :

P is a point on the plane section of π with ξ

$\Leftrightarrow$ $P \in \pi \cap \xi$ $\Leftrightarrow$ $P \in \pi$ and $P \in \xi$

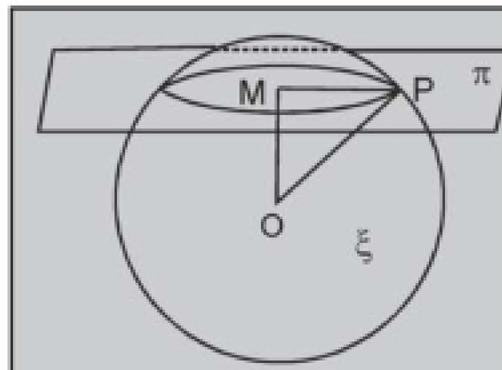


Fig. 65

Theorem. ***A plane section of a sphere of radius a (> 0) is a circle.***

Proof. (Fig 65). Let ξ be a sphere with centre O, Radius of ξ is a (> 0).

Let π be a plane making a plane section on ξ.

Let M be the foot of the perpendicular from O to the plane π so that M and O are different.

Case (*i*). Let $M \neq P$. Then $OM \perp MP$.

Since $\angle OMP = 90^0$, $MP^2 = OP^2 - OM^2 = a - OM^2$.

Now O and M are fixed points and hence OM is fixed.

$\therefore$ MP is a constant for all $P \in \xi \cap \pi$.

$\therefore$ The plane section is a circle with centre M and radius $\sqrt{a^2 - OM^2}$.

Case (*ii*). Let M = P. The case is trivial and the plane section is a point circle.

If M = O, then MP = OP = a (constant)

i.e., if the plane π passes through the centre of the sphere, the plane section is a circle with centre O and radius a.

6. 7. GREAT CIRCLE, SMALL CIRCLE

Definition. *If a plane π passes through the centre of a sphere ξ, then the plane section of the sphere is called a great circle.* ***(N. U. 07)***

The centre and radius of the great circle are respectively the centre and radius of the sphere.

Definition. *If a plane π does not pass through the centre of a sphere ξ and intersects the sphere ξ, then the plane section is called a small circle.*

The centre of the small circle is the foot of the perpendicular from the centre of the sphere to the plane π and the radius of the small circle is less than the radius of the sphere ξ.

Note. One and only one circle passes through three non-collinear points. The circle through three given points lies entirely on any sphere through the same three points. Thus if a circle lies on a sphere, then the sphere passes through any three points on the circle.

6. 8. CONDITION FOR A PLANE TO INTERSECT A SPHERE

There exist points of intersection of the sphere and a plane if and only if the distance of the centre of the sphere from the plane is less than or equal to the radius of the sphere.

Thus : the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ will intersect the plane $lx + my + nz = p$ if and only if

$$\left| \frac{l(-u) + m(-v) + n(-w) - p}{\sqrt{l^2 + m^2 + n^2}} \right| \leq \sqrt{u^2 + v^2 + w^2 - d}$$

i.e. if and only if $(lu + mv + nw + p)^2 \leq (l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d)$.

6. 9. INTERSECTION OF TWO SPHERES (FIG. 66).

Let ξ_1, ξ_2 be two spheres. If $\xi_1 \cap \xi_2 \neq \phi$, we say that the spheres ξ_1 and ξ_2 intersect or cut and ξ_1, ξ_2 are called intersecting spheres. If $\xi_1 \cap \xi_2 = \phi$, we say that the sphere ξ_1 and ξ_2 do not intersect or cut and ξ_1, ξ_2 are called non-intersecting spheres.

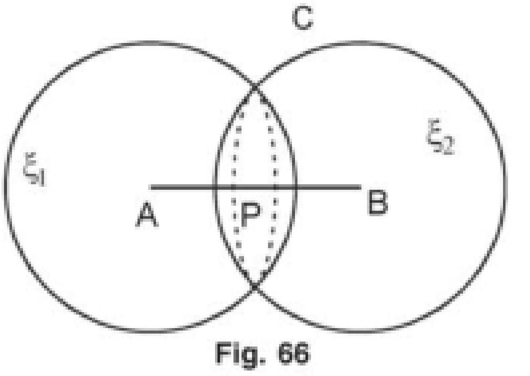


Fig. 66

Theorem. ***If two spheres intersect, then the locus of the set of points of intersection is a circle.***

Proof. Let S = 0, S' = 0 be two intersecting spheres where

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0,$$
$$S' \equiv x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$$
$$S - S' \equiv 2(u - u')x + 2(v - v')y + 2(w - w')z + (d - d') = 0$$

Clearly S − S' = 0, being of first degree, is a plane. Also the coordinates of the points common to two spheres satisfy both S = 0, S' = 0 and therefore S − S' = 0.

Thus the points common to two spheres are the same as those any one of them and this plane and, therefore, they determine a circle.

Note 1. S = 0, S' = 0 are two spheres intersecting in a circle C. Then equations to C are S = 0, S − S' = 0 or S' = 0, S − S' = 0.

2. U = 0 is a plane and S = 0 is a sphere such that the plane section of the sphere is the circle C. Then the equations S = 0, U = 0 together represent the circle C.

The equations $x^2 + y^2 + 2gx + 2fy + c = 0, z = 0$ represent a circle.

3. A, B are the centres of two intersecting spheres.

Let C be the circle (not a point circle) with centre P. Then (*i*) the line $\overleftrightarrow{AB}$ is perpendicular to the plane of the circle C and (*ii*) the line $\overleftrightarrow{AB}$ passes through the centre P i.e. A, P, B are collinear.

SOLVED PROBLEMS

Ex. 1. *Find the radius of the circle* $x^2 + y^2 + z^2 - 2x + 4y - 6z - 2 = 0, z = 0$.

Sol. For the sphere : centre = (1, −2, 3) and radius $= \sqrt{(1 + 4 + 9 + 2)} = 4$

Distance of (1, −2, 3) from the plane $z = 0$ is 3.

$\therefore$ radius of the circle $= +\sqrt{16 - 9} = +\sqrt{7}$.

Ex. 2. *Prove that the circle* $x^2 + y^2 + z^2 - 4x - 2y + 5z + 6 = 0, x + y + 2z + 2 = 0$ *is a great circle.*

Sol. Centre of the sphere (2, 1, −5/2) lies on the plane $x + y + 2z + 2 = 0$.

Ex. 3. *Are there points of intersection of the sphere* $x^2 + y^2 + z^2 + 3x + 5y - 2z + 9 = 0$ *with the plane* $x - y + 3z + 6 = 0$ *?*

Sol. For the sphere : centre $= \left(-\frac{3}{2}, \frac{-5}{2}, 1\right)$ and radius $= \sqrt{\frac{9}{4} + \frac{25}{4} + 1 - 9} = \frac{1}{\sqrt{2}}$

Distance of the centre from the plane $x - y + 3z + 6 = 0$

$$= \frac{-\frac{3}{2} - \left(-\frac{5}{2}\right) + 3\,(1) + 6}{\sqrt{\frac{9}{4} + \frac{25}{4} + 1}} = \frac{20}{\sqrt{38}} > \text{radius} \left(= \frac{1}{\sqrt{2}}\right)$$

$\therefore$ There are no points of intersection of the sphere with the plane (i.e. the sphere is not intersected by the plane).

Ex. 4. *If the spheres* $x^2 + y^2 + z^2 - 2x - 4y - 11 = 0$ ...(1)
$x^2 + y^2 + z^2 + 2x - y + 12z + 5 = 0$ ...(2) *intersect in a circle, find its equation.*

Sol. The plane of intersection of the spheres is $-4x - 3y - 12z - 16 = 0$

i.e. $4x + 3y + 12z + 16 = 0$ ...(3)

$\therefore$ Equation to the circle of intersection of (1) and (2) is (1), (3) or (2), (3).

6. 10. Theorem. ***If AB is a diameter of a sphere ξ with centre O, then $P\,(\neq A, \neq B) \in \xi \Rightarrow \angle APB = 90^o$.***

Proof. (Fig. 67). Let O = (0, 0, 0), P = (x_1, y_1, z_1),
A = (x_2, y_2, z_2).
Since O is the mid point of AB, B = $(-x_2, -y_2, z_2)$
$\therefore$ $\overline{\text{P}} = (x_1, y_1, z_1)$, $\overline{\text{A}} = (x_2, y_2, z_2)$,
Dr's of $\overline{\text{PA}} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$
Dr's of $\overline{\text{PB}} = (-x_2 - x_1, -y_2 - y_1, -z_2 - z_1)$
Also radius of the sphere $|\overline{\text{OP}}| = |\overline{\text{OA}}|$.

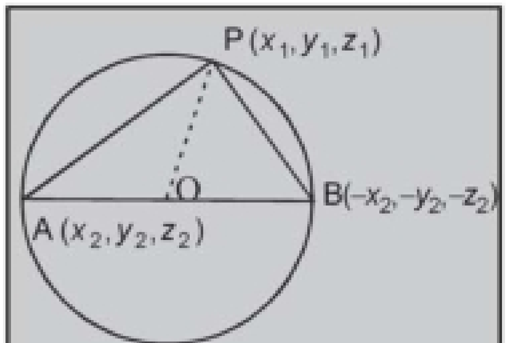


Fig. 67

$$= x_1^2 + y_1^2 + z_1^2 = x_2^2 + y_2^2 + z_2^2$$

Consider $(x_2 - x_1)(-x_2 - x_1) + (y_2 - y_1)(-y_2 - y_1) + (z_2 - z_1)(-z_2 - z_1)$

$$= -(x_2^2 + y_2^2 + z_2^2) + (x_1^2 + y_1^2 + z_1^2) = 0. \qquad \therefore \angle \text{APB} = 90^0.$$

Note. If $\overleftrightarrow{\text{PAB}}$ is the plane section of ξ then the plane section is a great circle and hence $\angle$ APB = 90^0 from the properties of circles.

6.11.Theorem. ***Equation to the sphere having A = (x_1, y_1, z_1) and B = (x_2, y_2, z_2) as the ends of the diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$***

(*N. U. 89*)

First Proof. Let ξ be the sphere with AB as diameter.

Let P = $(x, y, z) \in \xi$. Given A = (x_1, y_1, z_1), B = (x_2, y_2, z_2).

If P $\neq$ A, and P $\neq$ B then $\angle$ APB = 90^0

The Dc's of PA, PB are proportional to $(x - x_1, y - y_1, z - z_1)$ and $(x - x_2, y - y_2, z - z_2)$ respectively.

PA is perpendicular to PB

$\Leftrightarrow$ $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$

$\therefore$ Equation to the required sphere is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0.$$

Second Proof. Let ξ be the sphere with AB's diameter.

Let $P \in \xi$. Let $P = \bar{r} = (x, y, z)$.

Given $A = \bar{a} = (x_1, y_1, z_1)$ and $B = \bar{b} = (x_2, y_2, z_2)$.

If $P \neq A$ and $P \neq B$ then $\angle APB = 90^0$ and hence $\overline{PA}.\overline{PB}=0$

If $P = A$, then $\overline{PA} = 0 \Rightarrow \overline{PA}.\overline{PB} = 0$ and If $P = B$, then $\overline{PB} = 0 \Rightarrow \overline{PA}.\overline{PB} = 0$

$P \in \xi \Leftrightarrow \overline{PA}.\overline{PB} = 0 \Rightarrow \overline{AP}.\overline{BP} = 0 \Rightarrow (\bar{r}-\bar{a}).(\bar{r}-\bar{b}) = 0$

$\Leftrightarrow (x-x_1, y-y_1, z-z_1).(x-x_2, y-y_2, z-z_2) = 0$

$\Leftrightarrow (x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$

$\therefore$ Equation to the required sphere is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

e.g. Equation of the sphere having the segment joining $(2, 3, -1)$ and $(1, -2, -1)$ as a diameter is $(x-2)(x-1)+(y-3)(y+2)+(z+1)(z+1)=0$ ***(N. U. M 98)***

Note. Vector equation of the sphere having the points $\bar{a}, \bar{b}$ as the ends of a diameter is $(\bar{r}-\bar{a}).(\bar{r}-\bar{b}) = 0$.

SPHERE THROUGH A GIVEN CIRCLE.

6.12. Theorem. ***If the plane U = 0 intersects the sphere S = 0, in a circle C, then for all real values of λ, S + λ U = 0 represents the equation to a sphere passing through the circle C.***

Proof. Let $S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0$ be the sphere and

$U \equiv lx+my+nz-p=0$ be the plane.

For the sphere, centre $= (-u, -v, -w)$ and radius $= \sqrt{u^2+v^2+w^2-d}$

Plane U = 0 intersects the sphere S = 0

$\Leftrightarrow$ Distance of the centre from the plane $\leq$ radius

$$\Leftrightarrow \left| \frac{l(-u)+m(-v)+n(-w)-p}{\sqrt{l^2+m^2+n^2}} \right| \leq \sqrt{u^2+v^2+w^2-d}$$

$$\Leftrightarrow (lu+mv+nw+p)^2 \leq (l^2+m^2+n^2)(u^2+v^2+w^2-d) \qquad ...(1)$$

$S+\lambda U \equiv x^2+y^2+z^2+2ux+2vy+2wz+d+\lambda(lx+my+nz-p)$

$$\equiv x^2+y^2+z^2+2\left(u+\frac{\lambda l}{2}\right)x+2\left(v+\frac{\lambda m}{2}\right)y+2\left(w+\frac{\lambda n}{2}\right)z+(d-\lambda p)=0$$

Now $\left(u+\frac{\lambda l}{2}\right)^2+\left(v+\frac{\lambda m}{2}\right)^2\left(w+\frac{\lambda n}{2}\right)^2-(d-\lambda p)$

$=\frac{1}{4}(l^2+m^2+n^2)\lambda^2+(lu+mv+nw+p)\lambda+(u^2+v^2+w^2-d) \geq 0$, for all real values of λ.

Since from (1), $(lu+mv+nw+p)^2-4.\frac{1}{4}(l^2+m^2+n^2)(u^2+v^2+w^2-d) \leq 0$

$\therefore$ $S+\lambda U = 0$ represents a sphere through the circle C, for all real values of λ.

$\Leftrightarrow$ Plane U = 0 intersects the sphere S = 0 in a circle C.

Note 1. If S = 0, S' = 0 are two distinct intersecting spheres, then $\lambda_1 S + \lambda_2 S' = 0$ (for real values of λ_1, λ_2 and $\lambda_1 + \lambda_2 \neq 0$) represents a system of spheres passing through the circle of intersection of the spheres S = 0, S' = 0.

Also $S + \lambda(S - S') = 0$, λ being an arbitrary constant, represents a sphere through the circle of intersection of the spheres S = 0, S' = 0.

2. If the equation $x^2 + y^2 + 2gx + 2fy + c = 0, z = 0$ represents a circle then the equation to any sphere through the circle is $x^2 + y^2 + z^2 + 2gx + 2fy + kz + c = 0, k$ being a parameter.

SOLVED PROBLEMS

Ex. 1. *The plane of equation* $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ *meets the axes in A, B, C. Find the equation of the circumcircle of* ΔABC *and hence find its centre.* **(N. U. O 88)**

Sol. Given plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ ...(1) meets the axes in A, B, C.

$\therefore$ A = $(a, 0, 0)$, B = $(0, b, 0)$, C = $(0, 0, c)$.

Let $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ be the equation of the sphere through O, A, B, C.

$\therefore$ $d = 0$, $a^2 + 2ua = 0$ i.e. $2u = -a$

Similarly $2v = -b$, $2w = -c$.

$\therefore$ Equation to the sphere through O, A, B, C is

$$x^2 + y^2 + z^2 - ax - by - cz = 0 \qquad ...(2)$$

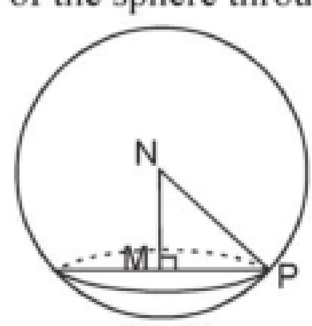


Fig. 68

Since A, B, C are common points to (1) and (2) and (1) is a plane intersecting (2), through A, B, C a circle passes and it is the circumcircle of Δ ABC.

Let N be the centre of the sphere and M be the centre of the circle (Fig. 67).

$\therefore$ $\overleftrightarrow{MN} \perp \overleftrightarrow{ABC}$ $\qquad \therefore N = \left(\dfrac{a}{2}, \dfrac{b}{2}, \dfrac{c}{2}\right)$ and d.rs. of $\overleftrightarrow{MN}$ are $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$

$\therefore$ Equation to $\overleftrightarrow{MN}$ are $\dfrac{x - a/2}{1/a} = \dfrac{y - b/2}{1/b} - \dfrac{z - c/2}{1/c}$ (= t say).

Let $M = \left(\dfrac{t}{a} + \dfrac{a}{2}, \dfrac{t}{b} + \dfrac{b}{2}, \dfrac{t}{c} + \dfrac{c}{2}\right)$

$\therefore$ $M \in \overleftrightarrow{ABC} \Rightarrow \dfrac{t}{a^2} + \dfrac{1}{2} + \dfrac{t}{b^2} + \dfrac{1}{2} + \dfrac{t}{c^2} + \dfrac{1}{2} = 1 \Rightarrow t = \dfrac{-1}{2(a^{-2} + b^{-2} + c^{-2})}$

Equations to circumcircle are (2) and (1) and the centre

$$M = \left(\frac{a}{2} + \frac{t}{a}, \frac{b}{2} + \frac{t}{b}, \frac{c}{2} + \frac{t}{c}\right) \text{ where } t = \frac{-1}{2(a^{-2} + b^{-2} + c^{-2})}$$

Ex. 2. *Show that the four points* $(-8, 5, 2)$, $(-5, 2, 2)$, $(-7, 6, 6)$, $(-4, 3, 6)$ *are concyclic.* **(A. N. U. M15, O. U. 07)**

Sol. Let A = $(-8, 5, 2)$, B = $(-5, 2, 2)$, C = $(-7, 6, 6)$, D = $(-4, 3, 6)$

Let l, m, n be d.rs. of normal to the plane $\overleftrightarrow{ABC}$.

$\therefore \quad \left.\begin{array}{l} 3l-3m+0n=0 \\ l+m+4n=0 \end{array}\right\} \frac{l}{-12}=\frac{m}{-12}=\frac{n}{6} \Rightarrow \frac{l}{2}=\frac{m}{2}=\frac{n}{-1}$

$\therefore$ Equation to $\overleftrightarrow{ABC}$ is $2(x+8)+2(y-5)-1(z-2)=0$

i.e. $2x+2y-z+8=0$ ...(1)

Let the equation of the sphere ξ through O, A, B, C be

$$x^2+y^2+z^2+2ux+2vy+2wz=0 .$$

$\therefore$ $A \in \xi \Rightarrow 64+25+4-16u+10v+4w=0$

$\Rightarrow -16u+10v+4w=-93$ ...(2)

$B \in \xi \Rightarrow 25+4+4-10u+4v+4w=0$

$\Rightarrow -10u+4v+4w=-33$ ...(3)

$C \in \xi \Rightarrow 49+36+36-14u+12v+12w=0$

$\Rightarrow -14u+12v+12w=-121$ ...(4)

$(2)-(3): -6u+6v=-60 \Rightarrow u-v=10$ ...(5)

$-3 \times (3)+(4): 16u=-22 \Rightarrow u=-11/8$

$\therefore$ From (5), $\frac{-11}{8}-v=10 \Rightarrow v=\frac{-91}{8}$. From (3), $\frac{110}{8}-\frac{364}{8}+4w=-33$

$\Rightarrow -254+32w=-264 \Rightarrow 32w=-10 \Rightarrow w=-5/16$

$\therefore$ Equation to the sphere ξ is $x^2+y^2+z^2-\frac{22}{8}x-\frac{182}{8}y-\frac{5}{8}z=0$

$\Rightarrow 8(x^2+y^2+z^2)-22x-182y-5z=0$ ...(6)

Now D = (– 4, 3, 6) satisfies (1) and (6).

$[\because \; -8,+6-6+8=0,\; 8(16+9+36)+88-546-30=0]$

$\therefore$ D is the concyclic with the points A, B, C and the equation to the circle is given by (1) and (6).

Ex. 3. *Find the equations of the spheres passing through the circle $x^2+y^2=4$, $z=0$ and is intersected by the plane $x+2y+2z=0$ in a circle of radius 3.*

(A. U. A11, 12, K.U. S. V. U. A 97, S. K. U. O 01, S.K.U M18)

Sol. Let the required sphere be ξ (Fig. 69)

Let its equation be $x^2+y^2+z^2-4+\lambda z=0$

i.e. $x^2+y^2+z^2+\lambda z-4=0$

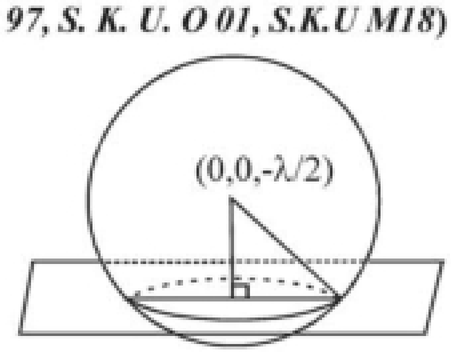


Fig. 69

Centre of ξ is $(0, 0, -\lambda/2)$ and radius of ξ is $\sqrt{\frac{\lambda^2}{4}+4}$.

Given plane is $x+2y+2z=0$.

$\therefore$ Perpendicular distance of the centre of the sphere from the plane

$=\left|\frac{0+0-2 \,.\, \lambda/2}{\sqrt{1+4+4}}\right|=\frac{|\lambda|}{3}$ $\qquad \therefore \frac{\lambda^2}{9}+9=\frac{\lambda^2}{4}+4 \Rightarrow \lambda=\pm 6$

$\therefore$ Equation to the required spheres are $x^2+y^2+z^2 \pm 6z-4=0$.

Ex. 4. *Prove that the plane $x+2y-z=4$ intersects the sphere $x^2+y^2+z^2-x+z-2=0$ in a circle of radius unity. Also find the equation of the sphere which has this circle for one of the great circles.* ***(S.V.U., Meerat 97, Auadh 99, 01)***

Sol. Given plane is $x + 2y - z = 4$ ...(1)

and given sphere is $x^2 + y^2 + z^2 - x + z - 2 = 0$ ...(2)

Let C be the centre and a be the radius of (2)

$\therefore \quad C = \left(\frac{1}{2},\ 0,\ -\frac{1}{2}\right)$ and $a = \sqrt{\left(\frac{1}{4} + 0 + \frac{1}{4} + 2\right)} = \sqrt{(5/2)}$

Let M be the centre of the circle given by (1) and (2).

$$\therefore \quad CM = \left| \frac{\frac{1}{2} + 2.0 + \frac{1}{2} - 4}{\sqrt{(1+4+1)}} \right| = \frac{3}{\sqrt{6}}.$$

Let r be the radius of the circle. $\quad \therefore CM^2 + r^2 = a^2 \Rightarrow \frac{9}{6} + r^2 = \frac{5}{2} \Rightarrow r = 1$

Let a sphere through the circle given by (1) and (2) be

$$x^2 + y^2 + z^2 - x + z - 2 + \lambda(x + 2y - z - 4) = 0$$

i.e. $\quad x^2 + y^2 + z^2 + (\lambda - 1)x + 2\lambda y\ (1 - \lambda)z - 2 - 4\lambda = 0$ ..(3)

Centre of (3) is $\left(-\frac{\lambda - 1}{2},\ -\lambda,\ -\frac{1-\lambda}{2}\right)$

If the circle is a great circle then the centre of (3) must lie on (1).

$\therefore \quad -\frac{\lambda - 1}{2} - 2\lambda + \frac{1 - \lambda}{2} = 4 \Rightarrow \lambda = -1$

$\therefore$ Equation to the required sphere is (from (3))

$$x^2 + y^2 + z^2 - 2x - 2y + 2z + 2 = 0 \quad ...(4)$$

Note. Centre of the great circle = centre of (4) = $(1,\ 1,\ -1)$ and radius of great circle = radius of (4) = $\sqrt{(1 + 1 + 1 - 2)} = 1$

Ex. 5. *Show that the two circles* $x^2 + y^2 + z^2 - y + 2z = 0, x - y + z = 2$ ...(1) $x^2 + y^2 + z^2 + x - 3y + z - 5 = 0,\ 2x - y + 4z - 1 = 0$ ...(2) *lie on the same sphere and find its equation.*

(*S. K. U. A11; O. U. 07; A.U.01; A. N.U.M 98, 04, A11, M 14; S. V. U A 02, M06, A.K.N.U M18*)

Sol. A sphere through the circle (1) is $x^2 + y^2 + z^2 - y + 2z + \lambda(x - y + z - 2) = 0$

i.e. $\quad x^2 + y^2 + z^2 + \lambda x - (1 + \lambda)y + (\lambda + 2)z - 2\lambda = 0$ ...(3)

A sphere through the circle (2) is

$$x^2 + y^2 + z^2 + x - 3y + z - 5 + \mu(2x - y + 4z - 1) = 0$$

i.e. $\quad x^2 + y^2 + z^2 + (2\mu + 1)x - (3 + \mu)y + (4\mu + 1)z - 5 - \mu = 0$ ...(4)

But circles (1) and (2) lie on the same sphere.

(3) and (4) represent the same sphere

$$\Rightarrow \left.\begin{array}{ll} \lambda = 2\mu + 1 & i.e. \quad \lambda - 2\mu - 1 = 0 \\ \lambda + 1 = 3 + \mu & i.e. \quad \lambda - \mu - 2 = 0 \end{array}\right\} \lambda = 3, \mu = 1$$

$$\left.\begin{array}{l}\lambda+2=4\mu+1\\ -2\lambda=-5-\mu\end{array}\right\}$$ These two are satisfied by $\lambda = 3, \mu = 1$.

$\therefore$ (3) and (4) represent the same sphere.

$\therefore$ Equation to the required sphere is $x^2 + y^2 + z^2 + 3x - 4y + 5z - 6 = 0$.

Ex. 6. *Find the equation of the sphere through the circle* $x^2+y^2+z^2+2x+3y+6=0$ $x-2y+4z-9=0$ *and the centre of the sphere* $x^2+y^2+z^2-2x+4y-6z+5=0$

(K. U. M 13)

Sol. Given circle is $x^2+y^2+z^2+2x+3y+6=0=x-2y+4z-9=0$ ... (1)

Any sphere through this circle will be of the form $S+\lambda\pi=0$

(i.e.,) $(x^2+y^2+z^2+2x+3y+6)+\lambda(x-2y+4z-9)=0$ ... (2)

where λ is a parameter.

Given sphere is $x^2+y^2+z^2-2x+4y-6z+5=0$ ... (3)

The centre $=(1,-2,3)$

By data (2) passes through this point.

Substituting this point in (2), we get $(1+4+9+2-6+6)+\lambda(1+4+12-9)=0$

$\Rightarrow$ $16+8\lambda=0 \Rightarrow \lambda=-2$

Substituting this value in (2), we get

$$(x^2+y^2+z^2+2x+3y+6)-2(x-2y+4z-9)=0$$

$\Rightarrow$ $x^2+y^2+z^2+7y-8z+24=0$. This is the required sphere.

EXERCISE 6 (*b*)

1. Find the equation of the sphere through the circle

   (*i*) $x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$ and the point (1, 2, 3). **(S. V. U. 08, M 13, M15)**

   (*ii*) $x^2+y^2+z^2+2x+3y+6=0=x-2y+4z-9$ and the centre of the sphere $x^2+y^2+z^2-2x+4y-6z+5=0$ **(K. U. A11)**

2. (*i*) Find the equation of the plane which cuts the sphere $x^2 + y^2 + z^2 = a^2$ in a circle whose centre is (α, β, γ).

   (*ii*) Find the equation of the circle lying on the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z - 3 = 0$ with centre at (2, −3, 4).

3. Find the centre and radius of the circle

   (*i*) $x^2 + y^2 + z^2 = 25$, $2x + 3y + 2z = 9$ **(S.K.U. M18)**

   (*ii*) $x^2 + y^2 + z^2 - 2y - 4z - 11 = 0$, $x + 2y + 2z - 15 = 0$

   (S.K.U. A11, O.U. M15, M 97, S.U.M M18, V.S.P.V M18)

   (*iii*) $x^2 + y^2 + z^2 + 2x - 2y - 4z - 19 = 0$, $x + 2y + 2z + 7 = 0$ **(S. V. U.)**

4. If *r* be the radius of the circle $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, $lx + my + nz = 0$, prove that

   $(r^2 + d)(l^2 + m^2 + n^2) = (mw - nv)^2 + (nu - lw)^2 + (lv - mu)^2$.

5. Find the equation of the sphere through the circle

(*i*) $x^2 + y^2 = a^2,\ z = 0$ and the point (x_1, y_1, z_1). **(*S. K. U. M15*)**

[**Hint.** Consider the circle $x^2 + y^2 = a^2,\ z = 0$ as the plane section of the sphere $x^2 + y^2 + z^2 = a^2$ with the plane z = 0.]

(*ii*) $x^2 + y^2 + z^2 + 2x + 3y + 6 = 0,\ x - 2y + 4z = 9$ and the centre of the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0$ (or) **(*K. U. A11, A12*)**

Find the equation of the sphere through the circle

$x^2 + y^2 + z^2 + 2x + 3y + 6 = 0,\ x - 2y + 4z = 9 = 0$ and the point (1, 2, 3). **(*V.S.P.U M18*)**

(*iii*) $x^2 + y^2 + z^2 = 9, 2x + 3y + 2z = 5$ and the point $(-1, -2, 3)$.

6. Find the centre of the circle through the points (a, 0, 0), (0, a, 0), (0, 0, a).

7. Show that the points (5, 0, 2), (2, −6, 0), (7, −3, 8), (4, −9, 6) are concyclic.

8. Find the equation of the sphere with (1, 2, 3) and (2, 3, 4) as the ends of a diameter.

9. Find the equation of the sphere with (2, −1, 4) and (−2, 2, −2) as the ends of a diameter. Hence find the area of the circle of intersection of the sphere with the plane $2x + y - z - 3 = 0$.

10. Show that the centres of all sections of the sphere $x^2 + y^2 + z^2 = a^2$ by planes through the points (α, β, γ) lie on the sphere $x(x - \alpha) + y(y - \beta) + z(z - \gamma) = 0$.

11. Show that the radius of the circle $x^2 + y^2 + z^2 + x + y + z - 4 = 0,\ x + y + z = 0$ is 2.

12. Three mutually perpendicular rays $\overrightarrow{OP}, \overrightarrow{OQ}, \overrightarrow{OR}$ whose d.cs. are $l_1, m_1, n_1;\ l_2, m_2, n_2;\ l_3, m_3, n_3$ intersect the sphere $x^2 + y^2 + z^2 = a^2$ in P, Q, R. Prove that the equation to $\overleftrightarrow{PQR}$ is $(l_1 + l_2 + l_3)x + (m_1 + m_2 + m_3)y + (n_1 + n_2 + n_3)z = a$.

Hence show that the radius of the circle through P, Q, R is $\sqrt{2}a/\sqrt{3}$ units.

13. Find the equation of the sphere with centre on the plane $4x - 5y - z = 3$ and passing through the circle $x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 = 0 = x^2 + y^2 + z^2 + 4x + 5y - 6z + 2$.

14. Find whether the following circle is a great circle or a small circle.

(*i*) $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0,\ x + y + z = 3$ **(*S. V. U. A11*)**

(*ii*) $x^2 + y^2 + z^2 + 12x - 12y - 16z + 111 = 0,\ 2x + 2y + z = 17$

15. Find the equation of the sphere for which the circle

(*KU 04, 98; NU 96; OU 02, SKU 02, V.S.P.U. M18*)

(*i*) $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0,\ 2x + 3y + 4z = 8$ is a great circle.

(*ii*) $x^2 + y^2 + z^2 = 9,\ x - 2y + 2z - 5 = 0$ is a great circle. Also find its centre and radius.

16. Find the condition for the sphere $x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$ to intersect the sphere $x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$ in a great circle. **(*N. U. S 93*)**

17. Prove that the circles $x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0;\ 5y + 6z + 1 = 0;$ $x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0,\ x + 2y - 7z = 0$ lie on the same sphere and find its equation. Also find the value of '*a*' for which $x + y + z = a\sqrt{3}$ touches the sphere. **(*K. U. 2008*)**

18. Find the conditions that the circles

$$x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0,\ l_1x + m_1y + n_1z = p_1;$$
$$x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0,\ l_2x + m_2y + n_2z = p_2$$

should lie on the same sphere.

ANSWERS

1. (*i*) $3(x^2+y^2+z^2)-2x-3y-4z-22=0$ (*ii*) $x^2+y^2+z^2+7y-8z+24=0$

2. (*i*) $\alpha(x-\alpha)+\beta(y-\beta)+\gamma(z-\gamma)=0$

(*ii*) $x^2+y^2+z^2-2x+4y-6z-3=0,\ x-y+z=9$

3. (*i*) $\left(\frac{18}{17},\frac{27}{17},\frac{18}{17}\right),\sqrt{\frac{344}{17}}$ (*ii*) $(1,3,4),1$ (*iii*) $\left(\frac{-7}{3},\frac{-5}{3},\frac{-2}{3}\right),3$

5. (*i*) $3x^2+3y^2+3z^2-2x-3y-4z-22=0$

(*ii*) $(x^2+y^2+z^2-a^2)z_1+(a^2-x_1^2-y_1^2-z_1^2)z=0$ (*iii*) $x^2+y^2+z^2+7y-8z+24=0$

6. $(a/3,a/3,a/3)$ **8.** $x^2+y^2+z^2-3x-5y-7z+20=0$

9. $x^2+y^2+z^2-y-2z-14=0;\ \frac{317\pi}{24}$ sq. units **13.** $x^2+y^2+z^2+7x+9y-11z-1=0$

14. (*i*) Great circle (*ii*) Small circle **15.** (*i*) $x^2+y^2+z^2-2x+4y-6z+10=0$

(*ii*) $9(x^2+y^2+z^2)-10x+20y-20z-31=0,\left(\frac{5}{9},\frac{-10}{9},\frac{10}{9}\right),\frac{\sqrt{504}}{9}$

16. $2(u_1u_2+v_1v_2+w_1w_2)-d_1=2(u_2^2+v_2^2+w_2^2)-d_2$

17. $x^2+y^2+z^2-2x-2y-2z-6=0,\ a=(3\mp3\sqrt{3})/\sqrt{3}$

18. $\begin{vmatrix} 2(u-u_2) & 2(v_1-v_2) & 2(w_1-w_2) \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0,\quad \begin{vmatrix} 2(u_1-u_2) & 2(v_1-v_2) & d_1-d_2 \\ l_1 & m_1 & -p_1 \\ l_2 & m_2 & -p_2 \end{vmatrix}=0,$

$\begin{vmatrix} 2(u_1-u_2) & 2(w_1-w_2) & d_1-d_2 \\ l_1 & n_1 & -p_1 \\ l_2 & n_2 & -p_2 \end{vmatrix}=0,\quad \begin{vmatrix} 2(v_1-v_2) & 2(w_1-w_2) & d_1-d_2 \\ m_1 & n_1 & -p_1 \\ m_2 & n_2 & -p_2 \end{vmatrix}=0$

6. 13. NOTATION

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d \equiv F(x, y, z)$$
$$S_1 \equiv xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d$$
$$S_2 \equiv xx_2 + yy_2 + zz_2 + u(x + x_2) + v(y + y_2) + w(z + z_2) + d$$
$$S_{11} \equiv x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d \equiv F(x_1, y_1, z_1)$$

6. 14. INTERSECTION OF A SPHERE AND A LINE

Let $S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ be the equation to the sphere ξ with centre $C = \bar{c} = (-u, -v, -w)$ and radius $= a = \sqrt{u^2 + v^2 + w^2 - d}$

Let the equation to the sphere be

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad ...(1) \quad \text{Let } B = (\alpha, \beta, \gamma)$$

Let the equation to the line be $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$

The coordinates of a point on the line which is at a distance r from (α, β, γ) are $(\alpha + lr,\ \beta + mr,\ \gamma + nr)$. If this point lies on the given sphere (1), then,

$$(\alpha + lr)^2 + (\beta + mr)^2 + (\gamma + nr)^2 + 2u(\alpha + lr) + 2v(\beta + mr) + 2w(\gamma + nr) + d = 0$$

$$\Rightarrow r^2(l^2 + m^2 + n^2) + 2r[l(\alpha + u) + m(\beta + v) + n(r + w)] +$$

$$(\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d) = 0 \quad ...(3)$$

This quadratic equation in r gives two values of r viz. r_1 and r_2 corresponding to which we get two points common to the sphere and the line. Therefore, a line meets a sphere always in two points. These points may be real, coincident or imaginary. These are

$P\ (\alpha + lr_1,\ \beta + mr_1,\ \gamma + nr_1), Q\ (\alpha + lr_2,\ \beta + mr_2,\ \gamma + nr_2)$

Let the points P, Q are coincident at T.

Note. BP . BQ = $|S_{11}|$ and $BT^2 = |S_{11}| \Rightarrow$ BP . PQ = BT^2.

6. 15. Definition. *If line L through a given point B has only one common point T with a given sphere, then L (= $\overleftrightarrow{BT}$) is called a tangent line to the sphere from B. T is called the point of contact of the tangent line $\overleftrightarrow{BT}$ with the sphere. $\overleftrightarrow{BT}$ is said to touch the sphere at T and is called a **tangent line** to the sphere at T.*

Also if C is the centre of the sphere, then $\overleftrightarrow{CT} \perp \overleftrightarrow{BT}$ i.e. line joining the point of contact to the centre of the sphere is perpendicular to the tangent line.

6. 16. Definition. *B is a point and ξ is a sphere with centre C and radius a.*

(i) If BC > a, we say that B is an external point to the sphere ξ and the set of points B such that BC > a is called the exterior of the sphere ξ.

(ii) If BC < a, we say that B is an internal point to the sphere ξ and the set of points B such that BC < a is called the interior of the sphere ξ.

Let $B = \bar{b} = (x_1, y_1, z_1)$. Let the sphere ξ be S = 0.

B is an external point to the sphere ξ.

$\Leftrightarrow BC > a \quad \Leftrightarrow |\bar{c} - \bar{b}| > a \quad \Leftrightarrow (\bar{c} - \bar{b})^2 > a^2$

$\Leftrightarrow \bar{b}^2 - 2\bar{b}.\bar{c} + \bar{c}^2 - a^2 > 0 \quad \Leftrightarrow F(\bar{b}) > 0 \quad \Leftrightarrow S_{11} > 0.$

B is an internal point to the sphere $\xi \Leftrightarrow S_{11} < 0$.

Except the point of contact all other points on a tangent line belong to the exterior of the sphere.

For, the distance of every point on the tangent line from the centre is greater than or equal to the radius.

Also a tangent line to a sphere does not pass through an interior point.

Definition. If a line through B intersects a sphere ξ in two distinct points P and Q, then PQ is called a chord of the sphere and $\overleftrightarrow{PQ}$ is called a *secant line* of the sphere ξ.

Note. r is the radius and C is the centre of the sphere ξ. The distance of C from the chord PQ is d. Then length of PQ is $2\sqrt{r^2 - d^2}$.

6. 17. LENGTH OF THE TANGENT LINE FROM A POINT

From an external point B to a sphere ξ, always there exists a tangent line through B to the sphere ξ.

Through B by taking different directions (d.cs.) we can draw infinitely many tangent lines to the sphere.

Definition. If $\overleftrightarrow{BT}$ is a tangent line from an external point B to a sphere touches the sphere at T, then BT is called the **length of the tangent line** to the sphere from B.

If $B = (x_1, y_1, z_1)$, we know that $BT^2 = |S_{11}|$.

$\therefore$ Length of the tangent line from $B(x_1, y_1, z_1)$ to the sphere $\sqrt{[S_{11}]}$

Note. From an internal point of a sphere, there exists no tangent line to the sphere.

Definition. Normal.

Let S be the sphere and P be a point on S. Then the line through P and perpendicular to the tangent line to S at the point is called The normal to the sphere S at P. The point P is called the foot of the normal at P.

6. 18. Theorem. ***The locus of the tangent line at a point on a sphere of non-zero radius is a plane.***

Proof. Let the equation to the sphere be

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \text{...(1)}$$

Let $B = (x_1, y_1, z_1)$ be a point on it.

Equations of any line through B are $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n} = r$ ...(2)

Any point on (2) at a distance r from (x_1, y_1, z_1) is $(x_1 + lr,\ y_1 + mr,\ z_1 + nr)$

If this point lies on (1), we get $(x_1 + lr)^2 + (y_1 + mr)^2 + (z_1 + nr)^2 + 2u(x_1 + lr)$
$+ 2v(y_1 + mr) + 2w(z_1 + nr) + d = 0$

$$\Rightarrow \quad r^2(l^2 + m^2 + n^2) + 2r[l(x_1 + u) + m(y_1 + v) + n(z_1 + w)]$$
$$+ (x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d) = 0 \qquad \text{...(3)}$$

Since $B(x_1, y_1, z_1) \in (1)$ we have $x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$

$\Rightarrow$ one root of (3) is zero.

Now (3) $\Rightarrow$ $r(l^2 + m^2 + n^2) + 2[l(x_1 + u) + m(y_1 + v) + n(z_1 + w)]$

If (2) is a tangent line to the sphere (1) then the points of intersection of (1) and (2) should coincide with (x_1, y_1, z_1).

$\Rightarrow$ The second root of (3) should also be zero.

$$\Rightarrow \quad l(x_1 + u) + m(y_1 + v) + n(z_1 + w) = 0 \qquad \text{...(6)}$$

$\Rightarrow$ Eliminating l, m, n from equations (2) and (6) we have

$$(x - x_1)(x_1 + u) + (y - y_1)(y_1 + v) + (z - z_1)(z_1 + w) = 0$$

$\Rightarrow \quad xx_1 + yy_1 + zz_1 + ux + vy + wz - ux_1 - vy_1 - wz_1 - (x_1^2 + y_1^2 + z_1^2) = 0$

$\Rightarrow \quad xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0 \qquad ...(7)$

since $x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$.

Equation (7) is a plane which gives the required locus.

$\therefore$ All the tangent lines at B pass through B and a plane having BC as its normal.

6. 19. TANGENT PLANE.

Definition. The locus of the tangent lines at a point B on a sphere ξ of non-zero radius is a plane called the tangent plane to the sphere ξ at B. B is called the point of contact of the plane with the sphere ξ.

Theorem. ***Equation to the tangent plane to the sphere $S = 0$ at (x_1, y_1, z_1) on the sphere is $x(x_1 + u) + y(y_1 + v) + z(z_1 + w) + ux_1 + vy_1 + wz_1 + d = 0$.***

Proof. We here proved in theorem vide 6.18 that the locus of the tangent lines at a point $B(x_1, y_1, z_1)$ is a plane. This plane is defined as the tangent plane at B and we have proved that the equation to this plane is

$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$

This is the required equation to the tangent plane at (x_1, y_1, z_1).

Note 1. Tangent plane at B is perpendicular to BC.

2. The line through B and perpendicular to the tangent plane at B is called the normal line to the sphere at B. The normal line at B passes through the centre C.

3. Perpendicular distance of any tangent line from centre C is equal to the radius a.

4. Perpendicular distance of any tangent line from C is equal to a.

5. If $lx + my + nz = p$ is the tangent plane, then

$$\left| \frac{l(-u) + m(-v) + n(-w) - p}{\sqrt{l^2 + m^2 + n^2}} \right| = \sqrt{u^2 + v^2 + w^2 - d}$$

i.e. $(lu + mv + nw + p)^2 = (l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d)$

6. Except the point of contact all other points on the tangent plane belong to the exterior of the sphere.

7. If B is a point in the exterior of the sphere ξ, then there exists a tangent plane to ξ through B.

8. ξ is a sphere and π is a plane.

ξ, π have only point in common $\Leftrightarrow \pi$ is a tangent plane to ξ.

e.g. Equation to the sphere with centre at $(1, -2, 3)$ and touching the plane $6x + 3y + 2z = 4$ is

$$(x-1)^2 + (y+2)^2 + (z-3)^2 = \left| \frac{6(1) - 3(-2) + 2(3) - 4}{\sqrt{36 + 9 + 4}} \right|$$

$\Rightarrow \quad x^2 + y^2 + z^2 - 2x + 4y - 6z + 12 = 0$.

6. 20. TOUCHING SPHERES

Definition. *ξ, ξ' are two spheres with only one point P in common. Then ξ, ξ' are said to touch at P. ξ, ξ' are called touching spheres and P is called the point of contact of the spheres ξ, ξ'.*

The tangent plane at P to ξ is equal to the tangent plane at P to ξ'. This plane is called the common tangent plane at P to ξ, ξ'.

If A, B are the centres of ξ, ξ' respectively, then A, P, B are collinear.

Theorem. ***S = 0, S' = 0 are the equations of two spheres touching at P. Then the equation to the common tangent plane at P to the two spheres is S − S' = 0.***

Proof. S = 0 and S' = 0 are two spheres with one and only one common point P.

$\therefore$ Their centres are not equal *i.e.* $(-u, -v, -w) \neq (-u', -v', -w')$

i.e. $(u-u', v-v', w-w') \neq (0, 0, 0)$.

Now $S - S' \equiv 2(u-u')x + 2(v-v')y + 2(w-w')z + (d-d') = 0$ is a plane to which all the points common to the spheres S = 0, S' = 0 belong. But S = 0, S' = 0 have only one point P in common.

$\therefore$ P is the only point common to S = 0, S' = 0 and which lies on the plane S − S' = 0.

$\therefore$ S − S' = 0 is the tangent plane at P to S as well as to S' = 0.

$\therefore$ S − S' = 0 is the common tangent plane at P to the spheres S = 0, S' = 0.

6. 21. Definition. *A, B are the centres and r_1 (> 0), r_2 (>0) are the radii of two spheres ξ, ξ' touching at P.*

(*i*) If A – P – B, the spheres are said to **touch *externally*** *at P*.

(*ii*) If A – P – B or B – A – P, the spheres are said to **touch *internally*** *at P*.

We have AB = $r_1 + r_2 \Rightarrow$ spheres ξ, ξ' **touch *externally*** at *P*.

Further, P is an internal point to AB and (P; A, B) = $r_1 : r_2$

Again we have, AB = $|r_1 - r_2| \Rightarrow$ spheres ξ, ξ' to **touch *internally*** at *P*.

Further, P is an internal point to AB and (P; A, B) = $r_1 : -r_2$.

SOLVED PROBLEMS

Ex. 1. *Find whether the points P = (3, 1, – 1), Q = (2, – 3, 1), R = (1, – 2, 0) belong to the exterior of ξ or to the interior of ξ or to ξ where ξ is the sphere* $x^2 + y^2 + z^2 - 3x + 5y + 7 = 0$.

Sol. Let the equation to the sphere ξ be S = 0 where $S \equiv x^2 + y^2 + z^2 - 3x + 5y + 7$

P = (3, 1, – 1). In this case $S_{11} = 9 + 1 + 1 - 9 + 5 + 7 = 14 > 0$

$\therefore$ P belongs to the exterior of ξ.

Q = (2, – 3, 1). In this case $S_{11} = 4 + 9 + 1 - 6 - 15 + 7 = 0$. $\therefore Q \in \xi$

R = (1, – 2, 0). In this case $S_{11} = 1 + 4 + 0 - 3 - 10 + 7 = -1 < 0$.

$\therefore$ R belongs to the interior of ξ.

Ex. 2. *Find the length of the tangent line from the point (3, 1, – 1) to the spheres* $x^2 + y^2 + z^2 - 3x + 5y + 7 = 0$.

Sol. If P = (x_1, y_1, z_1) and S = 0 is a sphere, then length of the tangent from P to the sphere is $\sqrt{S_{11}}$.

$\therefore$ Length of the tangent from (3, 1, – 1) to the sphere

$x^2 + y^2 + z^2 - 3x + 5y + 7 = 0$ is $\sqrt{9+1+1-9+5+7} = \sqrt{14}$.

Ex. 3. *Find the points of intersection of the line* $\frac{x-8}{4} = \frac{y}{1} = -(z-1)$ *and the sphere* $x^2 + y^2 + z^2 - 4x + 6y - 2z + 5 = 0$. **(S. V. U. A 93)**

Sol. Let any point on the given line be $(4t+8,\ t,\ 1-t)$.

If it belongs to the sphere $x^2 + y^2 + z^2 - 4x + 6y - 2z + 5 = 0$, then

$$(4t+8)^2 + t^2 + (1-t)^2 - 4(4t+8) + 6t - 2(1-t) + 5 = 0$$

i.e. $t^2 + 3t + 2 = 0$ *i.e.* $t = -1, -2$.

$\therefore$ The point of intersection of the line with the spheres are (4, – 1, 2), (0, – 2, 3).

Ex. 4. *Show that the plane* $2x - 2y + z + 12 = 0$ *touches the sphere* $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ *and find the point of contact.*

(S.K.U. M15, A11, O.U. M 13, 10, 07, S.V.U. M15, M 06, A. U. M 13, K. U. M15)

Sol. Given sphere is $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ ...(1)

Its centre = (1, 2, – 1) and radius $= \sqrt{1+4+1+3} = 3$

Given plane is $2x - 2y + z + 12 = 0$ ...(2)

Distance of (1, 2, – 1) from (2) $= \left|\frac{2(1) - 2(2) + 1(-1) + 12}{\sqrt{4+4+1}}\right| = 3 =$ radius.

$\therefore$ The plane touches the sphere.

Let the line through the centre and perpendicular to (2) be

$$\frac{x-1}{2} = \frac{y-2}{-2} = \frac{z+1}{1} \ (= r \text{ say})$$

A point on this line is $(2r+1,\ -2r+2,\ r-1)$.

If it is the point of contact of (2) with (1), then this point must lie on (2).

$\therefore$ $2(2r+1) - 2(-2r+2) + (r-1) + 12 = 0 \Rightarrow r = -1$

$\therefore$ Point of contact = (– 1, 4, – 2).

Ex. 5. *Find the equation of the sphere which touches the sphere* $x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$ *at the point (1, 2, – 2) and passes through the origin.*

Sol. Let $S \equiv x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$ and P = (1, 2, – 2)

$\therefore$ Equation to the tangent plane ($S_1 = 0$) at P is

$1.x + 2.y + (-2)z + x + 1 - 3y - 3.2 + 1 = 0$ *i.e.* $2x - y - 2z - 4 = 0$

Let the sphere touching the sphere S = 0 at P and passing through the origin be

$$x^2 + y^2 + z^2 + 2x - 6y + 1 + \lambda(2x - y - 2z - 4) = 0 \quad ...(1)$$

$\therefore$ $0 + 0 + 0 + 0 - 0 + 1 + \lambda(0 - 0 - 0 - 4) = 0 \Rightarrow \lambda = 1/4$

$\therefore$ Equation to the required sphere is

$4(x^2 + y^2 + z^2) + 10x - 25y - 2z = 0$ [from (1)]

Since (0, 0, 0) belong to the exterior of S = 0, the required sphere touches S = 0 externally at (1, 2, – 2).

Note. Even if (0, 0, 0) belongs to the interior of S = 0, the same method can be applied to find the touching sphere which touches internally.

Ex. 6. *Find the equations of the tangent line to the circle* $x^2 + y^2 + z^2 + 5x - 7y + 2z - 8 = 0,\ 3x - 2y + 4z + 3 = 0$ *at the point* $(-3, 5, 4)$.

Sol. The tangent line to a circle is the line of intersection of the tangent plane to the sphere at the given point and the plane of the circle.

Given sphere is $x^2 + y^2 + z^2 + 5x - 7y + 2z - 8 = 0$ ...(1)

and plane of the circle is $3x - 2y + 4z + 3 = 0$ ...(2)

$\therefore$ Equation to the tangent plane to (1) at $(-3, 5, 4)$ is

$$x(-3) + y(5) + z(4) + \frac{5}{2}x + \frac{5}{2}(-3) - \frac{7}{2}y - \frac{7}{2}(5) + z + 4 - 8 = 0$$

i.e. $x - 3y - 10z + 58 = 0$ ...(3)

$\therefore$ Equations to the tangent line to the circle $(-3, 5, 4)$ are

$3x - 2y + 4z + 3 = 0 = x - 3y - 10z + 58$

Note. Equations to the tangent line can be put in the symmetrical form

$$\frac{x+3}{32} = \frac{y-5}{34} = \frac{z-4}{-7}.$$

Ex. 7. *Find the equations of the tangent planes to the sphere* $x^2 + y^2 + z^2 + 2x - 4y + 6z - 7 = 0$ *which intersect in the line* $6x - 3y - 23 = 0 = 3z + 2$.

Sol. Let the plane through the line $6x - 3y - 23 = 0 = 3z + 2$ and touching the sphere $x^2 + y^2 + z^2 + 2x - 4y + 6z - 7 = 0$ be

$6x - 3y - 23 + \lambda(3z + 2) = 0$ i.e. $6x - 3y + 3\lambda z + 2\lambda - 23 = 0$

$$\therefore \quad = \left| \frac{6(-1) - 3(2) + 3\lambda(-3) + 2\lambda - 23}{\sqrt{(36 + 9 + 9\lambda^2)}} \right| = \sqrt{(21)} \Rightarrow 2\lambda^2 - 8\lambda - 4 = 0 \Rightarrow \lambda = 4,\ -\frac{1}{2}$$

$\therefore$ Required tangent planes are $2x - y + 4z - 5 = 0,\ 4x - 2y - z - 16 = 0$.

Ex. 8. (*i*) *Find the equations of spheres touching the coordinate planes. How many such spheres can be had ?*

(*ii*) *find the equations of the spheres touching the three coordinate axes. How many such spheres can be had ?*

Sol. (*i*) Let a sphere touching the coordinate planes be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \ (d > 0)$ ...(1)

(1) touches the YZ plane *i.e.* $x = 0 \Rightarrow \left| \frac{-u}{1} \right| = \sqrt{u^2 + v^2 + w^2 - d}$

$\Rightarrow \quad v^2 + w^2 = d$ ...(2)

Similarly, $w^2 + u^2 = d$ ...(3) $u^2 + v^2 = d$ ...(4)

$\therefore$ (2) + (3) + (4) $\Rightarrow u^2 + v^2 + w^2 = \frac{3d}{2}$

$\therefore \quad u^2 = \frac{d}{2}, v^2 = \frac{d}{2}, w^2 = \frac{d}{2}$ *i.e.* $u = \pm\sqrt{\frac{d}{2}} = v = w$

$\therefore$ From (1), equations of the spheres touching the coordinate planes are

$x^2 + y^2 + z^2 \pm \sqrt{(2d)}\, x \pm \sqrt{(2d)}\, y \pm \sqrt{(2d)}\, z + d = 0$ ...(2)

Since $d > 0$, there exist an infinite number of spheres touching the coordinate planes.

However, for given $d > 0$, there exist only eight spheres touching the coordinate planes.

(*ii*) Let a sphere touching the coordinate axes be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \ (d > 0) \qquad ...(1)$$

(1) touches the x-axis i.e. $y = 0$, $z = 0$ at points whose x-coordinates are given by $x^2 + 2ux + d = 0$ i.e. $4u^2 - 4d = 0$ ($\because$ (1) is touching) i.e. $u^2 = d$.

Similarly $v^2 = d, w^2 = d$. $\qquad \therefore u = v = w = \pm\sqrt{d}$.

$\therefore$ From (1), equations of the sphere touching the coordinate axes are

$x^2 + y^2 + z^2 \pm 2\sqrt{d}\, x \pm 2\sqrt{d}\, y \pm 2\sqrt{d}\, z + d = 0$. Since $d > 0$, there exist an infinite number of spheres touching the coordinate axes.

However, for a given $d > 0$, there exists only eight spheres touching the coordinate axes.

Ex. 9. *A sphere is inscribed in the tetrahedron with faces $x = 0$, $y = 0$, $z = 0$, $2x + 6y + 3z = 14$. Find the equation of the sphere.*

Sol. The plane $2x + 6y + 3z = 14$ meets the axes in (7, 0, 0), (0, 7/3, 0), (0, 0, 14/3).

Let the equation to the inscribed sphere be

$$x^2 + y^2 + z^2 - 2\sqrt{\frac{d}{2}}x - 2\sqrt{\frac{d}{2}}y - 2\sqrt{\frac{d}{2}}z + d = 0, \ \ d > 0$$

such that $\sqrt{\frac{d}{2}} < \frac{7}{3}\left(\text{smaller of } 7, \frac{7}{3}, \frac{14}{3}\right)$

Its centre $= \left(\sqrt{\frac{d}{2}}, \sqrt{\frac{d}{2}}, \sqrt{\frac{d}{2}}\right)$ and radius $= \sqrt{\left(\frac{d}{2} + \frac{d}{2} + \frac{d}{2} - d\right)} = \sqrt{\left(\frac{d}{2}\right)}$

Let $\sqrt{\frac{d}{2}} = k$ say $\qquad \therefore \left|\dfrac{2k + 6k + 3k - 14}{\sqrt{(4 + 36 + 9)}}\right| = k$

$\Rightarrow \quad (11k - 14)^2 = 49k^2 \Rightarrow 18k^2 - 77k + 49 = 0 \Rightarrow k = \frac{7}{9}, \frac{7}{2}$

Since $k = \sqrt{\left(\frac{d}{2}\right)} < \frac{7}{3}, k = \frac{7}{9}$ only. $\qquad \therefore$ Centre $= \left(\frac{7}{9}, \frac{7}{9}, \frac{7}{9}\right)$ and radius $= \frac{7}{9}$

Also $\sqrt{\frac{d}{2}} = \frac{7}{9}$ *i.e.* $d = \frac{98}{81}$

$\therefore$ Equation to the inscribed sphere is $x^2 + y^2 + z^2 - \frac{14}{9}(x + y + z) + \frac{98}{81} = 0$.

Ex. 10. *Show that the centres of the spheres which touch the lines $y = mx, z = c$, $y = -mx, z = -c$ lie upon the surface $mxy + cz(1 + m^2) = 0$.* ***(S. V. U., N. U. 92)***

Sol. Given lines are $y = mx, z = c \quad ...(1) \quad y = -mx, z = -c \qquad ...(2)$

(1) can be written as $\dfrac{x - 0}{1} = \dfrac{y - 0}{m}, z - c = 0$ and

(2) can be written as $\dfrac{x - 0}{1} = \dfrac{y - 0}{-m}, z - (-c) = 0$.

(Fig. 70). Let (α, β, γ) be the centre of the sphere which touch the lines (1) and (2). Sphere touches lines (1) and (2) $\Leftrightarrow$ (α, β, γ) is equidistance from the lines (1) and (2).

$$\Leftrightarrow \left| (\alpha, \beta, \gamma - c) \times \left(\frac{1}{\sqrt{(1+m^2)}}, \frac{m}{\sqrt{(1+m^2)}}, 0 \right) \right|$$

$$= \left| (\alpha, \beta, \gamma - c) \times \left(\frac{1}{\sqrt{(1+m^2)}}, \frac{-m}{\sqrt{(1+m^2)}}, 0 \right) \right|$$

$$\Rightarrow \quad \frac{1}{\sqrt{1+m^2}} |(\alpha, \beta, \gamma - c) \times (1, m, 0)|$$

$$\Leftrightarrow \quad |-m(\gamma - c), \gamma - c, \alpha m - \beta| = |(m(\gamma + c)\gamma + c, -\alpha m - \beta)|$$

$$\Leftrightarrow \quad m^2(\gamma - c)^2 + (\gamma - c)^2 + (\alpha m + \beta)^2 = m^2(\gamma + c)^2 + (\gamma + c)^2 + (\alpha m + \beta)^2$$

$$\Leftrightarrow \quad 4cm^2\gamma + 4c\gamma + 4m\alpha\beta = 0 \Leftrightarrow (m^2 + 1)c\gamma + m\,\alpha\beta = 0$$

$\therefore$ (α, β, γ) lies on the surface $(m^2 + 1)cz + m\,xy = 0$.

Ex. 11. *Find the locus of the middle points of the system of parallel chords of the spheres* $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

Sol. Let *l, m, n* be d.cs. of the system of parallel chords of the sphere and (x_1, y_1, z_1) be the middle point of a chord of the system.

Chord with the middle point (x_1, y_1, z_1) intersects the sphere in the points $(x_1 + lr, y_1 + mr, z_1 + nr)$.

$$\Rightarrow \quad (x_1 + lr)^2 + (y_1 + mr)^2 + (z_1 + nr)^2 + 2u(x_1 + lr) + 2v(y_1 + mr) + 2w(z_1 + nr) + d = 0$$

$$\Rightarrow \quad r^2(l^2 + m^2 + n^2) + 2[l(u + x_1) + m(v + y_1) + n(w + z_1)] + S_{11} = 0$$

$\Rightarrow$ Sum of the two values of r is zero

$$\Rightarrow \quad l(u + x_1) + m(v + y_1) + n(w + z_1) = 0$$

$\therefore$ Locus of the middle points of the system of parallel chords of S = 0 is

$$l(u + x) + m(v + y) + n(w + z) = 0.$$

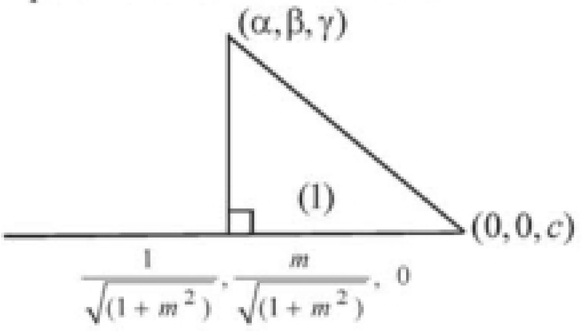


Fig. 70

Ex. 12. *Show that the spheres*

$$x^2 + y^2 + z^2 - 2x - 4y - 6z - 50 = 0,$$
$$x^2 + y^2 + z^2 - 10x + 2y + 18z + 82 = 0$$

touch externally at the point $\left(\frac{45}{13}, \frac{2}{13}, \frac{-57}{13} \right)$.

Sol. Let A, B be the centres and r_1, r_2 be the radii of the two spheres. (Fig. 71)

$\therefore$ $A = (1, 2, 3)$, $B = (5, -1, -9)$,

$$r_1 = \sqrt{(1+4+9+50)} = 8,$$
$$r_2 = \sqrt{(25+1+81-82)} = 5.$$

Now AB $= \sqrt{(4^2+3^2+12^2)} = 13 \therefore \text{AB} = r_1 + r_2$

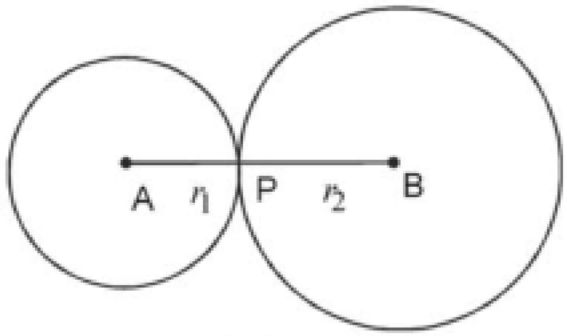


Fig. 71

$\therefore$ The two spheres touch externally, say, at P.

$\therefore$ (P; A, B) $= r_1 : r_2 = 8 : 5$

$$\therefore \text{P} = \left(\frac{45}{13}, \frac{2}{13}, \frac{-57}{13}\right)$$

EXERCISE 6 (*c*)

1. Find the points of intersection of the line

(*i*) $\dfrac{x+3}{4} = \dfrac{y+4}{3} = \dfrac{z-8}{-5}$ with the sphere $x^2+y^2+z^2+2x-10y = 23$.

(*ii*) $2x-1 = y+3 = 4-z$ with the sphere $x^2+y^2+z^2-6x+8y-4z+4 = 0$.

2. Prove that the sum of the squares of the intercepts made by a sphere on any three mutually perpendicular lines through a fixed point is constant. ***(N. U. A 89)***

3. Find the equation of the tangent plane to the sphere $x^2+y^2+z^2-2x-4y+2z-3 = 0$ at $(-1, 4, -2)$.

4. Show that the plane $4x+9y+14z-64 = 0$ touches the sphere $3(x^2+y^2+z^2)-2x-3y-4z-22 = 0$. Find also the point of contact. ***(S. V. U. A 11, S.K.U. M18)***

5. Find the values of a for which the plane $x+y+z = a\sqrt{3}$ touches the sphere $x^2+y^2+z^2-2x-2y-2z-6 = 0$. ***(K. U. A11)***

6. Find the equation of the sphere touching the sphere $x^2+y^2+z^2-x+3y+2z-3 = 0$ at $(1, 1, -1)$ and passing through the origin.

7. Find the equations of the spheres which pass through the circle ***(S. V. U. M 01)***

(*i*) $x^2+y^2+z^2 = 5,\ x+2y+3z-3 = 0$ and touch the plane $4x+3y-15 = 0$

(*ii*) $x^2+y^2+z^2-6x-2z+5 = 0,\ y = 0$ and touch the plane $3y+4z+5 = 0$.

(*iii*) $x^2+y^2+z^2-1 = 0, 2x+4y+5z = 6$ and touch the plane $z = 0$.

(S. V. U. M 13, O. U. M14)

8. Find the equation of the tangent plane to the sphere $3(x^2+y^2+z^2)-2x-3y-4z-22 = 0$ at the point $(1, 2, 3)$. Hence find the equations of the tangent line to the circle $3x^2+3y^2+3z^2-2x-3y-4z-22 = 0,\ 3x+4y+5z-26 = 0$ at the point $(1, 2, 3)$ in the symmetrical form. ***(K.U.)***

9. Find the equations of the tangent planes to the sphere $x^2+y^2+z^2 = 9$ which pass through the line (*i*) $\dfrac{x-5}{2} = \dfrac{y-1}{-2} = \dfrac{z-1}{1}$ ***(S. V. U.)***(*ii*) $x+y = 6,\ x-2z = 3$ ***(S.V.U.A12)***

10. Find the equation of the sphere which has its centre at the origin and touches the line $\dfrac{x+1}{1} = \dfrac{y-2}{-2} = \dfrac{z+3}{2}$ ***(S. V. U. 92)***

11. If the sphere $x^2 + y^2 + z^2 = r^2$ has a tangent plane which is making intercepts a, b, c on the coordinate axes, then show that $a^{-2} + b^{-2} + c^{-2} = r^{-2}$.

12. Find the tangent planes to the sphere $x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$ which are parallel to $2x + 2y - z = 0$. ***(N. U. S 98, S.V. U. 83, K. U. 89)***

13. Find the equations to the spheres through the points (4, 1, 0), (2, −3, 4), (1, 0, 0) and touching the plane $2x + 2y - z = 11$. ***(O. U. 88)***

14. (*i*) Find the equations of the spheres each with radius a (> 0) and which touch the coordinate axes.

(*ii*) Find the equation of the sphere inscribed in the tetrahedron $x = 0, y = 0, z = 0$ and $x + 2y + 2z = 1$.

15. Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at (1, −2, 1) and passes through the origin. ***(N. U. A 89, K.U. M18, S.U.M M18)***

16. Show that the spheres given below touch each other externally :

$x^2 + y^2 + z^2 + 2x - 4y - 6z + 10 = 0;\ x^2 + y^2 + z^2 - 6x - 4y - 12z + 40 = 0$.

17. Show that the spheres $x^2 + y^2 + z^2 - 25 = 0, x^2 + y^2 + z^2 - 24x - 40y - 18z + 225 = 0$ touch externally at the point $(12/5, 4, 9/5)$. ***(A.U M18, O. U. A 10)***

18. Show that the spheres $x^2 + y^2 + z^2 - 64 = 0, x^2 + y^2 + z^2 - 12x + 4y - 6z + 48 = 0$ touch internally at the point $\left(\frac{48}{7}, \frac{-16}{7}, \frac{24}{7}\right)$. ***(A. U. A11)***

19. Show that the locus of the centres of spheres which pass through the fixed point (0, 0, c) and touch the plane Z = 0 is $x^2 + y^2 - 2cz + c^2 = 0$.

20. Show that the locus of the centres of spheres which pass through the fixed point (0, 0, a) and touch the line x - axis is $x^2 - 2az + a^2 = 0$.

ANSWERS

1. (*i*) $(1,-1,3), (5,2,-2)$ (*ii*) $\left(\frac{-1}{3}, \frac{-14}{3}, \frac{17}{3}\right), \left(\frac{7}{3}, \frac{2}{3}, \frac{1}{3}\right)$ **3.** $2x - 2y + z + 12 = 0$

4. $(1,2,3)$ **5.** $k = \sqrt{3} \pm 3$ **6.** $2(x^2 + y^2 + z^2) - 3x + y + 4z = 0$ and it touches internally.

7. (*i*) $x^2 + y^2 + z^2 + 2x + 4y + 6z - 11 = 0,\ 5(x^2 + y^2 + z^2) - 4x - 8y - 12z - 13 = 0$

(*ii*) $x^2 + y^2 + z^2 - 6x - 4y - 2z + 5 = 0,\ 4(x^2 + y^2 + z^2) - 24x - 11y - 8z + 20 = 0$

(*iii*) $x^2 + y^2 + z^2 - 2x - 4y - 5z + 5 = 0, 5(x^2 + y^2 + z^2) - 2x - 4y - 5z + 1 = 0$

8. $\frac{x-1}{1} = \frac{y-2}{-2} = \frac{z-3}{1}$ **9.** (*i*) $x + 2y + 2z = 9, 2x + y - 2z = 2,$

(*ii*) $2x + y - 2z = 0, x + 2y + 2z = 9$ **10.** $9(x^2 + y^2 + z^2) = 5$

12. $2x + 2y - z + 10 = 0, 2x + 2y - z - 8 = 0$

13. $x^2 + y^2 + z^2 - 6x + 2y - 4z + 5 = 0$; $16(x^2 + y^2 + z^2) - 102x + 50y - 49z + 86 = 0$

14. (*i*) $2(x^2 + y^2 + z^2)\, 2\sqrt{2}\, ax \pm 2\sqrt{2}\, ay \pm 2\sqrt{2}\, az + a^2 = 0$

(*ii*) $x^2 + y^2 + z^2 - \frac{1}{4}x - \frac{1}{4}y - \frac{1}{4}z + \frac{1}{32} = 0$ **15.** $x^2 + y^2 + z^2 - 11x - 2y + z = 0$

6. 22. PLANE OF CONTACT

Definition. *Through an external point B planes are drawn touching a sphere. The locus of contact of the tangent planes of the sphere is a plane, called the plane of contact of the point B w.r.t. the sphere. If B is a point on the sphere, then the tangent plane at B to the sphere is called the plane of contact of B w.r.t. the sphere.*

Theorem. ***Equation of the plane of contact of the point (x_1, y_1, z_1) w.r.t. the sphere $S = 0$ of non-zero radius is $S_1 = 0$.***

Proof. Let $S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ be the equation to the sphere ξ with centre $C = \bar{c} = (-u, -v, -w)$ and radius $= a\,(> 0) = \sqrt{u^2 + v^2 + w^2 - d}$

Let $B = (x_1, y_1, z_1)$.

Equation to the tangent plane at $T\ (\alpha, \beta, \gamma)$ is

$x(\alpha + u) + y(\beta + v) + z(\gamma + w) + u\alpha + v\beta + w\gamma + d = 0$.

If this passes through the point B.

$x_1(\alpha + u) + y_1(\beta + v) + z_1(\gamma + w) + u\alpha + v\beta + w\gamma + d = 0$ which is the condition for the point (α, β, γ) to lie on the plane.

$x_1(x + u) + y_1(y + v) + z_1(z + w) + ux + vy + wz + d = 0 \Rightarrow S_1 = 0$

This is the equation to the plane of contact of the point (x_1, y_1, z_1) *i.e.* $S_1 = 0$

Note 1. Plane of contact is perpendicular to the line joining the given point with the centre of the sphere.

2. The locus of points of contact is the circle in which the plane of contact cuts the sphere.

3. The plane of contact of any point w.r.t. a sphere does not pass through the centre of the sphere.

4. If B is an interior point to sphere $S = 0$, there does not exist plane of contact of B w.r.t. $S = 0$.

5. From now on we take spheres which are not point spheres unless otherwise stated.

e.g. The plane of contact of the point $(3, 1, -1)$ w.r.t. the sphere

$2(x^2 + y^2 + z^2) - 6x + 10y + 7 = 0$ is

$x \cdot 3 + y \cdot 1 + z(-1) - \frac{3}{2}(x+3) + \frac{5}{2}(y+1) + \frac{7}{2} = 0$ *i.e.* $3x + 7y - 2z + 3 = 0$.

6. 23. POLAR PLANE, POLE OF THE POLAR PLANE

Definition. *ξ is a sphere and B is a point. The locus of the points, so that the plane of contact of each point w.r.t. ξ passes through B, is a plane called the polar plane of B w.r.t. ξ. B is called the pole of the polar plane.*

Theorem. ***The equation to the polar plane of the point (x_1, y_1, z_1) w.r.t. the sphere $S = 0$ is $S_1 = 0$.*** *(N. U. 07)*

Proof. Let $S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ be the equation to the sphere with centre $C = \bar{c} = (-u, -v, -w)$ and radius $= a = \sqrt{u^2 + v^2 + w^2 - d}$

Let $B = (x_1, y_1, z_1)$

Let $P = (\alpha, \beta, \gamma)$ be a point so that its plane of contact w.r.t. the sphere $S = 0$ passes through B.

Plane of contact of P is $\alpha(x+u)+\beta(y+v)+\gamma(z+w)+ux+vy+wz+d=0$

this passes through B

$\Rightarrow \quad \alpha(x_1+u)+\beta(y_1+v)+\gamma(z_1+w)+(ux_1+vy_1+wz_1+d)=0$

$\Rightarrow$ Locus of P is $x(x_1+u)+y(y_1+v)+z(z_1+w)+(ux_1+vy_1+wz_1+d)=0$

But by def. the locus of P is the polar plane of B.

$\therefore$ Equation to the polar plane of B is

$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0 \Rightarrow S_1=0$

Note 1. If B lies on the sphere, then the polar plane of B is tangent plane at B to the sphere. If B is exterior to the sphere the polar plane of B w.r.t. the sphere is the plane of contact of B w.r.t. the sphere.

2. Polar plane of B is perpendicular to CB since d.rs. of $\overleftrightarrow{CB}$ are

$x_1+u,\ y_1+v,\ z_1+w$.

e.g. The polar plane of the point (1, 3, 4) w.r.t. the sphere

$x^2+y^2+z^2-6x-2z+5=0$ is $x.1+y.3+z.4-3(x+1)-(z+4)+5=0$

i.e. $2x-3y+3z+2=0$.

6. 24. Theorem. ***ξ is a sphere. A lies on the polar plane of B w.r.t. ξ if and only if B lies on the polar plane A w.r.t. ξ.***

Proof. Let S = $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ ...(1)

be the equation to the sphere. Let A(x_1, y_1, z_1) and B(x_2, y_2, z_2).

The Polar plane of A is

$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$ ...(2)

If this passes through B(x_2, y_2, z_2) then

$x_1x_2+y_1y_2+z_1z_2+u(x_1+x_2)+v(y_1+y_2)+w(z_1+z_2)+d=0$ ...(3)

Evidently this is the condition for the plane of B to pass through A.

$\Rightarrow$ A lies on the polar plane of B.

Note. If S = 0 is the equation to ξ and A = $\bar{a}=(x_1, y_1, z_1)$, B = $\bar{b}=(x_2, y_2, z_2)$, then the condition for A to lie in the polar plane of B is

$(x_2, y_2, z_2)\,.\,(x_1, y_1, z_1)\,.\,(-u, -v, -w)+(-u, -v, -w)\,.\,(-u, -v, -w)-a^2=0$

i.e. $x_1x_2+y_1y_2+z_1z_2+(x_1+x_2)u+(y_1+y_2)v+(z_1+z_2)w+$

$u^2+v^2+w^2-a^2=0$

If S $\equiv x^2+y^2+z^2-a^2=0$, then the condition for A to lie on the polar plane of B is $x_1x_2+y_1y_2+z_1z_2=a^2$.

6. 25. CONJUGATE POINTS, CONJUGATE PLANES

Definition. *ξ is a sphere. If A, B are two points such that the polar plane of B w.r.t. ξ passes through A, then A, B are called conjugate points w.r.t. ξ.* ***(O. U. 08)***

The polar planes of A and B are called conjugate planes.

e.g. The polar plane of the point P (1, – 1, 2) w.r.t. the sphere.

$x^2+y^2+z^2-9=0$ is $x.1+y.(-1)+z.2-9=0$ i.e. $x-y+2z-9=0$ ...(1)

The polar plane of the point Q (5, 2, 3) w.r.t. the sphere

$x^2 + y^2 + z^2 - 9 = 0$ is $x.5 + y.2 + z.3 - 9 = 0$ i.e. $5x + 2y + 3z - 9 = 0$ ...(2)

Clearly the polar plane of P passes through Q and the polar plane of Q passes through P. Thus P, Q are conjugate points. Also the polar planes (1) and (2) are conjugate planes.

6.26. Theorem. ***If $x^2 + y^2 + z^2 - a^2 = 0$ is a sphere, then the pole of the plane $lx + my + nz = p$ $(p \neq 0)$ is $\left(\frac{a^2 l}{p}, \frac{a^2 m}{p}, \frac{a^2 n}{p}\right)$*** **(S. V. U. A 93)**

Proof. Let $P(x_1, y_1, z_1)$ be the pole of the plane $lx + my + nz - p = 0$ ...(1) w.r.t. the sphere S = 0.

$\therefore$ Polar plane of P w.r.t. S = 0 is $S_1 = 0$ *i.e.* $xx_1 + yy_1 + zz_1 - a^2 = 0$ ...(2)

Since (1) and (2) represent the same polar plane $x_1 : l = y_1 : m = z_1 : n = a^2 : p$

$\Rightarrow$ $x_1 = \frac{la^2}{p}, y_1 = \frac{ma^2}{p}, z_1 = \frac{na^2}{p} (p \neq 0)$. $P = \left(\frac{a^2 l}{p}, \frac{a^2 m}{p}, \frac{a^2 n}{p}\right)$

Note. If the line $\overleftrightarrow{OP}$ intersects the polar plane of P at Q, then OP . OQ = a^2.

For : Polar plane of $P(x_1, y_1, z_1)$ w.r.t. S = 0 is $xx_1 + yy_1 + zz_1 - a^2 = 0$ and the polar plane is perpendicular to $\overleftrightarrow{OP}$.

$\therefore$ Distance of O from the polar plane of P

$$= OQ = \left| \frac{-a^2}{\sqrt{x_1^2 + y_1^2 + z_1^2}} \right| = \frac{a^2}{OP} \text{ since } OP = \sqrt{x_1^2 + y_1^2 + z_1^2} \qquad \therefore OP \,.\, OQ = a^2$$

Observe that if P is interior (exterior) to the sphere then Q is exterior (interior) to the sphere.

6.27. Theorem. ***$S \equiv x^2 + y^2 + z^2 - a^2 = 0$ is a sphere. If $l_1 x + m_1 y + n_1 z = p_1$ $(\neq 0)$...(1), $l_2 x + m_2 y + n_2 z = p_2 (\neq 0)$ ...(2) are conjugate planes w.r.t. S = 0, then $a^2 (l_1 l_2 + m_1 m_2 + n_1 n_2) = p_1 p_2$.***

Proof. Pole of (1) w.r.t. S = 0 is $\left(\frac{a^2 l_1}{p_1}, \frac{a^2 m_1}{p_1}, \frac{a^2 n_1}{p_1}\right)$. But this point lies on (2).

$$\therefore \frac{l_2 a^2 l_1}{p_1} + \frac{m_2 a^2 m_1}{p_1} + \frac{n_2 a^2 n_1}{p_1} = p_2 \Rightarrow a^2(l_1 l_2 + m_1 m_2 + n_1 n_2) = p_1 p_2$$

e.g. Prove that the planes $5x - y - 6z + 25 = 0$ ...(1) $x - 2y - 3z + 25 = 0$ ...(2) and conjugate planes w.r.t. the sphere $x^2 + y^2 + z^2 = 25$ ...(3) **(O. U. M 97)**

Sol. Let the pole of (1) be $P(x_1, y_1, z_1)$ w.r.t. (3)

$\therefore$ Polar plane of P w.r.t. (3) is $xx_1 + yy_1 + zz_1 - 25 = 0$ ...(4)

Since (4) and (1) represent the same plane,

$$\frac{x_1}{5} = \frac{y_1}{-1} = \frac{z_1}{-6} = \frac{-25}{25} \Rightarrow P(x_1, y_1, z_1) = (-5, 1, 6)$$

Clearly P lies on (2). Similarly the pole of (2) w.r.t. (3) can be obtained as Q (– 1, 2, 3).

Clearly Q lies on (1). $\therefore$ Planes (1) and (2) are conjugate w.r.t. the sphere (3).

6.28. Theorem. ***S = 0 is a sphere. Then the polar planes of all points on the line L (not passing through C) w.r.t. S = 0 pass through another line L'.***

Proof. Let $S \equiv x^2 + y^2 + z^2 - a^2 = 0$ be the given sphere.

Let the equation to L be $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n} = t$ (say)

$\therefore$ Any point P on L is $(x_1 + lt,\ y_1 + mt,\ z_1 + nt)$.

$\therefore$ Equation to the polar plane of P w.r.t. S = 0 is

$$x\ (x_1 + lt) + y\ (y_1 + mt) + z\ (z_1 + nt) - a^2 = 0$$

i.e. $(x_1x + y_1y + z_1z - a^2) + t(lx + my + nz) = 0$

$\therefore$ The polar plane of any point on L w.r.t. S = 0 passes through the line L' of intersection of the planes $x_1x + y_1y + z_1z = a^2, lx + my + nz = 0$ ...(1)

(The two planes are not parallel since L is not passing through the origin).

Note 1. The polar plane of any point P on L passes through every point of L'. So the polar plane of every point of L', passes through the point P on L. As P can be any point on L, the polar plane of every point of L' passes through L.

2. D.rs. (1) are $(ny_1 - mz_1,\ lz_1 - nx_1,\ mx_1 - ly_1)$.

Since $l\,(ny_1 - mz_1) + m\,(lz_1 - nx_1) + n\,(mx_1 - ly_1) = 0$ the line L is perpendicular to its polar line L'.

6. 29. CONJUGATE LINES OR POLAR LINES

Definition. *ξ is a sphere L, L' are two lines such that the polar plane of every point on L w.r.t. ξ passes through L', then, L, L' are called conjugate lines.*

Some authors call L, L' as polar lines.

SOLVED PROBLEMS

Ex.1. *Find the pole of the plane $x + 2y + 3z = 7$ w.r.t the sphere $x^2 + y^2 + z^2 - 2x - 4y - 6z + 11 = 0$.* ***(SKU 98)***

Sol. Given plane is $x + 2y + 3z - 7 = 0$ ... (1)

Let $P(x_1, y_1, z_1)$ be the pole of (1) w.r.t. the sphere

$x^2 + y^2 + z^2 - 2x - 4y - 6z + 11 = 0$ ... (2)

Polar plane of $P(x_1, y_1, z_1)$ w.r.t (2) is

$xx_1 + yy_1 + zz_1 - (x + x_1) - 2(y + y_1) - 3\,(z + z_1) + 11 = 0$

$\Rightarrow x(x_1 - 1) + y(y_1 - 2) + z(z_1 - 3) - (x_1 + 2y_1 + 3z_1 - 11) = 0$ ...(3)

Since (1) and (2) represent the same polar plane,

$$\frac{x_1 - 1}{1} = \frac{y_1 - 2}{2} = \frac{z_1 - 3}{3} = \frac{(x_1 + 2y_1 + 3z_1 - 11)}{7} = t \text{ (say)}$$

$(x_1, y_1, z_1) = (t + 1, 2t + 2, 3t + 3)$ and $x_1 + 2y_1 + 3z_1 - 11 = 7t$

$\Rightarrow (t+1+4t+4+9t+9-11) = 7t \quad \Rightarrow 7t = -3 \Rightarrow t = -3/7$

$\Rightarrow$ Pole of (1) $= (x_1, y_1, z_1) = \left(+\frac{4}{7}, \frac{8}{7}, \frac{12}{7}\right)$

Ex. 2. *Show that the planes of contact of all points on the line* $\frac{x}{2} = \frac{y-a}{3} = \frac{z+3a}{4}$ *w.r.t. the sphere* $x^2 + y^2 + z^2 = a^2$ *pass through the line* $\frac{2x+3a}{-13} = \frac{y-a}{3} = \frac{z}{1}$.

Sol. Given line is $\frac{x}{2} = \frac{y-a}{3} = \frac{z+3a}{4}$ (= r say) ...(1)

and given sphere is $x^2 + y^2 + z^2 = a^2$ ...(2)

Let $\frac{2x+3a}{-13} = \frac{y-a}{3} = \frac{z}{1}$ *i.e,* $\frac{x + \frac{3a}{2}}{-13/2} = \frac{y-a}{3} = \frac{z}{1}$ (= t say) ...(3)

Any point on (1) is P $(2r,\ 3r + a,\ 4r - 3a)$.

$\therefore$ Plane of contact of P w.r.t. (2) is $x \,.\, 2r + y\,(3r + a) + z\,(4r - 3a) = a^2$

Any point on (3) is Q $\left(\frac{-13t - 3a}{2},\ 3t + a,\ t\right)$

Substituting Q in the L.H.S. of (4) we have $\frac{(-13t-3a)}{2}.\ 2r + (3t+a)\ (3r+a) + t\ (4r-3a)$

$= -13tr - 3ar + 9tr + 3ar + 3at + a^2 + 4tr - 3at = a^2$ = R.H.S.

Also $\frac{-13}{2}.\ 2r + 3\ (3r+a) + 1\ (4r-3a) = -13r + 9r + 3a + 4r - 3a = 0$

$\therefore$ (3) lies in (4) *i.e.,* the plane of contact of all points on (1) w.r.t. the sphere (2) pass through the line (3).

Ex. 3. *Find the polar line of* $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ ...(1)

w.r.t. the sphere $x^2 + y^2 + z^2 = 16$ ...(2) ***(A.K. U. M18)***

Sol. Any point P on (1) is $(2r + 1, 3r + 2, 4r + 3)$ say. Polar plane of P w.r.t. (2) is

$x\,(2r + 1) + y\,(3r + 2) + z\,(4r + 3) = 16$

i.e. $(x + 2y + 3z - 16) + r\,(2x + 3y + 4z) = 0$

$\therefore$ For all values of r, the polar plane passes through the line

$2x + 3y + 4z = 0 = x + 2y + 3z - 16$, which is the required polar line.

Ex. 4. *Find the locus of points whose polar planes w.r.t. the sphere* $x^2 + y^2 + z^2 = a^2$ *touch the sphere* $(x - \alpha)^2 + (y - \beta)^2 + z^2 = r^2$.

Sol. Let P = (x_1, y_1, z_1). The polar plane of P w.r.t. $x^2 + y^2 + z^2 = a^2$ is

$$xx_1 + yy_1 + zz_1 - a^2 = 0 \quad ...(1)$$

Centre and radius of the sphere $(x - \alpha)^2 + (y - \beta)^2 + z^2 = r^2$ are $(\alpha, \beta, 0)$ and r.

(1) touches the sphere $(x - \alpha)^2 + (y - \beta)^2 + z^2 = r^2$

$\Leftrightarrow$ distance of $(\alpha, \beta, 0)$ from (1) = r

$$\Leftrightarrow \left|\frac{\alpha x_1 + \beta y_1 + 0.\,z_1 - a^2}{\sqrt{x_1^2 + y_1^2 + z_1^2}}\right| = r \Leftrightarrow (\alpha x_1 + \beta y_1 - a^2)^2 = r^2(x_1^2 + y_1^2 + z_1^2)$$

$\therefore$ The locus of P is $(\alpha x + \beta y - a^2)^2 = r^2(x^2 + y^2 + z^2)$.

EXERCISE 6 (*d*)

1. (*i*) Find the plane of contact of the point (3, -1, 5) w.r.t. the sphere $x^2 + y^2 + z^2 - 2x + 4y + 6z - 11 = 0$.
   (*ii*) Find the plane of coontact of the point $(2,-1,1)$ with the respect to the sphere $2(x^2+y^2+z^2)+10x+6y+4z+5=0$ **(*A. U. M 13*)**
2. Find the polar plane of the point (0, -1, 1) w.r.t. the sphere $x^2 + y^2 + z^2 - 2x + 4y + 6z - 11 = 0$.
3. Find the pole of the plane $x - y + 5z - 3 = 0$ w.r.t. the sphere $x^2 + y^2 + z^2 = 9$.
   (*K. U. A12, A.U.96, N.U.2002, A.K.N.U M18*)
4. Find the pole of the plane $x - y - z + 9 = 0$ w.r.t. the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0$.
5. Prove that the points P (1, -1, 2) and Q (-2, 0, 4) are conjugate points w.r.t. the sphere $x^2 + y^2 + z^2 - 6x + 2y - 3z + 1 = 0$.
6. Find the polar plane of the point P (-2, 3, 0) w.r.t. the sphere $x^2 + y^2 + z^2 + 4x - 5z - 3 = 0$. Also show that the points P (-2, 3, 0) and Q (3, 4, 2) are conjugate points w.r.t. the sphere.
7. Show that the polar plane of the origin w.r.t. the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ touches the sphere $x^2 + y^2 + z^2 = a^2$ if $d^2 = a^2(u^2 + v^2 + w^2)$.
8. Prove that the distances of two points from the centre of a sphere are proportional to the distances of each from the polar plane of the other.
9. Find the locus of points whose polar planes w.r.t. the spheres $x^2 + y^2 + z^2 - 2x - 6y - 7 = 0$ and $x^2 + y^2 + z^2 - 4x - 4z = 0$ are mutually perpendicular.
10. (*i*) Show that the polar line of $\frac{x+1}{2} = \frac{y-2}{3} = z+3$ with respect to the sphere $x^2+y^2+z^2=1$ is the line $\frac{7x+3}{11} = \frac{2-7y}{5} = \frac{z}{-1}$ **(*O.U. 08, A.U M18*)**
    (*ii*) Find the polar lines of $6(x + 3) = 3(y + 1) = 2(z - 2)$ w.r.t. the sphere $x^2 + y^2 + z^2 = 1$ in the symmetrical form. Hence show that the two lines are perpendicular.

ANSWERS

1. $2x+y+8z-1=0$ **2.** $x-y-4z+10=0$ **3.** $(3,-3,15)$ **4.** $(0,-1,4)$

6. $6y-6z-14=0$ **9.** $x^2+y^2+z^2-3x-3y-2z+2=0$ **10.** (*ii*) $\frac{x}{1} = \frac{7y+3}{-11} = \frac{7z-2}{5}$

6. 30. ANGLES OF INTERSECTION OF SPHERES, ORTHOGONAL SPHERES

Definition. *P is common point to two spheres ξ_1, ξ_2. Any angle θ between the tangent planes at P to two spheres is called an angle of intersection of the spheres ξ_1, ξ_2 at P. The other angle between the spheres is $\pi - \theta$.*

If $\theta = \pi/2$ the spheres are said to intersect orthogonally at P and the spheres are called *orthogonal spheres*.

Theorem. *ξ_1, ξ_2 are two intersecting spheres (not touching). r_1, r_2 are their respective radii and d is the distance between their centres. If P is a common point to ξ_1, ξ_2 then an angle θ of intersection of the spheres ξ_1, ξ_2 at P is given by*

$$\cos\theta = \pm\left(\frac{r_1^2 + r_2^2 - a^2}{2r_1r_2}\right)$$ ***(O. U. A 10)***

Proof. (Fig. 72). Let A, B be the centres of the spheres ξ_1, ξ_2. The tangent planes at P to ξ_1, ξ_2 are perpendicular to AP, BP respectively. Since the angle between the planes is the angle between their normals, $\angle APB = \theta$ or $\pi - \theta$.

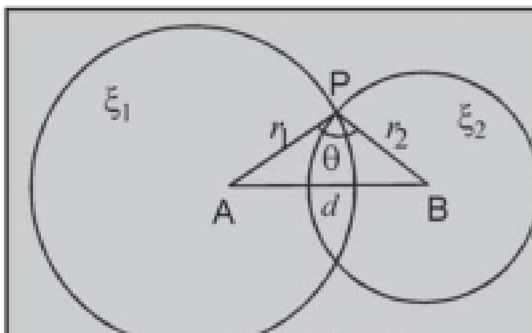


Fig. 72

AP = r_1, BP = r_2 and AB = d.

From Δ APB,

$AB^2 = AP^2 + PB^2 - 2\,AB.PB \cos\angle APB$

i.e., $d^2 = r_1^2 + r_2^2 \pm 2r_1r_1 \cos\theta$

i.e., $\cos\theta = \pm\dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$

Note 1. Since the value of cos θ is independent of P, then angle between two spheres ξ_1, ξ_2 can be found to be the same at any point of their intersection.

2. Spheres ξ_1, ξ_2 cut orthogonally $\Leftrightarrow \theta = 90^0 \Leftrightarrow r_1^2 + r_2^2 = d^2$.

In this case the tangent plane to ξ_1 at P passes through centre of ξ_2 and the tangent plane to ξ_2 at P passes through the centre of ξ_1.

6. 31. Theorem. $S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0,$ ***(S.K.U.2002)***

$S' \equiv x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$ *are two orthogonal spheres*

$\Leftrightarrow 2uu' + 2vv' + 2ww' = d + d'.$ ***(O. U. A12, M 14, S. V. U., N. U. O 88, 89, 91)***

Proof. Let A, B be the centres and r_1, r_2 be the radii of the spheres S = 0, S' = 0.

Spheres S = 0, S' = 0 cut orthogonally

$\Leftrightarrow AB^2 = r_1^2 + r_2^2$

$\Leftrightarrow (u'-u)^2 + (v'-v)^2 + (w'-w)^2 = u^2 + v^2 + w^2 - d + u'^2 + v'^2 + w'^2 - d'$

$\Leftrightarrow -2uu' - 2vv' - 2ww' = d + d' \Leftrightarrow 2uu' + 2vv' + 2ww' = d + d'$

Theorem. *If r_1, r_2 are the radii of two orthogonal spheres, then the radius of the circle of their intersection is* $\dfrac{r_1r_2}{\sqrt{r_1^2 + r_2^2}}$. ***(K.U. M18)***

(S. V. U. M15, N. U. A11, 89, 90, 06, 07; O. U, 08, M 01; S. K. U. M 13; K. U. M 14)

Proof. (Fig. 73) A, B are the centres of the two orthogonal spheres. M is the centre and a is the radius of the circle common to the spheres.

$\therefore$ A, M, B are colinear and MP $\perp$ AB.

P is a common point of intersection of the spheres,

$AP = r_1$, $BP = r_2$, $\angle APB = 90^0 \Rightarrow AB^2 = r_1^2 + r_2^2$

$\Rightarrow \quad (AM + MB)^2 = r_1^2 + r_2^2$

$\Rightarrow \quad (AM^2 + MB^2 + 2AM \,.\, MB = r_1^2 + r_2^2$

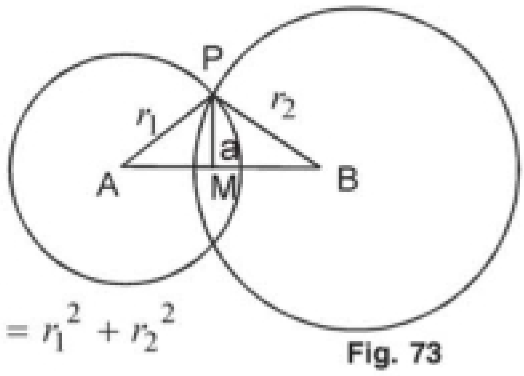


Fig. 73

$\Rightarrow \quad r_1^2 - a^2 + r_2^2 - a^2 + 2\sqrt{(r_1^2 - a^2)(r_2^2 - a^2)} = r_1^2 + r_2^2$

$\Rightarrow \quad 4(r_1^2 - a^2)(r_2^2 - a^2) = 4a^4$

$\Rightarrow \quad r_1^2 r_2^2 - a^2(r_1^2 + r_2^2) = 0 \Rightarrow a = \dfrac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$

Note. If S = 0, S' = 0 are the equations to the orthogonal spheres with radii r_1, r_2 then $2uu' + 2vv' + 2ww' = d + d'$ and $r_1^2 = u^2 + v^2 + w^2 - d$, $r_2^2 = u'^2 + v'^2 + w'^2 - d'$

$\therefore \quad r_1^2 + r_2^2 = u^2 + v^2 + w^2 + u'^2 + v'^2 + w'^2 - (2uu' + 2vv' + 2ww')$

$= (u - u')^2 + (v - v')^2 + (w - w')^2$

$\therefore \quad a = \dfrac{\sqrt{u^2 + v^2 + w^2 - d}\,.\sqrt{u'^2 + v'^2 + w'^2 - d'}}{(u - u')^2 + (v - v')^2 + (w - w')^2}$

6. 32. POWER OF A POINT

Definition. *B is a point on a line L intersecting a sphere ξ with centre C and radius = a in P, Q. Then the power of the point B w.r.t. the sphere ξ is*

(*i*) *BP . BQ if B is an external point to ξ*

(*ii*) *−BP . BQ if B is an internal point to ξ* (*iii*) *0 if B is on ξ.*

If $B = \bar{b} = (x_1, y_1, z_1)$ and the equation to the sphere ξ is S, then the power of the point B w.r.t. ξ is S_{11}. (Art. 12.14, Note).

i.e., $x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = \overline{CB}^2 - a^2 = CB^2 - (\text{radius of } \xi)^2$

If B is an external point to ξ and $\overleftrightarrow{BT}$ is a tangent line to ξ at T, then $BT^2 = S_{11}$ = Power of the point B w.r.t. ξ.

Note that : Power of the point B is independent of the *d.cs.l,m,n*.

Ex. *If the powers of a point w.r.t. two given spheres are in a constant ratio, show that the locus of the point is a sphere.*

6. 33. RADICAL PLANE

Definition. *The locus of points each of whose powers w.r.t. two non-concentric spheres are equal is a plane called the radical plane (R.P.) of the two spheres.* **(*N. U. 07*)**

ξ, ξ' are two non-concentric spheres and π is their radical plane.

P is a point on the radical plane π. $\Leftrightarrow$ power P w.r.t. ξ_1 = Power of P w.r.t. ξ_2.

Theorem. ***Equation to the radical plane of spheres S = 0, S' = 0 is S − S' = 0.***

(*S. V. U.*)

Proof. $S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

$S' \equiv x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$

$\therefore \quad (-u, -v, -w) \neq (-u', -v', -w')$

$B(x_1, y_1, z_1)$ is a point whose powers w.r.t. the spheres are equal.

$\Leftrightarrow \quad S_{11} = S'_{11} \quad \Leftrightarrow x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d$
$= x_1^2 + y_1^2 + z_1^2 + 2u'x_1 + 2v'y_1 + 2w'z_1 + d'$

$\Leftrightarrow \quad 2(u-u')x_1 + 2(v-v')y_1 + 2(w-w')z_1 + (d-d') = 0$

$\therefore$ Locus of B is $2(u-u')x + 2(v-v')y + 2(w-w')z + (d-d') = 0$

which is a plane.

But locus of B is the radical plane of the spheres S = 0, S' = 0.

$\therefore$ Radical plane of the spheres S = 0, S' = 0 is

$2(u-u')x + 2(v-v')y + 2(w-w')z + d - d' = 0$ *i.e.*, $S - S' = 0$

Note. D.rs. of the line of centres of the spheres are $u-u'$, $v-v'$, $w-w'$.

$\therefore$ Radical plane is perpendicular to the line of centres of the spheres.

Note 1. The line of centres and it is perpendicular to the radical plane.

$\therefore$ Radical plane is perpendicular to the line of centres of the spherers.

2. If two spheres intersect the plane of their circle of intersection is their radical plane.

3. If two spheres touch, their radical plane is the tangent plane at the point of contact to either of the spheres.

6. 34. RADICAL LINE

Definition. *If ξ, ξ', ξ'' are three spheres with non-collinear centres then the three radical planes of the spheres taken in pairs pass through a unique line, called the radical line.* **(N. U. 07, 06)**

6. 35. Theorem. *S = 0, S'= 0, S'' = 0 are three spheres whose centres are non-collinear, then the three radical planes of the spheres taken in pairs pass through a unique line.*

Proof. Let A, B, C be the centres of the spheres S = 0, S' = 0, S" = 0.

The radical plane (π) of S = 0, S' = 0 is S – S' = 0 and is perpendicular to AB.

The radical plane (π') of S' = 0, S" = 0 is S' – S" = 0 and is perpendicular to BC.

The radical plane (π'') of S" = 0, S = 0 is S" – S = 0 and is perpendicular to CA.

Since lines $\overleftrightarrow{AB}, \overleftrightarrow{BC}$ intersect, the planes π, π' have a line, say L in common.

All the points on L = 0 lie on the plane (S – S') + 1 (S' – S") = 0

i.e. S – S" = 0 *i.e.* S" – S = 0 *i.e.* L lies in the plane S" – S = 0

$\therefore \ \pi, \pi', \pi''$ pass through the line L.

Note. L is the radical line of the spheres S = 0, S' = 0, S" = 0

and its equation is S – S' = 0, S' – S" = 0 *i.e.* S = S' = S".

6. 36. RADICAL CENTRE

(N. U. 07)

Definition. *The four radical lines of four spheres with non-coplanar centres, taken three by three intersect at a unique point, called the radical centre of the spheres.*

Theorem. ***S = 0, S′ = 0, S″ = 0, S‴ = 0 are four spheres whose centres are non-coplanar, then the four radical lines of four spheres taken three by three intersect at a unique point.***

Proof. The radical plane of S = 0, S' = 0 is S – S' = 0

i.e. $2(u-u')x+2(v-v')y+2(w-w')z+(d-d')=0$ ...(1)

The radical planes of S = 0, S" = 0 is S – S" = 0

i.e. $2(u-u'')x+2(v-v'')y+2(w-w'')z+(d-d'')=0$ ...(2)

The radical plane of S = 0, S‴ = 0 is S – S‴ = 0

i.e. $2(u-u''')x+2(v-v''')y+2(w-w''')z+(d-d''')=0$ ...(3)

Since the centres $(-u, -v, -w)$, $(-u', -v', -w')$, $(-u'', -v'', -w'')$, $(-u''', -v''', -w''')$ are non-coplanar.

$$\begin{vmatrix} -u & -v & -w & 1 \\ -u' & -v' & -w' & 1 \\ -u'' & -v'' & -w'' & 1 \\ -u''' & -v''' & -w''' & 1 \end{vmatrix} \neq 0 \Rightarrow \begin{vmatrix} -u & -v & -w & 1 \\ u-u' & v-v' & w-w' & 0 \\ u-u'' & v-v'' & w-w'' & 0 \\ u-u''' & v-v''' & w-w''' & 0 \end{vmatrix} \neq 0$$

$$\Rightarrow \begin{vmatrix} u-u' & v-v' & w-w' \\ u-u'' & v-v'' & w-w'' \\ u-u''' & v-v''' & w-w''' \end{vmatrix} \neq 0$$

$\Rightarrow$ Radical planes (1), (2), (3) pass through a unique point P.

$\therefore$ P lies on the radical planes of S = 0, S'= 0, S' = 0, S" = 0,

$\Rightarrow$ P lies on the radical line of S = 0, S'= 0, S" = 0

Similarly P lies on the radical line of S = 0, S'= 0, S‴ = 0,

P lies on the radical line of S = 0, S" = 0, S‴ = 0.

P lies on the radical line of S' = 0, S" = 0, S‴ = 0.

$\therefore$ The four radical lines of four spheres taken three by three intersect at the unique point P.

Note. P is the radical centre of the spheres S = 0, S' = 0, S" = 0, S‴ = 0.

6. 37. Theorem. ***The centre of the sphere ξ which intersects two spheres ξ_1, ξ_2 orthogonally lies on the radical plane of the sphere ξ_1, ξ_2.***

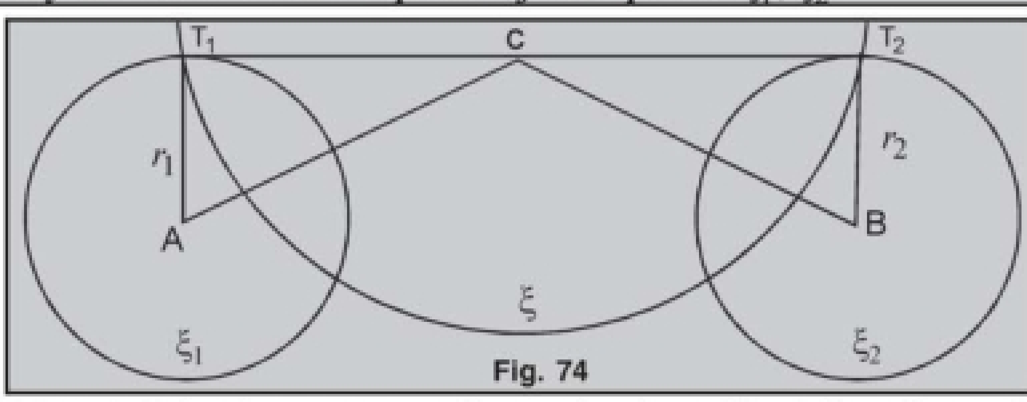


Fig. 74

Proof. Let A, B be the centres and r_1, r_2 be the radii of the spheres ξ_1, ξ_2 (Fig. 74).

Let C be the centre of the sphere ξ. Let T_1, T_2 be the respective common points to ξ_1, ξ_2 and ξ_2, ξ. (Fig. 74).

ξ intersects ξ_1 orthogonally

$\Rightarrow$ $CT_1 \perp AT_1$ $\Rightarrow AC^2 - AT_1^2 = CT_1^2$

$\Rightarrow$ $AC^2 - (\text{radius of } \xi_1)^2 = CT_1^2$ Art. 6.32

$\Rightarrow$ Power of the point C w.r.t. $\xi_1 = CT_1^2 = (\text{radius of } \xi)^2$.

Similarly power of point C w.r.t. $\xi_2 = CT_2^2 = (\text{radius of } \xi)^2$.

$\therefore$ Power of the point C w.r.t. ξ_1 = Power of the point C w.r.t. ξ_2.

$\therefore$ C lies on the radical plane of ξ_1, ξ_2.

OR

Let S' = 0, S" = 0 be the equations of the spheres ξ_1, ξ_2 and S = 0 be the equation of ξ.
Radical plane of s' = 0, S" = 0 is S' − S" = 0

i.e. $2(u'-u'')x + 2(v'-v'')y + 2(w'-w'')z + d' - d'' = 0$

ξ intersects ξ_1 orthogonally $\Rightarrow 2uu' + 2vv' + 2ww' = d + d'$ ...(1)

ξ intersects ξ_1 orthogonally $\Rightarrow 2uu'' + 2vv'' + 2ww'' = d + d''$ ... (2)

(1) − (2) : $2(u'-u'')x + 2(v'-v'')y + 2(w'-w'')z + d' - d'' = 0$

$\Rightarrow 2(u'-u'')(-u) + 2(v'-v'')(-v) + 2(w'-w'')(-w) + d' - d'' = 0$

$\therefore$ The centre $(-u, -v, -w)$ of S = 0 clearly lies on the radical plane of S' = 0, S" = 0.

i.e. the centre of ξ lies on the radical plane ξ_1, ξ_2.

SOLVED PROBLEMS

Ex. 1. *Find the equation of the sphere through the circle* $x^2 + y^2 + z^2 - 2x + 3y - 4z + 6 = 0,\ 3x - 4y + 5z - 15 = 0$ *and cutting the sphere* $x^2 + y^2 + z^2 + 2x + 4y - 6z + 11 = 0$ *orthogonally.*

Sol. Let a sphere through the given circle and cutting the sphere

$x^2 + y^2 + z^2 + 2x + 4y - 6z + 11 = 0$ ...(1) orthogonally be

$x^2 + y^2 + z^2 - 2x + 3y - 4z + 6 + \lambda(3x - 4y + 5z - 15) = 0$

i.e., $x^2 + y^2 + z^2 + (3\lambda - 2)x + (-4\lambda + 3)y + (5\lambda - 4)z + (-15\lambda + 6) = 0$

$$\therefore \quad \frac{2(3\lambda - 2)}{2} + \frac{4(-4\lambda + 3)}{2} - \frac{6(5\lambda - 4)}{2} = 11 - 15\lambda + 6 \quad i.e.\ \lambda = -1/5$$

$\therefore$ Equation to the required sphere is $5(x^2 + y^2 + z^2) - 13x + 19y - 25z + 45 = 0$.

Ex. 2. *Find the equation of the sphere which touches the plane* $3x + 2y - z + 2 = 0$ *at (1, −2, 1) and cuts orthogonally the sphere* $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$.

(A. U. A12, M14; K. U. 08, M13; N. U. S 98; S. V. U. 93, A10, A.K.N.U M18, A.U M18, V.S.P.U M18)

Sol. For the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$ ...(1)

centre = (2, −3, 0), radius $= \sqrt{4 + 9 - 4} = 3$

Since the plane $3x + 2y - z + 2 = 0$ at (1, −2, 1) is the tangent plane to the required sphere, equations to the normal at (1, −2, 1) is the tangent plane to the required sphere, equations to the normal at (1, −2, 1) are $\dfrac{x-1}{3} = \dfrac{y+2}{2} = \dfrac{z-1}{-1}$ (= t say)

$\therefore$ Centre of the required sphere can be taken as $(3t + 1, 2t - 2, -t + 1)$

and radius $= \sqrt{(3t+1-1)^2+(2t-2+2)^2+(-t+1-1)^2} = \sqrt{14}\,|t|$

Since the required sphere cuts orthogonally (1),

$(3t+1-2)^2+(2t-2+3)^2+(-t+1-0)^2 = 14t^2+9$ *i.e.,* $t=-3/2$.

$\therefore$ Centre of the required sphere $\left(-\frac{7}{2}, -5, \frac{5}{2}\right)$

$\therefore$ Equation to the required sphere is $\left(x+\frac{7}{2}\right)^2+(y+5)^2+\left(z-\frac{5}{2}\right)^2=\left(\frac{3}{2}\times\sqrt{14}\right)^2$

i.e. $x^2+y^2+z^2+7x+10y-5z+12=0$.

Ex. 3. *Find the radical centre of the sphere*

$x^2+y^2+z^2+4y=0$ ...(1) $x^2+y^2+z^2+2x+2y+2z+2=0$ ...(2)

$x^2+y^2+z^2+3x-2y+8z+6=0$ ...(3)

$x^2+y^2+z^2-x+4y-6z-2=0$ ...(4) ***(A. U. A11, O. U. A11, A12)***

Sol. R.P. of (1) and (2) is $2x-2y+2z+2=0$ *i.e.,* $x-y+z+1=0$ ...(5)

R.P. of (1) and (3) is $3x-6y+8z+6=0$ ...(6)

R.P. of (1) and (4) is $x+6z+2=0$ ...(7)

R.P. of (3) and (4) is $4x-6y+14z+8=0$ *i.e.* $2x-3y+7z+4=0$ ...(8)

$\therefore$ Radical line of the spheres (1), (2), (3) is $x-y+z+1=0$ ...(5)

$3x-6y+8z+6=0$ ...(6)

and radical line of the spheres (1), (3), (4) is $x+6z+2=0$ ...(7)

$2x-3y+7z+4=0$ ...(8)

The point of the intersection of these radical lines is the radical centre of the spheres.

$3\times(5)-(8): x-4z-1=0$ ...(9)

$(7)-(9): 10z+3=0 \Rightarrow z-\frac{3}{10}$ $\therefore x=-\frac{1}{5}, y=\frac{1}{2}$

$\therefore$ Radical centre of the spheres $=\left(-\frac{1}{5},\frac{1}{2},-\frac{3}{10}\right)$.

Ex. 4. *Show that all spheres through the origin and each set of points where the planes parallel to the plane* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ *cut the axes, form a system of spheres which are cut orthogonally by the sphere*

$x^2+y^2+z^2+2fx+2gy+2hz=0$ ...(1) *if* $af+bg+ch=0$.

Sol. Let a plane parallel to $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\lambda$ $(\lambda\neq 0)$

Let it meet the axes A, B, C.

$\therefore$ A = $(\lambda a, 0, 0)$, B = $(0, \lambda b, 0)$, C = $(0, 0, \lambda c)$,

$\therefore$ Equation to the sphere through O, A, B, C is $x^2+y^2+z^2-\lambda ax-\lambda by-\lambda cz=0$

This intersects (1) orthogonally if $2.f.\left(\frac{-\lambda a}{2}\right)+2g\left(\frac{-\lambda by}{2}\right)+2h\left(\frac{-\lambda cz}{2}\right)=0$

i.e., $af+bg+ch=0$ $(\because \lambda\neq 0)$.

EXERCISE 6 (e)

1. Show that the spheres $x^2 + y^2 + z^2 + 6y + 2z + 8 = 0$, $x^2 + y^2 + z^2 + 6x + 8y + 4z + 20 = 0$ are orthogonal. ***(A. U. M 14, A.K.U M18, V.S.P.U M18)***

2. Prove that the equation of the sphere which cuts orthogonally each of the spheres $x^2 + y^2 + z^2 = a^2 + b^2 + c^2$, $x^2 + y^2 + z^2 + 2ax = a^2$, $x^2 + y^2 + z^2 + 2by = b^2$, $x^2 + y^2 + z^2 + 2cz = c^2$ is

$$x^2 + y^2 + z^2 + \frac{b^2 + c^2}{a}x + \frac{c^2 + a^2}{b}y + \frac{a^2 + b^2}{c}z + a^2 + b^2 + c^2 = 0.$$

3. Prove that every sphere through the circle $x^2 + y^2 + z^2 - 2ax + r^2 = 0, z = 0$ intersects orthogonally every sphere through the circle $x^2 + z^2 = r^2, y = 0$.

4. Prove that the sphere which cuts two spheres S' = 0 and S" = 0 at right angles will cut the sphere λ S' + μ S" = 0 at right angles.

5. Two points A and B are conjugate w.r.t. a sphere S = 0. Prove that the sphere on AB at diameter cuts the sphere S = 0 orthogonally.

6. Show that the equation to the system of spheres through the points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ is $x^2 + y^2 + z^2 - ax - by - cz - \lambda\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1\right) = 0$. Find λ, if (1) intersects the sphere $x^2 + y^2 + z^2 - 2ax - 2by - 2cz = 0$ orthogonally. ***(NU 94)***

7. Show that the sphere intersecting the spheres $x^2 + y^2 + z^2 + x - 3z - 2 = 0$, $x^2 + y^2 + z^2 + \frac{1}{2}x + \frac{3}{2}y + 2 = 0$ orthogonally and passing through the points $(0, 3, 0)$, $(-2, -1, -4)$ is $x^2 + y^2 + z^2 + 2x - 2y + 4z - 3 = 0$.

8. Find the equation to the sphere with $(1, 2, -3)$, $(5, 0, 1)$ as the ends of one of its diameters. Also find as angle between it and the sphere $x^2 + y^2 + z^2 - 2x - 4y - 6z + 10 = 0$.

9. Find the equation to the sphere through the circle given by $x^2 + y^2 + z^2 - 2x - 4y - 11 = 0$, $x^2 + y^2 + z^2 + 2x - y + 12z + 5 = 0$ and through the point $(1, -1, -1)$.

10. Find equations to the radical line of the spheres $x^2 + y^2 + z^2 + 4y = 0$, $x^2 + y^2 + z^2 + 3x - 2y + 8z + 6 = 0$, $x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0$

11. Show that the locus of the points such that lengths of the tangents from each point to the three spheres $x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0$, $x^2 + y^2 + z^2 + 4x + 4y + 4z + 4 = 0$, $x^2 + y^2 + z^2 + x + 6y - 4z - 2 = 0$ are equal is $\frac{x}{2} = \frac{y-1}{5} = \frac{z}{3}$.

12. Show that the radical line for the spheres $x^2 + y^2 + z^2 - 4x + 3 = 0$, $x^2 + y^2 + z^2 - 6y + 3 = 0$, $x^2 + y^2 + z^2 + 4x + 2y - 4z + 3 = 0$ is $\frac{x}{3} = \frac{y}{2} = \frac{z}{7}$. ***(A. U. A12)***

ANSWERS

6. $-\frac{a^2 + b^2 + c^2}{2}$ **8.** $x^2 + y^2 + z^2 - 6x - 2y + 2z + 2 = 0, \text{Cos}^{-1}\left(-\frac{2}{3}\right)$

9. $5(x^2 + y^2 + z^2) + 14x - 2y + 72z + 41 = 0$ **10.** $x - y + z + 1 = 0 = 3x - 6y + 8z + 6$

6. 38. COAXAL SYSTEM OF SPHERES

Definition. *A system of spheres such that any two spheres of the system have the same radical plane is called a coaxal system of spheres.* ***(K.U.)***

$S = 0$, $S' = 0$ are two spheres of a coaxal system of spheres ξ.

$\Rightarrow$ $S - S' = 0$ is the radical plane of the coaxal system of spheres ξ.

Theorem. ***If $S = 0$ is a sphere and $U = 0$ is a plane, then the equation $S + \lambda U = 0$ (λ being real) represents a coaxal system of spheres with radical plane $U = 0$.***

Proof. Given that $S = 0$ is a sphere and $U = 0$ is a plane

Consider the equation $S + \lambda U = 0$, A being real ...(1)

Let $S + \lambda_1 U = 0$ ...(2) and $S + \lambda_2 U = 0$ ...(3)

$(\lambda_1 \neq \lambda_2)$ be two spheres of the system (1).

Radical plane of (2) and (3) is $(\lambda_1 \neq \lambda_2)$, $U = 0$

$\therefore$ $U = 0$ $(\because \lambda_1 \neq \lambda_2)$, independent of λ.

$\therefore$ Every two spheres of the system (1) have the same radical plane $U = 0$.

$\therefore$ $S + \lambda U = 0$ is the equation to coaxal system of spheres with radical plane $U = 0$.

Note. $S = 0$, $S' = 0$ are two non-concentric spheres. Then $S - S' = 0$ is the radical plane of $S = 0$, $S' = 0$.

$\therefore$ $S + \lambda(S - S') = 0$, λ being real, is a coaxal of system of spheres

i.e. $(1+\lambda) S + (-\lambda)S' = 0$ is a coaxal system of spheres

i.e. $\lambda_1 S + \lambda_2 S' = 0$, $(\lambda_1, \lambda_2) \neq (0, 0)$ and $\lambda_1 + \lambda_2 \neq 0$ represents a coaxal system of spheres with radical plane $S - S' = 0$.

6. 39. Theorem. ***The centres of the spheres of a coaxal system of spheres are collinear.***

Proof. Let ξ be a coaxal system of spheres with radical plane π.

Let ξ_1, ξ_2 be two spheres of the system ξ with centres A, B. $\therefore$ $AB \perp \pi$. ...(1)

Let ξ_3 be a sphere of the system ξ distinct from ξ_1, ξ_2 with centre C.

$\therefore$ ξ_1, ξ_3 have the same radical plane and $AC \perp \pi$. ...(2)

$\therefore$ From (1) and (2), A, B, C are collinear.

$\therefore$ All the centres of the spheres of the system lie on $\overleftrightarrow{AB}$ *i.e.* all the centres are collinear.

$\overleftrightarrow{AB}$ is called the line of centres of the coaxal system ξ.

Note. Radical plane of a coaxal system of spheres is perpendicular to the line of centres of the system.

6.40. A SIMPLIFIED FORM OF THE EQUATION TO A COAXAL SYSTEM OF SPHERES.

Theorem. ***A coaxal system of spheres can be reduced to the form $x^2 + y^2 + z^2 + 2\lambda x + d = 0$ where d is a constant and λ is a parameter.*** ***(S.K.U.97, S.V.U.99, 98)***

Proof. The line of centres of a coaxal system of spheres is perpendicular to the radical plane of the system.

Let the point of intersection of the line of centres with the radical plane be origin O, the line of centres be X-axis and the radical plane be YZ plane.

With this frame of reference, let a sphere of the coaxal system be

$x^2 + y^2 + z^2 + 2ux + d = 0$ ...(1) ($\because$ centre lies on x - axis)

where u, d are parameters. O is a point on the radical plane of the system.

$\therefore$ Power of O w.r.t. (1) = 0 + 0 + 0 2u (0) + d = d.

Since power of the point O w.r.t. any sphere of the system must be the same, d is a constant.

$\therefore$ Equation to the coaxal system of spheres can be taken as $x^2 + y^2 + z^2 + 2\lambda x + d = 0$ where λ (= u) is a parameter and d is a constant.

Note. The equation $x^2 + y^2 + z^2 + 2\lambda x + d = 0$ (λ is a parameter and d is a constant) represents a coaxal system with the line of centres as X - axis and the radical plane as YZ plane.

By giving values to λ and taking f as constant, we get spheres of the coaxal system.

6. 41. Consider a coaxal system of spheres $x^2 + y^2 + z^2 + 2\lambda x + d = 0$ ...(1)

where d is a constant and λ is a parameter.

Radical plane of the system is YZ plane *i.e.* $x = 0$ ...(2)

$\therefore$ (1) and (2) intersect in points given by $x = 0$, $y^2 + z^2 + d = 0$ *i.e.* $y^2 + z^2 = -d$.

(*i*) $d < 0$. All the points of intersection lie on the circle $x = 0$, $y^2 + z^2 = (\sqrt{-d})^2$ and every sphere of the system passes through the circle.

$\therefore$ The coaxal system is an intersecting type of coaxal system of spheres.

(*ii*) $d = 0$. $\therefore$ $x = 0, y^2 + z^2 = 0$ *i.e.* $x = 0, y = 0, z = 0$ *i.e.* (0, 0, 0) is a point common to (1) and (2). *i.e.* every pair of spheres of the system touch at (0, 0, 0) and the radical plane of the system is the tangent plane at (0, 0, 0) to every sphere of the system.

$\therefore$ The coaxal system is a touching type of coaxal system such that every pair touches at (0, 0, 0).

(*iii*) $d > 0$. There are no points common to (1) and (2) *i.e.* there are no points in common to any two spheres of the system.

$\therefore$ The coaxal system is a non-intersecting type of coaxal system of spheres.

6. 42. LIMITING POINTS

Definition. *Point spheres of a coaxal system of spheres are called limiting points of the system.* **(*K.U., N. U. 90*)**

Let $x^2 + y^2 + z^2 + 2\lambda x + d = 0$ where d is a constant and λ is a parameter, represent a coaxal system of spheres.

For any sphere of the system, radius $= \sqrt{(\lambda^2 - d)}$ and centre = $(-\lambda, 0, 0)$

For limiting points of the system, radius = 0

i.e. $\sqrt{(\lambda^2 - d)} = 0$ *i.e.* $\lambda = \pm\sqrt{d}$

(*i*) If $d = 0$ then $\lambda = 0$ and hence the system has only one limiting point and it is, (0, 0, 0). in this case the system is a touching type of coaxal system of spheres at (0, 0, 0).

(*ii*) If $d > 0$; then λ has two values $\pm\sqrt{d}$ and hence the system has two limiting points only. The limiting points are $(\sqrt{d}, 0, 0)$, $(-\sqrt{d}, 0, 0)$. In this case no two spheres of the system intersect.

(*iii*) If $d < 0$, the system has no limiting points. In this case the system is intersecting type.

Note 1. Equation to the limiting point $(\sqrt{d}, 0, 0)$ is

$$(x - \sqrt{d})^2 + y^2 + z^2 = 0 \quad i.e. \quad x^2 + y^2 + z^2 - 2\sqrt{d}x + d = 0$$

Equation to the limiting point $(-\sqrt{d}, 0, 0)$ is

$$(x + \sqrt{d})^2 + y^2 + z^2 = 0 \quad i.e. \quad x^2 + y^2 + z^2 + 2\sqrt{d}x + d = 0$$

2. If there is only one limiting point (0, 0, 0), its equation is $x^2 + y^2 + z^2 = 0$.

SOLVED PROBLEMS

Ex. 1. *Find the limiting points of the coaxal system defined by spheres*

$x^2 + y^2 + z^2 + 4x - 2y + 2z + 6 = 0$ ...(1)

and $x^2 + y^2 + z^2 + 2x - 4y - 2z + 6 = 0$ ...(2)

(*S. K. U. O 01; A. U. M 14, A.U M18, V.S.P.U M18*)

Sol. The R.P. of the sphere of coaxal system is $2x + 2y + 4z = 0$ *i.e.*, $x + y + 2z = 0$

$\therefore$ Equation to a sphere of coaxal system is

$$x^2 + y^2 + z^2 + 4x - 2y + 2z + 6 + \lambda(x + y + 2z) = 0$$

i.e. $x^2 + y^2 + z^2 + (4 + \lambda)x + (\lambda - 2)y + (2\lambda + 2)z + 6 = 0$

$\therefore$ centre $= \left(-\dfrac{4+\lambda}{2}, \dfrac{2-\lambda}{2}, -\lambda - 1\right)$ and

radius $= \sqrt{\left[\dfrac{(4+\lambda)^2}{4} + \dfrac{(2-\lambda)^2}{4} + (\lambda+1)^2 - 6\right]}$

For limiting points of the system, radius = 0.

$\therefore \quad \dfrac{(4+\lambda)^2}{4} + \dfrac{(2-\lambda)^2}{4} + (\lambda+1)^2 - 6 = 0 \quad$ *i.e.*, $\lambda^2 + 2\lambda = 0$ i.e. $\lambda = 0, -2$

$\therefore$ Limiting points are $(-2, 1, -1); (-1, 2, 1)$.

Ex. 2. *Find the equation of the sphere belonging to the coaxal system given by* $x^2 + y^2 + z^2 - 2ax - 2ay - 2az + 4a^2 + \lambda(x + y - z) = 0$ *and* $x^2 + y^2 + z^2 - 4ax - 4ay + 4a^2 = 0$ *and which cuts the sphere* $x^2 + y^2 + z^2 - 2ax = 0$ *orthogonally.*

Sol. R.P. of the given spheres is $2ax + 2ay - 2az = 0$ *i.e.*, $x + y - z = 0$

$\therefore$ Equation to a sphere of the coaxal system is

$$x^2 + y^2 + z^2 - 2ax - 2ay - 2az + 4a^2 + \lambda(x + y - z) = 0$$

If the sphere intersects $x^2 + y^2 + z^2 + 2ax = 0$ orthogonally, then

$$2\left(\frac{\lambda - 2a}{2}\right).a + 2\left(\frac{\lambda - 2a}{2}\right).0 + 2\left(\frac{-\lambda - 2a}{2}\right).0 = 4a^2 \qquad i.e., \lambda = 6a.$$

$\therefore$ Equation to the required sphere is $x^2 + y^2 + z^2 - 4ax - 4ay - 8az + 4a^2 = 0$.

Ex. 3. *Prove that every sphere through the limiting points of a coaxal system intersects every sphere of that system orthogonally.*

Sol. Let a coaxal system of spheres be $x^2 + y^2 + z^2 + 2\lambda x + d = 0$ ...(1)

$(d > 0)$

$\therefore$ Limiting points of the system are $(-\sqrt{d}, 0, 0), (\sqrt{d}, 0, 0)$

Let a sphere through the limiting points be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + c = 0 \qquad ...(2)$$

$\therefore \quad d - 2u\sqrt{d} + c = 0,\ d + 2u\sqrt{d} + c = 0 \qquad \therefore$ Solving, $u = 0, c = -d.$

$\therefore$ Equation to the system of spheres through the limiting points is

$$x^2 + y^2 + z^2 + 2vy + 2wz - d = 0 \qquad ...(3)$$

Since $2 \,.\, \lambda \,.\, 0 + 2 \,.\, 0 \,.\, v + 2 \,.\, 0 \,.\, w = d - d$ is true for any sphere of the system (1), every sphere of the system intersects every sphere of the system of spheres through the limiting points orthogonally.

EXERCISE 6 (*f*)

1. Find the limiting points of the coaxal system of spheres $x^2 + y^2 + z^2 - 20x + 30y - 40z + 29 + \lambda(2x - 3y + 4z) = 0$.

   (S. V. U. A11, O 98, 08, N. U. A11, A12, K. U. A11, S.K.U M18)

2. Find the equation to the spheres of the coaxal system $x^2 + y^2 + z^2 - 5 + \lambda(2x + y + 3z - 3) = 0$ which touch the plane $3x + 4y - 15 = 0$. ***(S. K. U. M 13; N. U. M 98)***

3. (*i*) Find the limiting points of the coaxal system of spheres of which two members are $x^2 + y^2 + z^2 + 3x - 3y + 6 = 0,\ x^2 + y^2 + z^2 - 6y - 6z + 6 = 0$

   (S. K. U. M15, K.U. 90, M 13; S. V. U. 90, 95, A12, N. U. 90, 92, A.K.N.U M18)

   (*ii*) Find the limiting points of the coaxal system determined by the two spheres whose equations are ***(S. K. U. M18)***

   $x^2 + y^2 + z^2 - 8x + 2y - 2z + 32 = 0,\ x^2 + y^2 + z^2 - 7x + z + 23 = 0$.

4. Find the equation of the sphere through the point (0, 1, 2) and belonging to the coaxal system defined by $x^2 + y^2 + z^2 + 3x - 3y + 2z = 0,\ x^2 + y^2 + z^2 + 2x - y - z + 10 = 0$.

   (S. K. U. M18)

5. A and B are two fixed points. P is a point such that PA = n . PB. Show that the locus of P is a coaxal system of spheres.

6. Find the equation of the radical plane of the coaxal system whose limiting points are (– 1, 2, 1) and (– 2, 1, – 1).

7. Show that the sphere which intersects two spheres orthogonally will intersect every member of the coaxal system determined by them orthogonally.

8. If (– 2, 1, – 1) is a limiting point of a coaxal system for which $x + y + 2z = 0$ is the radical plane, then show that the other limiting point is (– 1, 2, 1). ***(K. U. M18)***

9. Three spheres of radii r_1, r_2, r_3 have their centres A, B, C at the points $(a,0,0), (0,b,0), (0,0,c)$ and $r_1^2 + r_2^2 + r_3^2 = a^2 + b^2 + c^2$ fourth sphere passes through the origin and the points A, B, C. Show that the radical centre of the four spheres lies on the plane $ax + by + cz = 0$.

ANSWERS

1. $(2,-3,4), (-2,3,-4)$

2. $x^2 + y^2 + z^2 + 4x + 2y + 6z - 11 = 0$; $5(x^2 + y^2 + z^2) - 8x - 4y - 12z - 13 = 0$

3. (*i*) $(-2,1,-1), (-1,2,1)$ (*ii*) $(5,-3,4); (3,1,-2)$ **4.** $x^2 + y^2 + z^2 + 4x - 5y + 5z - 10 = 0$

6. $x + y + 2z = 0$ **8.** $(5,3,4), (3,-1,-2)$

UNIT - IV

7. **The Cone**

Definitions of a cone, vertex, guiding curve, generators. Equation of the cone with a given vertex and guiding curve. Enveloping cone of a sphere. Equations of cones with vertex at origin are homogenous. Condition that the general equation of the second degree should represent a cone. Condition that a cone may have three mutually perpendicular generators. Intersection of a line and a quadric cone. Tangent lines and tangent plane at a point. Condition that a plane may touch a cone. Reciprocal cones. Intersection of two cones with a common vertex. Right circular cone. Equation of the right circular cone with a given vertex, axis and semi-vertical angle.

7

THE CONE

7. 1. Definition. *The surface generated by a straight line that which passes through a fixed point and intersecting a given curve or touching a given surface, is called a cone.*

The fixed point is called the vertex and the given curve the guiding curve of the cone.

An individual straight line on the surface of a cone is called a generator.

Thus a coneis the set of lines called generators through a given point.

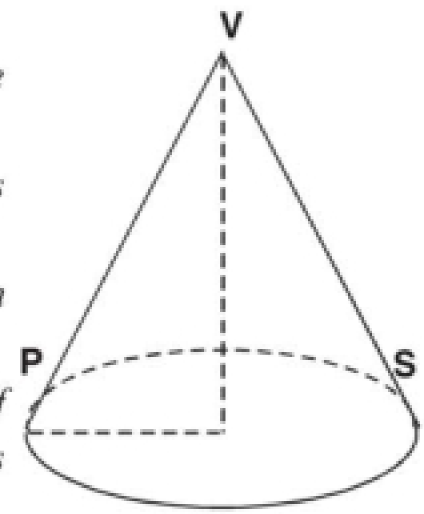


Fig. 1

Another Definition. *Let S be a set of points in space. If there exists a point V in S such that* $P \in S \Rightarrow \overleftrightarrow{VP} \subset S$ *then S is called the cone and V is said to be the vertex of the cone.*

$\overleftrightarrow{VP}$ *is called a generator of a cone.*

Note 1. If V is the vertex of the cone S and P is a point on 'S' then $\overleftrightarrow{VP}$ is a generator.

2. If L is a generator of the cone S then every point of L lies on S.

e.g. (*i*) The equation $2x^2+3y^2-z^2=0$ represents a cone with vertex as origin.

(*ii*) Intersecting pairs of planes form a cone with every point on the common line as vertex.

(*iii*) A plane is a cone with every point on it as vertex.

7. 2. Theorem. ***If $f(x, y, z)$ is a homogeneous polynomial of n^{th} degree then the surface S represented by $f(x, y, z)=0$ is a cone with vertex at the origin.***

Proof. Since $f(x, y, z)$ is a homogeneous polynomial of degree n, for a real number λ.

$f(\lambda x, \lambda y, \lambda z)=\lambda^n f(x, y, z)$.

Let P (x, y, z) be a point on the surface S. $\quad \therefore P \in S \Rightarrow f(x, y, z)=0$

$\Rightarrow \lambda^n f(x, y, z)=0 \Rightarrow f(\lambda x, \lambda y, \lambda z)=0$

$\Rightarrow$ every point on $\overleftrightarrow{OP}$ lies on the surface S. $\quad \therefore P \in S \Rightarrow \overleftrightarrow{OP} \subset S$.

Hence the homogeneous equation $f(x, y, z)=0$, represents a cone with vertex at the origin.

Corollary. The line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is $(\lambda l, \lambda m, \lambda n)$ where λ is a real number.

$\therefore \frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is a generator of the cone $\Leftrightarrow f(l, m, n)=0$

$\Leftrightarrow$ Any point on the generator $\in$ cone $\Leftrightarrow (\lambda l, \lambda m, \lambda n) \in$ cone

$\Leftrightarrow f(\lambda l, \lambda m, \lambda n)=0 \Leftrightarrow \lambda^n f(l, m, n)=0 \Leftrightarrow f(l, m, n)=0$

NOTE. If $f(x, y, z)$ is a homogeneous polynomial of degree n then the cone $f(x, y, z) = 0$ is called a cone of nth degree.

e.g. (*i*) $2x^3 - y^3 + 3x^2z + 2z^3 = 0$ is a cone of third degree.

(*ii*) $x^2 + y^2 - z^2 = 0$ is a cone of 2nd degree.

Cones of second degree are also called Quadric cones. In this chapter we deal with quadric cones only.

It will be seen that the degree of equation of a cone whose generators intersect a given conic or touch a given sphere is of the second degree.

7. 3. QUADRIC CONES WITH VERTEX AT THE ORIGIN

Theorem. ***The equation of a cone with vertex at the origin, is a homogeneous equation.***

Proof. Let the general equation of second degree

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$ represent cone S with vertex at (0, 0, 0).

Let P (x_1, y_1, z_1) be a point on the cone

$\therefore$ The equation of generator $\overleftrightarrow{OP}$ is $\frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1} (= \lambda)$

Any point $(\lambda x_1, \lambda y_1, \lambda z_1)$ on $\overleftrightarrow{OP}$ lies on the cone S.

$\Leftrightarrow \lambda^2(ax_1{}^2 + by_1{}^2 + cz_1{}^2 + 2fy_1z_1 + 2g_1z_1x_1 + 2hx_1y_1) + 2\lambda(ux_1 + vy_1 + wz_1) + d = 0$

This is true for all real values of λ.

$\Leftrightarrow ax_1{}^2 + by_1{}^2 + cz_1{}^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 = 0$ ...(*i*)

$ux_1 + vy_1 + wz_1 = 0$ ...(*ii*) $d = 0$ ...(*iii*)

The relation (*iii*) is obvious as the origin lies on the cone.

If $u, v, w,$ are not all zero, the equation (*ii*)

$\Rightarrow (\lambda x_1, \lambda y_1, \lambda z_1)$ lies on the plane $ux + vy + wz = 0$

Which is a contradiction. Thus we have $u = v = w = 0, d = 0$

Hence the equation to the cone S with vertex at the origin is given by the homogeneous equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ ...(I)

Conversely. Every homogeneous equation of the second degree represents a cone with its vertex at the origin.

Let the homogeneous equation of second degree be

$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

Let P (x_1, y_1, z_1) be a point on the cone.

Then $ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 = 0$ .... (I)

Multiplying by a real number λ^2 we have

$a\lambda^2.x_1^2 + b\lambda^2y_1^2 + c\lambda^2z_1^2 + 2f\lambda^2y_1z_1 + 2g\lambda^2z_1x_1 + 2h\lambda^2x_1y_1 = 0$

$\Rightarrow (\lambda x_1, \lambda y_1, \lambda z_1)$ lies on the surface.

Thus P lies on the surface $\Rightarrow$ Every point on OP lies on it.

$\therefore$ The surface is generated by lines through O and hence, by definition is a cone with its vertex at O.

Note. 1. Let $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

(i) If $\Delta = 0$, the equation I represents a pair of planes and S is called a degenerate cone.
(ii) If $\Delta \neq 0$, then the surface S is called a Quadric cone or a non-degenerate cone.

2. The equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ always represents a cone of second degree with its vertex at the origin.

3. As the above equation of the cone contains five arbitrary constants, we need five conditions to determine the cone.

4. The general equation of the cone with vertex at (α, β, γ) is

$$a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2 + 2f(y-\beta)(z-\gamma) + 2g(z-\gamma)(x-\alpha) + 2h(x-\alpha)(\gamma-\beta) = 0$$

This is a homogeneous equation in $(x-\alpha), (y-\beta)$ and $(z-\gamma)$.

SOLVED PROBLEMS

Ex. 1. *Find the equation of the cone whose generators pass through the point (α, β, γ) and have their direction cosines satisfying the relation $al^2 + bm^2 + cn^2 = 0$.*

Sol. Equation to the generator passing through (α, β, γ)

and having direction cosines (l, m, n) is $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n} = k$ (say)

$\Rightarrow l = \dfrac{x-\alpha}{k}, m = \dfrac{y-\beta}{k}$ and $n = \dfrac{z-\gamma}{k}$

But l, m, n satisfy $al^2 + bm^2 + cn^2 = 0 \Leftrightarrow \dfrac{1}{k}\left[(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2\right] = 0$

Hence the required to the cone is $(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = 0$

Ex. 2. *Show that $x = -y = -z$ is a generator of the cone $5yz + 8zx - 3xy = 0$.*

(*S.K.U. M18*)

Sol. $\dfrac{x}{1} = \dfrac{y}{-1} = \dfrac{z}{-1}$ is a generator of the cone $5yz + 8zx - 3xy = 0$ .... (1)

$\Leftrightarrow 5(-1)(-1) + 8(-1)(1) - 3(1)(-1) = 0 \Rightarrow 5 - 8 + 3 = 0$

Hence the give line is a generator of the cone (1)

EXERCISE 7 (*a*)

1. Find the equation of the cone whose vertex is at the origin and the direction cosines of whose generators satisfy the relation $3l^2 - 4m^2 + 5n^2 = 0$.
2. Show that the lines drawn through the point (α, β, γ) whose direction cosines satisfy $al^2 + bm^2 + cn^2 = 0$ generate the cone $a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2 = 0$.

ANSWERS

1. $3x^2 - 4y^2 + 5z^2 = 0$

7. 4. Theorem. ***Show that the general equation of the cone of the second degree which pass through the co-ordinate axes is $fyz+gzx+hxy=0$ (N.U.A.2006)***

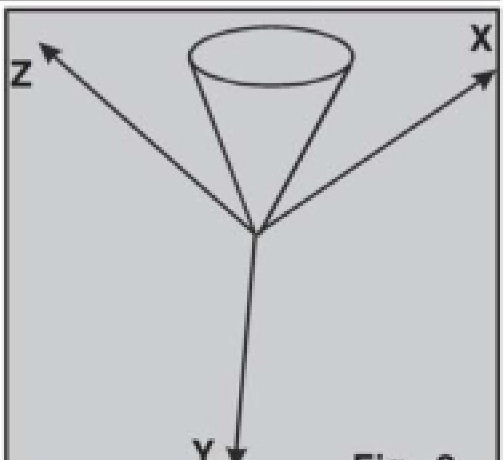


Fig. 2

Proof. The equation of the cone of the second degree is $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ ...(1)

X - axis is a generator of the cone.

$\Rightarrow$ the direction cocines (1, 0, 0) of the X - axis satisfies (1)

$\Rightarrow a=0$.

Similarly, Y - axis is a generator $\Rightarrow b=0$.

Similarly, Z - axis is a generator $\Rightarrow c=0$.

Hence the general equation of the cone containing the three axes is $fyz+gzx+hxy=0$

SOLVED PROBLEMS

Ex. 1. *Show that a cone can be found so as to contain any two given sets of three mutually perpendicular concurrent lines as generators.*

Sol. Let one set of three mutually perpendicular concurrent lines be taken as co-ordinate axes.

$\therefore$ The general equation of the cone through the coordinate axes is

$fyz+gzx+hxy=0$ **...**(1)

Let the other set of perpendicular lines be OA, OB, OC given by the equations

Let the other set of lines be OA, OB, OC given by the equations

$$\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1};\ \frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2};\ \frac{x}{l_3}=\frac{y}{m_3}=\frac{z}{n_3}$$

Then $m_1n_1+m_2n_2+m_3n_3=0$ ...(*i*) $n_1l_1+n_2l_2+n_3l_3=0$ ...(*ii*)

$l_1m_1+l_2m_2+l_3m_3=0$ ...(*iii*)

OA, OB, OC are the generators of cone

$\Rightarrow fm_1n_1+gn_1l_1+hl_1m_1=0$ ...(2) $fm_2n_2+gn_2l_2+hl_2m_2=0$ ...(3)

Adding (2) and (3) we get $f(m_1n_1+m_2n_2)+g(n_1l_1+n_2l_2)+h(l_1m_1+l_2m_2)=0$

i.e. $f(-m_3n_3)+g(-n_3l_3)+h(-l_3m_3)=0$ ...by (*iii*)

i.e. $fm_3n_3+gn_3l_3+hl_3m_3=0$

$\Rightarrow$ the line OC with direction ratio (l_3, m_3, n_3) lies on the cone $fyz+gzx+hxy=0$.

Hence the cone (1) contains two sets of mutually perpendicular generators.

Ex. 2. *Find the equation to the cone which passes through the three coordinate axes as well as the three lines* $\frac{1}{2}x=y=-z,\ x=\frac{1}{3}y=\frac{1}{5}z$ *and* $\frac{1}{8}x=-\frac{1}{11}y=\frac{1}{5}z$.

Sol. The equation to the cone passing through the three coordinate axes can be taken in the form $fyz+gzx+hxy=0$ .... (1)

The line $\frac{x}{2}=\frac{y}{1}=\frac{z}{-1}$ lies on (1)

$\Leftrightarrow f(1)(-1)+g(-1)(2)+h(2)(1)=0 \Rightarrow f+2g-2h=0$ .... (2)

The line $\frac{x}{1}=\frac{y}{3}=\frac{z}{5}$ lies on (1)

$\Leftrightarrow f(3)(5)+g(5)(1)+h(1)(3)=0 \Rightarrow 15f+5g+3h=0$ .... (3)

Solving (2) and (3) : $\frac{f}{6+10}=\frac{g}{-30-3}=\frac{h}{5-30} \Rightarrow \frac{f}{16}=\frac{g}{-33}=\frac{h}{-25}$

Hence the equation of the cone (1) is $16yz-33zx-25xy=0$

Clearly the line with d.r's $(8,-11,5)$ lies on it.

Ex. 3. *Find the equation of the cone which contains the three coordinate axes and the two lines through the origin with direction cosines* (l_1,m_1,n_1) *and* (l_2,m_2,n_2)

Sol. The equation to the cone containing the coordinate axes can be taken as

$fyz+gzx+hxy=0$ .... (1)

The two lines with d.c.'s (l_1,m_1,n_1) and (l_2,m_2,n_2) lie on cone (1)

$\Leftrightarrow fm_1n_1+gn_1l_1+hl_1m_1=0$ .... (2)

and $f(m_2n_2)+gn_2l_2+hl_2m_2=0$ .... (3)

Solving (2) and (3) : $\frac{f}{l_1l_2m_2n_1-l_1l_2m_1n_2}=\frac{g}{m_1m_2(n_2l_1)-m_1m_2n_1l_2}=\frac{h}{n_1n_2l_2m_1-n_1n_2l_1m_2}$

$$\Rightarrow \frac{f}{l_1l_2(m_2n_1-m_1n_2)}=\frac{g}{m_1m_2(n_2l_1-n_1l_2)}=\frac{h}{n_1n_2(l_2m_1-l_1m_2)}$$

Substituting in (1) the required equation of the cone is

$$l_1l_2(m_2n_1-m_1n_2)yz+m_1m_2(n_2l_1-n_1l_2)zx+n_1n_2(l_2m_1-l_1m_2)xy=0$$

$$\Rightarrow \sum l_1l_2(m_1n_2-m_2n_1)yz=0$$

EXERCISE 7 (*b*)

1. Find the equation to the cone which passes through the three coordinte axes and the lines $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ and $\frac{x}{2}=\frac{y}{1}=\frac{z}{1}$ ***(A. U. A' 08, S. V. U. A'06, A11, A.K.U M18, S.U.M M18)***

2. Find the equation of the quadric cone through the coordinate axes and the three lines $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$, $\frac{x}{-1}=\frac{y}{1}=\frac{z}{1}$ and $\frac{x}{5}=\frac{y}{4}=\frac{z}{1}$. ***(O. U. A12)***

3. Find the equation of the quadric cone which passes through the coordinate axes and the three lines $\frac{x}{3}=\frac{y}{5}=\frac{z}{1}$, $\frac{x}{1}=\frac{y}{-1}=\frac{z}{2}$ and $\frac{x}{-11}=\frac{y}{5}=\frac{z}{8}$.

ANSWERS

1. $3yz+16zx+15xy=0$ **2.** $5yz+8zx-3xy=0$

3. $33yz+25zx-16xy=0$

7. 5. CONE AND A PLANE THROUGH ITS VERTEX.

Find the angle between the lines of intersection of the plane $px+qy+ry=0$ and the cone $F(x,y,z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

Sol. Let a line of intersection of the plane with the cone be $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.

$\therefore$ The line lies in the given plane as well as on the cone $\Leftrightarrow pl+qm+rn=0$ and $al^2+bm^2+cn^2+2fmn+2gnl+2hlm=0$ .... (2)

Now substituting $n=-\frac{pl+qm}{r}$ in (2) we have

$$al^2+bm^2+c\left(\frac{pl+qm}{r}\right)^2+(2fm+2gl)\left(\frac{-pl+qm}{r}\right)+2hlm=0$$

$$\Rightarrow l^2(cp^2+ar^2-2grp)+2lm(cpq+hr^2-gpr-frp)+m^2(br^2+cq^2-2fqr)=0$$

$$\Rightarrow \frac{l^2}{m^2}(cp^2+ar^2-2grp)+\frac{2l}{m}(cpq+hr^2-gpr-frp)+(br^2+cq^2-2fqr)=0 \quad \text{... (3)}$$

This is a quadratic equation in $\frac{l}{m}$ and shows that the plane cuts the cone in two times. If (l_1,m_1,n_1) and (l_2,m_2,n_2) are the d.c's of the two lines then $\frac{l_1}{m_1},\frac{l_2}{m_2}$ are the roots of (3)

$$\frac{l_1}{m_1}\cdot\frac{l_2}{m_2}=\frac{br^2+cq^2-2fqr}{cp^2+ar^2-2grp}$$

$$\Rightarrow \frac{l_1l_2}{br^2+cq^2-2fqr}=\frac{m_1m_2}{cp^2+ar^2-2grp}=\frac{n_1n_2}{aq^2+bp^2-2hpq} \quad \text{by symmetry}$$

$$\text{each } = \frac{l_1l_2+m_1m_2+n_1n_2}{(b+c)p^2+(c+a)q^2+(a+b)r^2-2fqr-2grp-2hpq}$$

Also sum of the the roots of (3) is $\frac{l_1}{m_1}+\frac{l_2}{m_2}=-\frac{2(cpq+hr^2-gpr-frp)}{cp^2+ar^2-2grp}$

$$\Rightarrow \frac{l_1m_2+l_2m_1}{-2(cpq+hr^2-gpr-frp)}=\frac{m_1m_2}{cp^2+ar^2-2grp}=\frac{l_1l_2}{br^2+cq^2-2fqr}=\frac{n_1n_2}{aq^2+bp^2-2hpq}$$

$$\text{each } = \frac{[(l_1m_2+l_2m_1)^2-4l_1l_2m_1m_2]^{1/2}}{[4(cpq-gpr-frp+hr^2)-4(br^2+cq^2-2fqr)(cp^2+ar^2-2grp)]^{1/2}}$$

$$\Rightarrow \frac{l_1m_2-l_2m_1}{\pm 2rD} \quad \text{where } D^2=\begin{vmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & 0 \end{vmatrix}$$

$$\text{By symmetry } \frac{l_1m_2-l_2m_1}{\pm 2r\,D}=\frac{m_1n_2-m_2n_1}{\pm 2p\,D}=\frac{n_1l_2-n_2l_1}{\pm 2q\,D}=\frac{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}}{\pm 2D\sqrt{p^2+q^2+r^2}}$$

Let θ be the angle between the lines then $\tan\theta = \dfrac{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}{l_1l_2 + m_1m_2 + n_1n_2}$

$$\Rightarrow \tan\theta = \frac{2D\sqrt{(p^2+q^2+r^2)}}{(a+b+c)(p^2+q^2+r^2) - F(p,q,r)}$$

Cor. Condition of Perpendicularity.

If the lines of intersection of the plane $px+qy+rz=0$ and the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ is a right angle then $\theta = 90° \Rightarrow \tan\theta = \tan 90° = \infty$

$\Rightarrow (a+b+c)(p^2+q^2+r^2) - F(p,q,r) = 0$ which is the required condition.

SOLVED PROBLEMS

Ex. 1. *Find the equation of the lines of intersection of the plane* $2x+y-z=0$ *and the cone* $4x^2-y^2+3z^2=0$ **(A. N. U. M15, S. K. U. A11, S.U.M M18)**

Sol. Let a line intersection of the plane with cone be $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$ .... (1)

The line (1) lies on the plane and the cone $\Leftrightarrow 2l+m-n=0$ .... (2)

and $4l^2-m^2+3n^2=0$ .... (3) $\therefore\ n = 2l+m$

Substituting the value of n in (3) : $4l^2-m^2+3(2l+m)^2=0 \Rightarrow 8l^2+6lm+m^2=0$

$\Rightarrow (2l+m)(4l+m)=0 \Rightarrow 2l+m=0$ .... (4) and $4l+m=0$ ..... (5)

(*i*) when $2l+m=0$, we have from (2) : $n=0$

$\Rightarrow \dfrac{l}{1} = \dfrac{m}{-2} = \dfrac{n}{0}$

(*ii*) Solving $2l+m-n=0$ .... (2) and $4l+m+0.n=0$ .... (3)

$$\frac{l}{0+1} = \frac{m}{-4-0} = \frac{n}{2-4} \Rightarrow \frac{l}{1} = \frac{m}{-4} = \frac{n}{-2}$$

Hence the two lines are $\dfrac{x}{1} = \dfrac{y}{-2} = \dfrac{z}{0}$ and $\dfrac{x}{1} = \dfrac{y}{-4} = \dfrac{z}{-2}$

Ex. 2. *Find the equations of the lines of intersecton of the plane* $3x+4y+z=0$ *and the cone* $15x^2-32y^2-7z^2=0$.

Sol. Let a line of intersection be $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$

The line belongs to $3x+4y+z=0 \Leftrightarrow 3l+4m+n=0$ .... (1)

$\Rightarrow n = -(3l+4m)$

Also the line lies on the given cone $15x^2-32y^2-7z^2=0$

$\Leftrightarrow 15l^2-32m^2-7n^2=0$ .... (2)

Substituting the value of n in (2) : $15l^2-32m^2-7(3l+4m)^2=0$

$\Rightarrow 2l^2-7lm+6m^2=0 \Rightarrow (l+2m)(2l+3m)=0$

$\Rightarrow l+2m=0$ .... (3) and $2l+3m=0$ .... (4)

(*i*) Solving $3l+4m+n=0$ .... (1) and $l+2m+0.n=0$ .... (3)

$$\frac{l}{0-2}=\frac{m}{1-0}=\frac{n}{6-4}\Rightarrow\frac{l}{-2}=\frac{m}{1}=\frac{n}{-2}$$

(*ii*) Again solving $3l+4m+n=0$ .... (1) and $2l+3m+0.n=0$ ... (4)

$$\frac{l}{0-3}=\frac{m}{2-0}=\frac{n}{9-8}\Rightarrow\frac{l}{-3}=\frac{m}{2}=\frac{n}{1}$$

$\therefore$ The two lines of intersection are $\frac{x}{-2}=\frac{y}{1}=\frac{z}{-2}$ and $\frac{x}{-3}=\frac{y}{2}=\frac{z}{1}$

Ex. 3. *Find the angle between the lines of intersection of the plane* $x-3y+z=0$ *and the cone* $x^2-5y^2+z^2=0$. **(*K. U. M15, A. U. A12, N. U. A10, S. V. U. A' 06*)**

Sol. Given cone $x^2-5y^2+z^2=0$. and the plane $x-3y+z=0$.

Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be one of the common lines of the cone and the plane.

$\therefore\ l^2-5m^2+n^2=0$ ...(1) and $l-3m+n=0\Rightarrow\frac{l+n}{3}=m$ ...(2)

Substituting (2) in (1) $l^2-5\left(\frac{l+n}{3}\right)^2+n^2=0$

$\Rightarrow\ 2l^2-5l\,n+2n^2=0\Rightarrow(2l-n)(l-2n)=0$

$\Rightarrow\ 2l-n=0$ ...(3) and $l-2n=0$ ...(4)

Now solving (2) and (3) *i.e.* $l-3m+n=0$

$2l+0.m-n=0$

we have $\frac{l}{3-0}=\frac{m}{2+1}=\frac{n}{0+6}\Rightarrow\frac{l}{1}=\frac{m}{1}=\frac{n}{2}$

Again solving (2) and (4) *i.e.* $l-3m+n=0$, $l+0.m-2n=0$

We have $\frac{l}{6-0}=\frac{m}{1+2}=\frac{n}{0+3}\Rightarrow\frac{l}{2}=\frac{m}{1}=\frac{n}{1}$

Hence the direction ratios of the two lines of intersection are (1, 1, 2) and (2, 1, 1).

$\therefore$ The equations of the lies of intersection are $\frac{x}{1}=\frac{y}{1}=\frac{z}{2}$ and $\frac{x}{2}=\frac{y}{1}=\frac{z}{1}$

If θ is the angle between the lines, then $\cos\theta=\frac{1(2)+1(1)+2(1)}{\sqrt{(1+1+4)}.\sqrt{(4+1+1)}}=\frac{5}{6}$

$$\Rightarrow\theta=\cos^{-1}\frac{5}{6}$$

Ex. 4. *Show that the equation of the quadric cone which contains the three coordinate axes and the lines in which the plane* $x-5y-3z=0$ *cuts the cone* $7x^2+5y^2-3z^2=0$ *is* $yz+10zx+18xy=0$.

Sol. Let the plane $x-5y-3z=0$ cut the cone $7x^2+5y^2-3z^2=0$

along the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$

$\Rightarrow l-5m-3n=0$ .... (1) and $7l^2+5m^2-3n^2=0$ .... (2)

Eliminating l from (1) and (2) : $7(5m+3n)^2+5m^2-3n^2=0$

$\Rightarrow 6m^2+7mn+2n^2=0 \Rightarrow (2m+n)(3m+2n)=0$

$\Rightarrow 2m+n=0$ .... (3) and $3m+2n=0$ .... (4)

(*i*) Solving $l-5m-3n=0$ .... (1) and $l.0+2m+n=0$ .... (3)

$$\frac{l}{-5+6}=\frac{m}{0-1}=\frac{n}{2-0} \Rightarrow \frac{l}{1}=\frac{m}{-1}=\frac{n}{2}$$

(*ii*) Solving $l-5m-3n=0$ .... (1) and $l.0+3m+2n=0$ ... (4)

$$\frac{l}{-10+9}=\frac{m}{0-2}=\frac{n}{3-0} \Rightarrow \frac{l}{-1}=\frac{m}{-2}=\frac{n}{3}$$

$\therefore$ The two lines of intersection are $\frac{x}{1}=\frac{y}{-1}=\frac{z}{2}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{-3}$

Let the cone containing the coordinate axes be $fyz+gzx+hxy=0$ .... (5)

The two above lines are the generators of (5)

$\Leftrightarrow f(2)(-1)+g(2)(1)+h(1)(-1)=0 \Rightarrow 2f-2g+h=0$ .... (6)

and $f(2)(-3)+g(-3)(1)+h(1)(2)=0 \Rightarrow 6f+3g-2h=0$ .... (7)

Solving (6) and (7) : $\frac{f}{4-3}=\frac{g}{6+4}=\frac{h}{6+12} \Rightarrow \frac{f}{1}=\frac{g}{10}=\frac{h}{18}$

Hence the required cone is $yz+10zx+18xy=0$

Ex. 5. *Prove that the angle between the lines of intersection of the plane* $x+y+z=0$ *with the cone* $ayz+bzx+cxy=0$ *is* $\pi/3$ *if* $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$. ***(S.V. U. M15, S.V.M M18, A.U M18)***

Sol. Let a line of intersection be $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ .... (1)

(1) lies on the given plane and cone

$\Leftrightarrow l+m+n=0$ .... (2) and $amn+bnl+clm=0$ .... (3)

Eliminating n from (2) and (3)

$-am(l+m)-bl(l+m)+clm=0 \Rightarrow bl^2+(a+b-c)lm+am^2=0$

$$\Rightarrow b\left(\frac{l}{m}\right)^2+(a+b-c)\frac{l}{m}+a=0$$

This is a quadratic in $\frac{l}{m}$. Let the roots be $\frac{l_1}{m_1}, \frac{l_2}{m_2}$

$\therefore \frac{l_1}{m_1}\cdot\frac{l_2}{m_2}=\frac{a}{b} \Rightarrow \frac{l_1l_2}{a}=\frac{m_1m_2}{b}=\frac{n_1n_2}{c}$ by symmetry

each $=\frac{l_1l_2+m_1m_2+n_1n_2}{a+b+c}=k$

Again $\frac{l_1}{m_1}+\frac{l_2}{m_2}=\frac{c-b-a}{b} \Rightarrow \frac{l_1m_2+l_2m_1}{c-b-a}=\frac{m_1m_2}{b}=k$ (say)

Now $(l_1m_2-l_2m_1)^2=(l_1m_2+l_2m_1)^2-4l_1l_2m_1m_2$

$$=k^2(c-b-a)^2-4\,(ak)\,(bk)=k^2[(c-b-a)^2-4ab]$$

$$=k^2(a^2+b^2+c^2-2ab-2bc-2ca)$$

Now $\tan\theta=\frac{\sqrt{\Sigma(l_1m_2-l_2m_1)^2}}{l_1l_2+m_1m_2+n_1n_2}=\frac{\sqrt{3k^2(a^2+b^2+c^2-2bc-2ca-2ab)}}{k\,(a+b+c)}$

If $\theta=\frac{\pi}{3}$, $\tan^2\frac{\pi}{3}=(\sqrt{3})^2=\frac{3\,(a^2+b^2+c^2-2bc-2ac-2ab)}{(a+b+c)^2}$

$\Rightarrow (a+b+c)^2=a^2+b^2+c^2-2bc-2ca-2ab$

$\Rightarrow 4(bc+ca+ab)=0 \Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$

Ex. 6. *Prove that if the angle between the lines of intersection of the plane* $x+y+z=0$ *and the cone* $ayz+bzx+cxy=0$ *is* $\frac{\pi}{2}$ *then* $a+b+c=0$.

Sol. Let a line of intersection be $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ ...(1)

(1) lies on the cone and the plane $\Leftrightarrow l+m+n=0$ ...(2)

$amn+bnl+clm=0$ ...(3)

Substituting $n=-l-m$ in (3)

$-am\,(l+m)-bl\,(l+m)+clm=0 \Rightarrow -alm-am^2-bl^2-blm+clm=0$

$\Rightarrow bl^2+(a+b-c)\,lm+am^2=0 \Rightarrow b\left(\frac{l}{m}\right)^2+(a+b-c)\frac{l}{m}+a=0$

This is a quadratic in $\frac{l}{m}$. Let the roots be $\frac{l_1}{m_1},\frac{l_2}{m_2}$

$\therefore \left(\frac{l_1}{m_1}\right)\left(\frac{l_1}{m_2}\right)=\frac{a}{b} \Rightarrow \frac{l_1\,l_2}{a}=\frac{m_1\,m_2}{b}$

By symmetry : $\frac{l_1\,l_2}{a}=\frac{m_1\,m_2}{b}=\frac{n_1\,n_2}{c}=K$ (say)

the angle between the lines with $d\,cs$ (l_1,m_1,n_1) and (l_2,m_2,n_2) is $\frac{\pi}{2}$

$\Leftrightarrow l_1l_2+m_1\,m_2+n_1\,n_2=0 \quad \Rightarrow ka+kb+kc=0 \quad \Rightarrow a+b+c=0$.

EXERCISE 7 (*c*)

1. Find the equations of the lines of intersection of the plane and the cone given below

   (*a*) $x+3y-2z=0,\ x^2+9y^2-4z^2=0$ (*b*) $x+7y-5z=0,\ 3yz+14zx-30xy=0$

2. Find the lines of intersection and angle between them.

   (*a*) $3x+y+5z=0,\ 6yz-2zx+5xy=0$ (*b*) $x+y+z=0, 6xy+3yz-2zx=0$ **(*K. U. M15*)**

(c) $10x+7y-6z=0, 20x^2+7y^2-108z^2=0$ (d) $x+y+z=0, x^2-yz+xy-3z^2=0$

(e) $4x-y-5z=0, 8yz+3zx-5xy=0$

3. Show that the condition that the plane $ux+vy+wz=0$ may cut the cone $ax^2+by^2+cz^2=0$ in perpendicular generators is $(b+c)u^2+(c+a)v^2+(a+b)w^2=0$.

4. Prove that the plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.

5. Prove that the plane $lx+my+nz=0$ cuts the cone $(b-c)x^2+(c-a)y^2+(a-b)z^2+2fyz+2gzx+2hxy=0$ in perpendicular lines if $(b-c)l^2+(c-a)m^2+(a-b)n^2+2fmn+2gnl+2hlm=0$.

6. Find the angle between the lines of intersection of the plane $x+y+z=0$ and the cone $\frac{yz}{b-c}+\frac{zx}{c-a}+\frac{xy}{a-b}=0$

ANSWERS

1. (a) $\frac{x}{2}=\frac{y}{0}=\frac{z}{1}, \frac{x}{0}=\frac{y}{2}=\frac{z}{3}$ (b) $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}, \frac{x}{3}=\frac{y}{1}=\frac{z}{2}$

2. (a) $\text{Cos}^{-1}(1/6)$ (b) $\pi/3$ (c) $\text{Cos}^{-1}(16/21)$ (d) $\pi/6$ (e) $\pi/2$ **6.** $\pi/3$

7. 6. CONE WITH A BASE CURVE

Definition. *Let S be the set of lies concurrent at V and C be a curve not containing V.*

If $P \in C \Rightarrow \overrightarrow{VP} \subset S$ then S is called the cone with vertex at V. C is called the base curve or guiding curve. $\overrightarrow{VP}$ is called a generator of the cone.

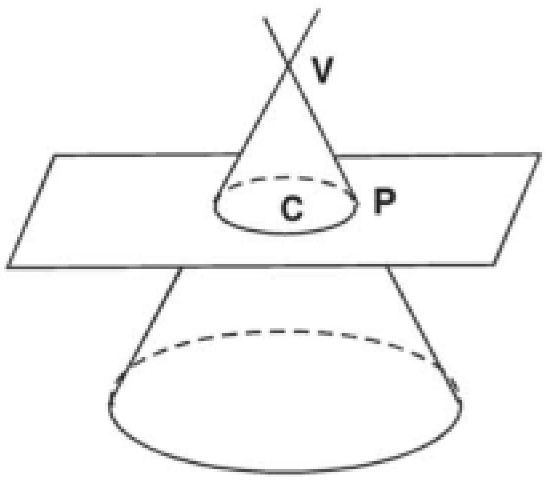


Fig. 3

7. 7. Theorem. ***The equation of a cone with vertex at $(\alpha, \beta, \gamma) \notin XY$ plane and the guiding curve $f(x, y)=0, z=0$ is***

$$(z-\gamma)^2 \cdot f\left(\alpha-\gamma\frac{X-\alpha}{z-\gamma}, \beta-\gamma\frac{y-\beta}{z-\gamma}\right)=0$$

Proof. Let the equation to a line through (α, β, γ) with direction ratios (l, m, n) be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad (=r) \qquad \text{.... (1)}$$

A point on the line is $(lr+\alpha, mr+\beta, nr+\gamma)$.

Let the equation to the curve be $f(x, y)=ax^2+2hxy+by^2+2gx+2fy+c=0,\ z=0$

The line passes through the conic.

$\Leftrightarrow$ The point $P(lr+\alpha, mr+\beta, nr+\gamma)$ lies on the $f(x,y)=0$ and on the plane $z=0$.

$z=0 \Rightarrow nr+\gamma \Rightarrow r=-\dfrac{\gamma}{n}$

Hence the point $P=\left(\alpha-\dfrac{l\gamma}{n}, \beta-\dfrac{m\gamma}{n}, 0\right)$ which will lie on the given conic,

$$\Leftrightarrow a\left(\alpha-\frac{l\gamma}{n}\right)^2+2h\left(\alpha-\frac{l\gamma}{n}\right)\left(\beta-\frac{m\gamma}{n}\right)+b\left(\beta-\frac{m\gamma}{n}\right)^2+2g\left(\alpha-\frac{l\gamma}{n}\right)+2f\left(\beta-\frac{m\gamma}{n}\right)+c=0 \quad \text{.... (3)}$$

This is the condition for the line to intersect the conic.

Now eliminating l, m, n between (1) and (3)

$$a\left(\alpha-\frac{x-\alpha}{z-\gamma}.\gamma\right)^2+2h\left(\alpha-\frac{x-\alpha}{z-\gamma}.\gamma\right)\left(\beta-\frac{y-\alpha}{z-\gamma}.\gamma\right)$$
$$+b\left(\beta-\frac{y-\beta}{z-\gamma}.\gamma\right)^2+2g\left(\alpha-\frac{x-\alpha}{z-\gamma}.\gamma\right)+2f\left(\beta-\frac{y-\beta}{z-\gamma}.\gamma\right)+c=0$$

$$\Rightarrow a(\alpha z-x\gamma)^2+2h(\alpha z-x\gamma)(\beta z-\gamma y)+b(\beta z-\gamma y)^2$$
$$+2g(\alpha z-\gamma x)(z-\gamma)+2f(\beta z-\gamma y)(z-\gamma)+c(z-\gamma)^2=0$$

$(z-\gamma)^2.f\left(\alpha-\dfrac{x-\alpha}{z-\gamma}.\gamma, \beta-\dfrac{y-\beta}{z-\gamma}.\gamma\right)=0$ which is the required equation of the conic.

Note. The guiding curve of the cone may be $f(y,z)=0, x=0$ or $f(z,x)=0, y=0$

SOLVED PROBLEMS

Ex. 1. *Find the equation of the cone whose vertex is the origin and whose base curve is* $x^2+y^2+z^2+2ux+d=0$

$px+qy+rz=k$ ...(C) ***(A. U. A11)***

Sol. Let $P(x_1, y_1, z_1)$ be a point on the cone.

$\therefore$ The equation to $\overleftrightarrow{OP}$ is $\dfrac{x}{x_1}=\dfrac{y}{y_1}=\dfrac{z}{z_1}$ $(=r)$. A point on $\overleftrightarrow{OP}$ is $(\lambda x_1, \lambda y_1, \lambda z_1)$.

$\overleftrightarrow{OP}$ intersects the base curve $C \Rightarrow (\lambda x_1, \lambda y_1, \lambda z_1) \in C$

$\Leftrightarrow \lambda^2(\lambda x_1{}^2, \lambda y_1{}^2, \lambda z_1{}^2)+2u\lambda\, x_1+d=0$ and $\lambda(px_1+qy_1+rz_1)=k$.

Eliminating λ from the above two relations.

We have $k^2(x_1{}^2+y_1{}^2+z_1{}^2)+2u\,k\,x_1(px_1+qy_1+rz_1)+d(px_1+qy_1+rz_1)^2=0$

Hence the equation to the locus of p is the curve

$k^2(x^2+y^2+z^2)+2u\,k\,x(px+qy+rz)+d(px+qy+rz)^2=0$

Alternative method :

Sol. The given curve is $x^2+y^2+z^2+2u\,x+d=0$ ...(1)

$px+qy+rz=k$ ...(2)

The required equation is the homogenous equation of second degree satisfied by the points common to the two equations.

$\therefore\ px+qy+rz=k$. $\qquad \dfrac{px+qy+rz}{k}=1 \qquad ...(3)$

Now homogeneuising the equation of (1) by (3) we have the homogeneous equation

$$x^2+y^2+z^2+2u\,x\left(\frac{px+qy+rz}{k}\right)+d\left(\frac{px+qy+rz}{k}\right)^2=0$$

Thus the equation to the homogeneous cone is

$$k^2\,(x^2+y^2+z^2)+2ukx\,(px+qy+rz)+d\,(px+qy+rz)^2=0$$

Ex. 2. *Find the equation of the cone whose vertex is (1, 1, 0) and whose guiding curve is* $y=0,\ x^2+z^2=4$ ***(A. U. A12)***

Sol. Let the equation to the generator through the vertex (1, 1, 0) be

$$\frac{x-1}{l}=\frac{y-1}{m}=\frac{z}{n}\ (=r) \qquad ...(1)$$

A point on the generator is $(lr+1, mr+1, nr)$

If this point lies on the curve $y=0,\ x^2+z^2=4$, then $mr+1=0,\ (lr+1)^2+(nr)^2=4$

i.e., $\quad r=-\dfrac{1}{m},\ (lr+1)^2+n^2r^2=4$

Eliminating r $\left(1-\dfrac{l}{m}\right)^2+\dfrac{n^2}{m^2}=4 \Rightarrow (m-l)^2+n^2=4m^2 \qquad ...(2)$

Eliminating l, m, n from (2) by using (1)

$$[(y-1)+(x-1)]^2+z^2=4\,(y-1)^2 \Rightarrow x^2-3y^2+z^2-2xy+8y-4=0$$

which is the equation to the required cone.

Ex. 3. *Find the equation of the cone with vertex* $(5,4,3)$ *and* $3x^2+2y^2=6, y+z=0$ *as base .* ***(S.K.U. M18)***

Sol. Let the equation to the generator be $\dfrac{x-5}{l}=\dfrac{y-4}{m}=\dfrac{z-3}{n}=k \qquad ...\ (1)$

Any point on (1) is $(lk+5, mk+4, nk+3)$

This point lies on the base $\Leftrightarrow 3\,(lk+5)^2+2\,(mk+4)^2=6 \qquad ...\ (2)$

and $mk+4+nk+3=0 \Rightarrow k\,(m+n)=-7 \qquad ...\ (3)$

Substituting (3) in (2) : $3\left(5-\dfrac{7l}{m+n}\right)^2+2\left(4-\dfrac{7m}{m+n}\right)^2=6$

$$\Rightarrow 3\,(5m+5n-7l)^2+2\,(4m+4n-7m)^2=6\,(m+n)^2$$

$$\Rightarrow 3\,[5\,(y-4)+5\,(z-3)-7\,(x-5)]^2+2\,[4\,(z-3)-3\,(y-4)]^2=6\,(y-4+z-3)^2$$

$$\Rightarrow 3\,(-7x+5y+5z)^2+2\,(-3y+4z)^2=6\,(y+z-3)^2$$

$$\Rightarrow 147x^2+87y^2+101z^2-210xy+90yz-210xy-294=0$$

Ex. 4. *Obtain the locus of the lines which pass through a point* (α, β, γ) *and through the points of the conic* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$

Sol. Let the equation line through (α,β,γ) be $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=k$ .... (1)

Any point on the line is $(\alpha+lk, \beta+mk, \gamma+nk)$

The point lies on the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$

$$\Leftrightarrow \frac{(\alpha+lk)^2}{a^2}+\frac{(\beta+mk)^2}{b^2}=1 \text{ and } \gamma+nk=0 \Rightarrow k=-\frac{\gamma}{n}$$

Substituting the value of k we have

$$\frac{1}{a^2}\left(\alpha-\frac{l\gamma}{n}\right)^2+\frac{1}{b^2}\left(\beta-\frac{m\gamma}{n}\right)^2=1 \Rightarrow \frac{(\alpha n-\gamma l)^2}{a^2n^2}+\frac{(\beta n-m\gamma)^2}{b^2n^2}=1$$

Eliminating l, m, n using (1)

$$\frac{1}{n^2a^2k^2}[\alpha(z-\gamma)-\gamma(x-\alpha)]^2+\frac{1}{n^2b^2k^2}[\beta(z-\gamma)-\gamma(y-\beta)]^2=1$$

$$\Rightarrow \frac{(\alpha z-\gamma x)^2}{a^2}+\frac{(\beta z-\gamma y)^2}{b^2}=(z-\gamma)^2$$

Ex. 5. *Find the equation of the cone whose vertex is* $(1,2,3)$ *and base* $y^2=4ax,\ z=0$.

(*A. N. U. A12, S. K. U. A11, K.U. M18*)

Sol. Let a line through $(1,2,3)$ be $\frac{x-1}{l}=\frac{y-2}{m}=\frac{z-3}{n}=k$ .... (1)

Any point on the line is $(1+lk, 2+mk, 3+nk)$

It lies on the given conic $\Leftrightarrow (2+mk)^2=4a(1+lk)$ and $3+nk=0 \Rightarrow k=-3/n$

Eliminating k we have $\left(2-\frac{3m}{n}\right)^2=4a\left(1-\frac{3l}{n}\right) \Rightarrow (2n-3m)^2=4a(n-3l)\,n$

Using (1) we get $[2(z-3)-3(y-2)]^2=4a[z-3-3(x-1)](z-3)$

$$\Rightarrow (2z-3y)^2=4a(y-3x)^2(z-3)$$

Ex. 6. *The section of a cone whose vertex is P and guiding curve the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$ *by the plane* $z=0$ *is a rectangular hyperbola. Show that the locus of P is* $\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$

Sol. Let the point $P=(x_1, y_1, z_1)$

Equation to a line through P be $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=k$

Any point on the line is $(x_1+lk,\ y_1+mk,\ z_1+nk)$

This lies on the conic

$$\Leftrightarrow \frac{(x_1+lk)^2}{a^2}+\frac{(y_1+mk)^2}{b^2}=1,\ z_1+nk=0 \Rightarrow k=-\frac{z_1}{n}$$

Eliminating k we have

$$\frac{1}{a^2}\left[x_1-\frac{lz_1}{n}\right]^2+\frac{1}{b^2}\left[y_1-\frac{mz_1}{n}\right]^2=1 \Rightarrow \frac{(nx_1-lz_1)^2}{a^2}+\frac{(ny_1-mz_1)^2}{b^2}=n^2$$

$$\Rightarrow \frac{1}{a^2}[x_1(z-z_1)-z_1(x-x_1)]^2+\frac{1}{b^2}[y_1(z-z_1)-z_1(y-y_1)]^2=(z-z_1)^2$$

$$\frac{(zx_1-xz_1)^2}{a^2}+\frac{(zy_1-yz_1)^2}{b^2}=(z-z_1)^2$$

Now this meets $x=0$ in a curve $\dfrac{z^2x_1^2}{a^2}+\dfrac{(zy_1-yz_1)^2}{b^2}=(z-z_1)^2,\ x=0$

This will be a rectangular hyperbola $\Leftrightarrow$ co. eft. of y^2+ co.eft of $z^2=0$

$\Rightarrow \dfrac{z_1^2}{b^2}+\left(\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}-1\right)=0$. Hence the locus of P is $\dfrac{x^2}{a^2}+\dfrac{y^2+z^2}{b^2}=1$

Ex. 7. *A cone has as base the circle* $x^2+y^2+2ax+2by=0, z=0$ *and passes through the fixed point* $(0,0,c)$. *If the section of the cone by ZX plane is a rectangular hyperbola, Prove that the vertex lies on a fixed circle.*

Sol. Let $P(x_1,y_1,z_1)$ be the vertex of the cone and base curve

$$x^2+y^2+2ax+2by=0,\ z=0$$

The line through P, $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$ .... (1)

meets the plane $z=0$ at the point given by

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \Rightarrow \text{ at } \left(x_1-\frac{lz_1}{n}, y_1-\frac{mz_1}{n}, 0\right)$$

This point lies on the circle $\Leftrightarrow \left(x_1-\dfrac{lz_1}{n}\right)^2+\left(y_1-\dfrac{mz_1}{n}\right)+2a\left(x_1-\dfrac{lz_1}{n}\right)+2b\left(y_1-\dfrac{mz_1}{n}\right)=0$

Eliminatig (l,m,n) using (1)

$$\left(x_1-\frac{x-x_1}{z-z_1}.\,z_1\right)^2+\left(y_1-\frac{y-y_1}{z-z_1}.\,z_1\right)^2+2a\left(x_1-\frac{x-x_1}{z-z_1}.\,z_1\right)+2b\left(y_1-\frac{y-y_1}{z-z_1}.\,z_1\right)=0$$

$$\Rightarrow (x_1z-xz_1)^2+(y_1z-yz_1)^2+2a\,(x_1z-xz_1)\,(z-z_1)+2b\,(y_1z-yz_1)(z-z_1)=0 \quad ...\ (1)$$

This cone passes through $(0,0,c)$

$$\Leftrightarrow (x_1c)^2+(y_1c)^2+2a\,(x_1c)\,(c-z_1)+2b\,(y_1c)\,(c-z_1)=0$$

$\Rightarrow (x_1^2 + y_1^2)\,c + (2ax_1 + 2by_1)\,(c - z_1) = 0$ .... (2)

Again the section of the cone (1) by the plane $y = 0$ is

$$(x_1 z - x z_1)^2 + (y_1 z)^2 + 2a\,(x_1 z - x z_1)\,(z - z_1) + 2by_1 z\,(z - z_1) = 0 \quad .... (3)$$

(3) represents a rectangular hyperbola

$\Leftrightarrow$ co. eft. of $z^2 +$ co.eft of $y^2 = 0 \Rightarrow x_1^2 + y_1^2 + 2ax_1 + 2by_1 + z_1^2 = 0$ .... (4)

Locus of P is given by (2) and (4) as $(x^2 + y^2)\,c + (2ax + 2by)\,(c - z) = 0$ .... (5)

and $x^2 + y^2 + z^2 + 2ax + 2by = 0$ .... (6)

Multiplying (6) by c and subtracting from (5), we get

$$cz^2 + 2azx + 2byz = 0 \Rightarrow 2ax + 2by + cz = 0 \quad .... (7)$$

Hence $x^2 + y^2 + z^2 + 2ax + 2by = 0$ and $2ax + 2by + cz = 0$ together represent a circle

EXERCISE 7 (*d*)

1. Find the equation to the cone with vertex at the origin and whose base curve is

(*a*) $z = 2, x^2 + y^2 = 4$ (*b*) $x^2 + y^2 + z^2 + x - 2y + 3z = 4, x^2 + y^2 + z^2 + 2x - 3y + 4z = 5$

(*c*) $x^2 + y^2 - z^2 - 2x + 1 = 0, z = 3$ (*d*) $ax^2 + by^2 + cz^2 = 1, lx + my + nz = p$

2. Show that the equation of the cone whose vertex is the origin and whose base is the circle through three points $(a,0,0),(0,b,0),(0,0,c)$ is $\sum a\,(b^2 + c^2)yz = 0$.

3. Show that the equation of the cone with vertex at the origin and base curve $z = k, f(x,y) = 0$ is $f\left(\frac{xk}{z}, \frac{yk}{z}\right) = 0$.

4. Find the equation of the cone with vertex at

(*a*) $(-1,1,2)$ guiding curve $3x^2 - y^2 = 1, z = 0$ (*b*) $(1,1,1)$ guiding curve $x^2 + y^2 = 4, z = 2$

(*V.S.P.U. M18*)

(*c*) $(1,2,3)$ guiding curve $x^2 + y^2 + z^2 = 4, x + y + z = 1$ **(*K. U. A' 08, O. U. A' 08*)**

(*d*) $(1,1,1)$ guiding curve $x^2 + y^2 + z^2 = 1, x + y + z = 1$

5. Find the equation of the cone whose vertex is (α,β,γ) and base $ax^2 + by^2 = 1, z = 0$.

(*N.U. A11*)

6. The vertex of the cone is (a,b,c) and the yz-plane cuts it in the curve $F(y,z) = 0, x = 0$, show that xz plane cuts it in the curve $y = 0, F\left[\frac{bx}{x-a}, \frac{cx - az}{x-a}\right] = 0$

ANSWERS

1 (*a*) $x^2 + y^2 = z^2$ (*b*) $x^2 + y^2 + z^2 + (x - 2y - 3z)\,(x - y + z) - 4(x - y + z)^2 = 0$

(*c*) $9x^2 + 9y^2 - 8z^2 - 6zx = 0$ (*d*) $\Sigma(ap^2 - l^2)\,x^2 - 2\sum lmxy = 0$

4. (*a*) $12x^2 - 4y^2 + z^2 + 4yz + 12zx + 4z - 4 = 0$

(*b*) $x^2+y^2-2z^2+2yz+2zx-4x-4y+4=0$

(*c*) $5x^2+3y^2+z^2-2xy-6yz-4zx+6x+8y+10z-26=0$

(*d*) $x^2+y^2+z^2-2(xy+yz+zx)+2(x+y+z)=3$

5. $a(\alpha z-\gamma x)^2+b(\beta z-\gamma y)^2=(z-\gamma)^2$

7. 8. ENVELOPING CONE

Definition. *Let S be a surface and P be a point not on the surface. The set of tangent lines to the surface S and passing through P form a cone with vertex at P.*

This is called the enveloping cone or the tangent cone of the given surface.

7. 9. Theorem. ***The enveloping cone of the sphere $x^2+y^2+z^2=a^2$ with vertex at (x_1, y_1, z_1) is $(xx_1+yy_1+zz_1-a^2)^2=(x^2+y^2+z^2-a^2)(x_1^2+y_1^2+z_1^2-a^2)$.***

Proof. Let $S=x^2+y^2+z^2-a^2=0$ **(*O. U. A12, M15*)**

$P(x_1, y_1, z_1) \notin S=0$

$\Rightarrow x_1^2+y_1^2+z_1^2-a^2 \neq 0$

Let Q (x, y, z) be a point on the enveloping cone C.

$\therefore \overleftrightarrow{PQ}$ is a tangent line to the sphere S = 0.

Let $R \in \overleftrightarrow{PQ}$ and (R ; P, Q) = $\lambda : 1$

$\therefore R=\left[\dfrac{\lambda x+x_1}{\lambda+1}, \dfrac{\lambda y+y_1}{\lambda+1}, \dfrac{\lambda z+z_1}{\lambda+1}\right]$

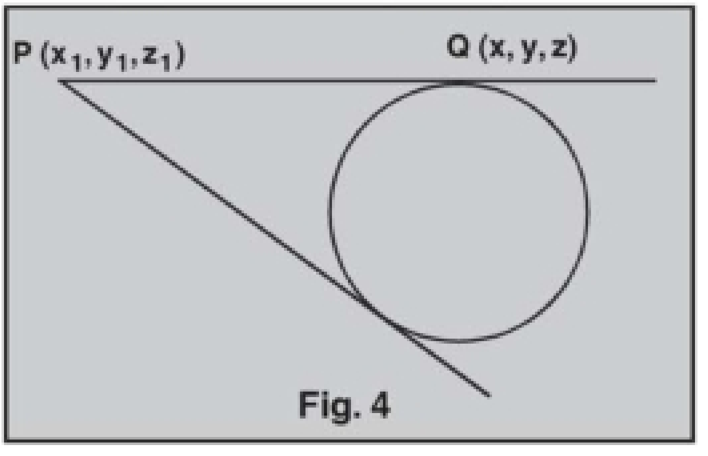


Fig. 4

$R \in S=0$

$\Rightarrow \left(\dfrac{\lambda x+x_1}{\lambda+1}\right)^2+\left(\dfrac{\lambda y+y_1}{\lambda+1}\right)^2+\left(\dfrac{\lambda z+z_1}{\lambda+1}\right)^2=a^2$

$\Rightarrow (\lambda x+x_1)^2+(\lambda y+y_1)^2+(\lambda z+z_1)^2=a^2(\lambda+1)^2$

$\Rightarrow \lambda^2(x^2+y^2+z^2-a^2)+2\lambda(xx_1+yy_1+zz_1-a^2)+(x_1^2+y_1^2+z_1^2-a^2)=0$

$\Rightarrow \lambda^2 S+2\lambda S_1+S_{11}=0$

If $\overleftrightarrow{PQ}$ is a tangent line to the sphere then the two roots of te equation (1) are equal

$\Rightarrow 4S_1^2-4SS_{11}=0 \Rightarrow S_1^2=SS_{11}$

Hence the equation to the enveloping cone C is $S_1^2=SS_{11}$

i.e., $(xx_1+yy_1+zz_1-a^2)^2=(x^2+y^2+z^2-a^2)(x_1^2+y_1^2+z_1^2-a^2)$

7. 10. RIGHT CIRCULAR CONE

Definition. *A right circular cone is a surface generator by a line which passes through afixed point, and makes a constant angle with a fixed line through the fixed point.*

Let S be a set of concurrent lines, concurrent at V. If there exists a line L passing through V such that for a line M, $M \in S \Rightarrow (L, M)=\theta$ the S is called a *right circular cone* with vertex at V.

The line L is called the *axis* and θ the *semi-vertical angle* of the cone.

Note. The section of a right circular cone by any plane perpendicular to the axis is a circle.

7. 11. Theorem. ***The equation of a right circular cone with vertex at (α,β,γ), semi- vertical angle θ and axis having direction ratios (l,m,n) is***

$$[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2=(l^2+m^2+n^2)[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]\cos^2\theta$$

(O. U. A11)

Proof. Let V be the vertex and **VL** be the axis of the cone. $V=(\alpha,\beta,\gamma)$ and the direction ratios of the axis VL are (l,m,n).

Let P (x,y,z) be a point on the cone.

D.r's of V.P are $(x-\alpha,\ y-\beta,\ z-\gamma)$

Semi vertical angle $\theta=(\overleftrightarrow{VL},\overleftrightarrow{VP})$

$$\cos\theta=\frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]},\sqrt{(l^2+m^2+n^2)}}$$

Hence the equation of the rigt circular cone is

$$[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2](l^2+m^2+n^2)\cos^2\theta$$

$$=[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2$$

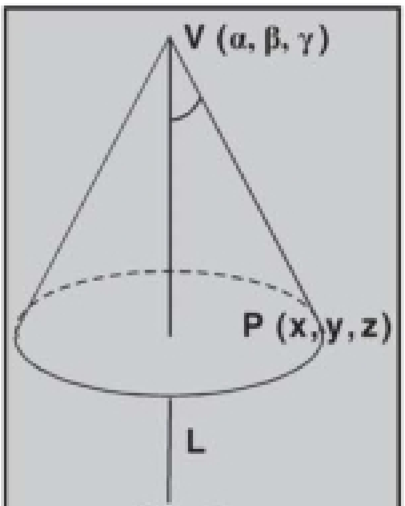


Fig. 5

Corollary 1. If the vertex be the origin then the equation of the cone becomes $(lx+my+nz)^2=(l^2+m^2+n^2)(x^2+y^2+z^2)\cos^2\theta$

Corollary 2. The equation of the right circular cone with vertex at (0, 0, 0) and whose axis is the z - axis and semi - vertical angle α is $x^2+y^2=z^2\tan^2\alpha$

Proof. Since d.r's of the z-axis are (0, 0, 1)

$l=0,\ m=0,\ n=1$

$\therefore$ The equation to the right circular cone is $(x^2+y^2+z^2)\cos^2\alpha=z^2$

$\Rightarrow(x^2+y^2)=z^2(\sec^2\alpha-1)\Rightarrow(x^2+y^2)=z^2(\tan^2\alpha)$

SOLVED PROBLEMS

Ex.1. *Find the enveloping cone of the sphere $x^2+y^2+z^2+2x-2y=2$, with its vertex at (1, 1, 1).* ***(A. N. U. M15)***

Sol. Given vertex $=(1,1,1)$.

Equation to the given sphere is $S=x^2+y^2+z^2+2x-2y-2=0$

Now $S_1=x.1+y.1+z.1+(x+1)-(y+1)-2=0=2x+z-2$

$S_{11}=1+1+1+2-2-2=1$

$\therefore$ The equation to the enveloping cone is $S_1{}^2=SS_{11}$

$(2x+z-2)^2=(x^2+y^2+z^2+2x-2y-2)(1)\Rightarrow3x^2-y^2+4zx-10x+2y-4z+6=0$

Ex. 2. *Find the equation to the right circular cone whose vertex is P(2, -3, 5), axis PQ which makes equal angles with the axis and which passes through A (1, -2, 3).*

(S. K. D. M15, S.K.U M18)

Sol. The axis of the cone makes equal angles θ with the coordinate axes

$\therefore$ d.r's of the axis are $(\cos\theta, \cos\theta, \cos\theta) \Rightarrow$ d.r's of the axis are (1, 1, 1)

Let α be the semi-vertical angle of the cone with vertex P (2, -3, 5)

$\therefore$ The equation to the cone is

$[(x-2)^2+(y+3)^2+(z-5)^2](1+1+1)\cos^2\alpha = [1.(x-2)+1.(y+3)+1(z-5)]^2$

The point A $(1,-2,3)$ lies on the cone

$\Leftrightarrow [(1-2)^2+(-2+3)^2+(3-5)^2]3\cos^2\alpha = [(1-2)+(-2+3)+(3-5)]^2 \Leftrightarrow \cos\alpha = \frac{\sqrt{2}}{3}$

$\therefore$ The equation to the required cone is

$[(x-2)^2+(y+3)^2+(z-5)^2]\times\frac{2}{3} = [(x-2)+(y+3)+(z-5)]^2$

Simplifying the equation $x^2+y^2+z^2+6(yz+zx+xy)-16x-36y-4z-28=0$

Ex. 3. *Find the equation of the right circular cone with vertex at* $(2,1,-3)$ *and whose axis is parallel to OY and whose semi vertical angle is* $45°$.

Sol. Axis is parallel to OY $\Rightarrow$ d.c.s of axis are $(0,1,0)$

Given semi vertical angle $\alpha = 45°$, vertex $=(2,1,-3)$.

$\therefore$ Equation to the cone is $[(x-2)^2+(y-1)^2+(z+3)^2](0+1+0)\cos^2 45$

$$= [0.(x-2)+1.(y-1)+0.(z+3)]^2$$

$\Rightarrow \frac{1}{2}[(x-2)^2+(y-1)^2+(z+3)^2] = (y-1)^2 \Rightarrow (x-2)^2-(y-1)^2+(z+3)^2 = 0$

$\Rightarrow x^2-y^2+z^2-4x+2y+6z+12=0$

Ex. 4. *Find the equation of the right circular cone whose vertex is the origin, axis as the line* $x=t, y=2t, z=3t$ *and whose semivertical angle is* $60°$.

(A. N.U. A12, S.K.U M18)

Sol. Vertex $(\alpha,\beta,\gamma)=(0,0,0)$

Equation to the axis $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}=t$

$\Rightarrow$ D. r's of the axis $(l,m,n)=(1,2,3)$

Semi vertical angle $=60°$

$\therefore$ Equation to the required cone is $[(x-0)^2+(y-0)^2+(z-0)^2][1^2+2^2+3^2]\cos^2 60°$

$$= [1.(x-0)+2.(y-0)+3.(z-0)]^2$$

$\Rightarrow \frac{14}{4}(x^2+y^2+z^2) = (x+2y+3z)^2$

$\Rightarrow 7(x^2+y^2+z^2) = 2(x^2+4y^2+9z^2+4xy+12yz+6zx)$

$\Rightarrow 5x^2-y^2-11z^2-24yz-12zx-8xy=0$

Ex. 5. *Show that the plane* $z=0$ *cuts the enveloping cone of the sphere* $x^2+y^2+z^2=11$ *which has its vertex at* $(2,4,1)$ *in a rectangular hyperbola.*

Sol. Let $S \equiv x^2+y^2+z^2-11=0$

Given point $P=(2,4,1)=(x_1,y_1,z_1)$

$S_1 \equiv xx_1+yy_1+zz_1-11 \equiv x(2)+y(4)+z(1)-11=2x+4y+z-11$

$S_{11} \equiv x_1^2+y_1^2+z_1^2-11 \equiv (2)^2+(4)^2+(1)^2-11=10$

$\therefore$ Equation to the enveloping cone is $SS_{11}=S_1^2$

$\Rightarrow (x^2+y^2+z^2-11)(10)=(2x+4y+z-11)^2$

where the plane $z=0$ cuts the cone, then the equation to the conic is

$10(x^2+y^2-11)=(2x+4y-11)^2 \Rightarrow 6x^2-6y^2-16xy+88y+44x-331$

In this equation coeft. of x^2+ co.eft. of $y^2=6-6=0$

$\Rightarrow$ The conic is a rectangular hyperbola

Ex. 6. *Find the equation of the cone generated by rotating the line* $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ *about the line* $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$ *as axis.*

Sol. Given lines pass through the origin $\Rightarrow$ vertex is the origin.

D.r's of axis are (a,b,c)

Semi vertical angle = angle between the generator and the axis

$$\Rightarrow \cos\theta=\frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}} \quad .... (1)$$

$\therefore$ Equation to the cone is $[(x-0)^2+(y-0)^2+(z-0)^2](a^2+b^2+c^2).\cos^2\theta$

$$=[a(x-0)+b(y-0)+c(z-0)]^2$$

Using (1) we have $\Rightarrow (x^2+y^2+z^2)(al+bm+cn)^2=(l^2+m^2+n^2)(ax+by+cz)^2$

Ex. 7. *If* α *is the semi vertical angle of a right circular cone which passes through the lines OY, OZ and* $x=y=z$. *Show that* $\cos\alpha=(9-4\sqrt{3})^{-1/2}$.

Sol. Let (l,m,n) be d.r.'s of the axis of the cone

D.r.'s of OY are $(0,1,0)$. D. r.'s of OZ are $(0,0,1)$

α is the angle between the axis and OY

$$\Rightarrow \cos\alpha=\frac{0.l+1.m+0.n}{\sqrt{0+1+0}\sqrt{l^2+m^2+n^2}}=\frac{m}{\sqrt{l^2+m^2+n^2}} \quad .... (1)$$

Also α is the angle between the axis and OZ

$$\Rightarrow \cos\alpha=\frac{0.l+0.m+1.n}{\sqrt{0+0+1}\sqrt{l^2+m^2+n^2}}=\frac{n}{\sqrt{l^2+m^2+n^2}} \quad .... (2)$$

From (1) and (2) $m=n$

Similarly angle between the axis and $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$ is

$$\cos\alpha = \frac{1.l+1.m+1.n}{\sqrt{1+1+1}\sqrt{l^2+m^2+n^2}} = \frac{l+m+n}{\sqrt{3(l^2+m^2+n^2)}} \quad .... (3)$$

Equating (1) and (3) $m = \dfrac{l+m+n}{\sqrt{3}}$

$$\Rightarrow l + m(1-\sqrt{3}) + n = 0 \Rightarrow l + m(1-\sqrt{3}) + m = 0 \qquad (\because m = u)$$

$$l = m(\sqrt{3}-2) \Rightarrow \frac{l}{\sqrt{3}-2} = \frac{m}{1} = \frac{n}{1}$$

$\therefore$ From (1) $\cos\alpha = \dfrac{1}{\sqrt{(\sqrt{3}-2)^2+1+1}} = \dfrac{1}{\sqrt{9-4\sqrt{3}}} = (9-4\sqrt{3})^{-1/2}$

Ex. 8. *Lines are drawn through the origin with direction ratios* $(1,2,2),(2,3,6)$ *and* $(3,4,12)$. *Find the direction ratios of the axis of the right circular cone and hence show that its semi vertical* angle is $\text{Cos}^{-1}(1/\sqrt{3})$. Also find the equation of the cone.

Sol. Let (l,m,n) be the direction ratios of the axis of the right circular cone

Let α be the semi vertical angle of the cone

$\therefore$ Each given line is at α with the axis

(*i*) $\cos\alpha = \dfrac{1.l+2.m+2.n}{\sqrt{1+4+4}\sqrt{l^2+m^2+n^2}}$ .... (1)

(*ii*) $\cos\alpha = \dfrac{2.l+3.m+6.n}{\sqrt{4+9+36}\sqrt{l^2+m^2+n^2}}$ .... (2)

(*iii*) $\cos\alpha = \dfrac{3.l+4.m+12.n}{\sqrt{9+16+144}\sqrt{l^2+m^2+n^2}}$ .... (3)

From (*i*) and (*ii*) : $\dfrac{l+2m+2n}{3} = \dfrac{2l+3m+6n}{7} \Rightarrow l+5m-4n=0$ ... I

From (*i*) and (*iii*) : $\dfrac{1}{3}(l+2m+2n) = \dfrac{1}{13}(3l+4m+12n) \Rightarrow 2l+7m-5n=0$ ... II

Solving I and II : $\dfrac{l}{-25+28} = \dfrac{m}{-8+5} = \dfrac{n}{7-10} \Rightarrow \dfrac{l}{3} = \dfrac{m}{-3} = \dfrac{n}{-3} \Rightarrow \dfrac{l}{1} = \dfrac{m}{-1} = \dfrac{n}{-1}$

$\therefore$ Direction cosines of the axis are $\left(\dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}\right)$

$\therefore$ The semi vertical angle is given by

From (1) $\cos\alpha = \dfrac{1(1)+2(-1)+2(-1)}{\sqrt{1+4+4}\sqrt{1+1+1}} = \dfrac{1}{\sqrt{3}} \Rightarrow \alpha = \text{Cos}^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

Equation the cone is $\cos\alpha = \dfrac{1}{\sqrt{3}} = \dfrac{1(x-0)-1(y-0)-1(z-0)}{\sqrt{1+1+1}\sqrt{x^2+y^2+z^2}}$

$$\Rightarrow (x-y-z)^2 = x^2+y^2+z^2 \Rightarrow yz-zx-xy=0$$

Ex. 9. *Find the equation of the cone formed by rotating the line* $2x+3y=6, z=0$ *about the Y - axis.*

Sol. The direction cosines of the axis are $(0,1,0)$

Given equation to the generator is $2x+3y=6, z=0$

$$2x=-3(y-2), z=0$$

$$\Rightarrow \frac{x}{3}=\frac{y-2}{-2}=\frac{z}{0} \qquad .... (1)$$

Also Y - axis meets the line $2x+3y=6$, $z=0$ at $(0,2,0)$

$\Rightarrow$ vertex of the plane $=(0,2,0)$

$\therefore$ Semi vertical angle = Ange between the line (1) and Y - axis.

$$\cos\alpha=\frac{0.3+1(-2)+0.0}{\sqrt{0+1+0}\,\sqrt{9+4+0}}=\frac{-2}{\sqrt{13}}$$

$\therefore$ Equation to the right circular cone with vertex $(0,2,0)$ and axis d.r.'s $(0,1,0)$ is

$$[0(x-0)+1(y-2)+0.z]^2(0+1+0)\cos^2\alpha=(0.x+1(y-2)+0.z)^2$$

$$[x+(y-2)+z]^2\frac{4}{13}=(y-2)^2 \Rightarrow 4x^2-9(y-2)^2+4z^2=0$$

EXERCISE 7 (*e*)

1. Find the enveloping cone with vertex at the origin and generators touching the sphere $x^2+y^2+z^2-2x+4z-1=0$.
2. Find the enveloping cone of the sphere $x^2+y^2+z^2+2x-4y=0$ with its vertex at $(1,1,1)$.
3. Find the equation of the right circular cone whose vertex is $P(2-3,5)$, axis PQ which makes equal angles with the axes and semi-vertical angle 30°. ***(S.K.U. A11)***
4. Find the right circular cone which passes through the point (1, 1, 2) and has vertex at the origin and the axis $\frac{x}{2}=\frac{y}{-4}=\frac{z}{3}$. ***(K. U. A12)***
5. Find the equation to the right circular cone whose vertex is the origin, the axis is along X - axis and semi vertical angle is α.
6. Find the equation to the right circular cone whose vertex is $(3,2,1)$, axis line $\frac{x-3}{4}=\frac{y-2}{1}=\frac{z-1}{3}$ and semi vertical angle 30°.
7. Find the equation to the right circular cone which passes through $(1,1,1)$, whose vertex is $(1,0,1)$ and axis of the cone makes equal angles with coordinate axes.
8. Find the equation to the right circular cone whose vertex is $(1,-2,-1)$, axis the line $\frac{x-1}{3}=\frac{y+2}{4}=\frac{z+1}{5}$ and semi vertical angle 60°. ***(N. U. A11, A.U M18)***
9. Find the equation to the right circular cone generated by the lines drawn from the origin to cut the circle through the three points $(1,2,2)$, $(2,1,-2)$ and $(2,-2,1)$.

ANSWERS

1. $4x^2+3y^2-5z^2-6yz-8x+16z-4=0$ **2.** $3x^2+2yz-4xy-6x+6y-2z+1=0$

3. $5(x^2+y^2+z^2)-8(xy+yz+zx)-4x+86y-58z+278=0$

4. $4x^2+40y^2+19z^2-72yz+36zx-48xy=0$ **5.** $y^2+z^2=x^2\tan^2\alpha$

7. $xy+yz+zx-x-2y-z+1=0$

8. $7x^2-7y^2-25z^2+48xy+80yz-60zx+22x+4y+17z+78=0$

9. $8x^2-4y^2-4z^2+5xy+yz+5zx=0$

7. 12. NOTATION

Let S represent the second degree general equation in x, y, z. The following notation is used in this chapter.

i.e. $S \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d$

$E = E(x,y,z) = ax^2+by^2+cz^2+2fyz+2gzx+2hxy$

$U = ax+hy+gz+u$; $V = hx+by+fz+v$; $W = gx+fy+cz+w$; $D = ux+vy+wz+d$

and $U_1 = ax_1+hy_1+gz_1+u$

$V_1 = hx_1+by_1+fz_1+v$; $W_1 = gx_1+fy_1+cz_1+w$; $D_1 = ux_1+vy_1+wz_1+d$

Then $S_1 = axx_1+byy_1+czz_1+f(yz_1+y_1z)+g(zx_1+z_1x)$

$$+h(xy_1+x_1y)+u(x+x_1)+v(y+y_1)+w(z+z_1)+d$$

$$=(ax_1+hy_1+gz_1+u)x+(hx_1+by_1+fz_1+v)y$$

$$+(gx_1+fy_1+cz_1+w)z+ux_1+vy_1+wz_1+d = U_1x+V_1y+W_1z+D_1$$

$S_{11} = U_1x_1+V_1y_1+W_1z_1+D_1$

7.13. Theorem. ***If*** (x_1, y_1, z_1) ***is the vertex of the cone*** $S = 0$ ***then*** $U_1 = V_1 = W_1 = D_1 = 0$.

Proof. Let the equation to the cone be

$S = ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$

Given vertex of the cone, $P = (x_1, y_1, z_1)$

Shifting the origin to the point P the new equation of the cone referred to vertex P as the new origin is $a(x+x_1)^2+b(y+y_1)^2+c(z+z_1)^2+2f(y+y_1)(z+z_1)$

$$+2g(x+x_1)(z+z_1)+2h(x+x_1)(y+y_1)+2u(x+x_1)$$

$$+2v(y+y_1)+2w(z+z_1)+d=0$$

$\Rightarrow ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2x(ax_1+hy_1+gz_1+u)$

$$+2y(hx_1+by_1+fz_1+v)+2z(gx_1+fy_1+cz_1+w)+ax_1^2$$

$$+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1x_1+2ux_1+2vy_1+2wz_1+d=0.$$

$\Rightarrow E(x,y,z)+2U_1x+2V_1y+2W_1z+S_{11}=0$

This must be homogeneous equation

$\Rightarrow$ $U_1=0$, $V_1=0$, $W_1=0$ and $S_{11}=0$.

But $S_{11}=U_1x_1+V_1y_1+W_1z_1+D_1=0$ $\Rightarrow D_1=0$ $\Rightarrow U_1=V_1=W_1=D_1=0$

Corollary 1. If the equation S = 0 represents a cone then the condition is

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0$$

Proof. Eliminating x_1, y_1, z_1 in the equations

$U_1 = ax_1 + hy_1 + gz_1 + u = 0$; $V_1 = hx_1 + by_1 + fz_1 + v = 0$; $W_1 = gx_1 + fy_1 + cz_1 + w = 0$

$D_1 = ux_1 + vy_1 + wz_1 + d = 0$

We get
$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0$$

This is the required condition that the equation S = o represents a cone.

Corollary 2. The vertex (x_1, y_1, z_1) satisfies the equations

$U \equiv ax + hy + bz + u = 0$ ...(1) $V \equiv hx + by + fz + v = 0$ ...(2)

$W \equiv gx + fy + cz + w = 0$ ...(3) $D \equiv ux + vy + wz + d = 0$ ...(4)

Thus the vertex is obtained by solcing any three of the above four equations

Note. Consider the homogeneous polynomial

$$S(x, y, z, t) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2uxt + 2vyt + 2wzt + dt^2$$

Now $\dfrac{\partial S}{\partial x} = 2(ax + hy + gz + ut)$

$\dfrac{\partial S}{\partial y} = 2(hx + by + fz + vt)$; $\dfrac{\partial S}{\partial z} = 2(gx + fy + cz + wt)$; $\dfrac{\partial S}{\partial t} = 2(ux + vy + wz + dt)$

Now equating $\dfrac{\partial S}{\partial x}, \dfrac{\partial S}{\partial y}, \dfrac{\partial S}{\partial z}, \dfrac{\partial S}{\partial t}$ each to zero and putting $t = 1$, we get

U = V = W = D = 0.

SOLVED PROBLEMS

Ex. 1. *If* $ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$ *represents a cone prove that* $\dfrac{u^2}{a} + \dfrac{v^2}{b} + \dfrac{w^2}{c} = d$.

Sol. Let (x_1, y_1, z_1) be the vertex of the given cone.

The given equation represents a cone if

$U_1 = 0 \Rightarrow ax_1 + u = 0 \Rightarrow x_1 = \dfrac{-u}{a}$; $V_1 = 0 \Rightarrow by_1 + v = 0 \Rightarrow y_1 = \dfrac{-v}{b}$

$W_1 = 0 \Rightarrow cz_1 + w = 0 \Rightarrow z_1 = \dfrac{-w}{c}$ and $D_1 = 0 \Rightarrow ux_1 + vy_1 + wz_1 + d = 0$

Substituting in $D_1 = 0$ the values of $x_1 + y_1 + z_1$ we get

$$u\left(\frac{-u}{a}\right) + v\left(\frac{-v}{b}\right) + w\left(\frac{-w}{c}\right) + d = 0 \Rightarrow \frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = d$$

Ex.2. *Find the vertex of the cone* $7x^2+2y^2+2z^2-10zx+10xy+26x-2y+2z-17=0$.

Sol. Consider the homogeneous equation *(O. U. A12, N.U.A.2006, S. V. U. M15)*

$$S(x,y,z,t)=7x^2+2y^2+2z^2-10zx+10xy+26xt-2yt+2zt-17t^2=0$$

$$\therefore \quad \frac{\partial S}{\partial x}=14x-10z+10y+26t=14x+10y-10z+26 \qquad (\because t=1)$$

$$\frac{\partial S}{\partial y}=4y+10x-2t=10x+4y-2$$

$$\frac{\partial S}{\partial z}=4z-10x+2t=-10x+4z+2\,; \qquad \frac{\partial S}{\partial t}=26x-2y+2z-34t=26x-2y+2z-34$$

Coordinates of vertex satisfy the equations

$14x+10y-10z+26=0$ ...(1) $\qquad 10x+4y-2=0$ ...(2)

$-10x+4z+2=0$ ...(3) $\qquad 26x-2y+2z-34=0$ ...(4)

Solving (1), (2) and (3) we get $x=1,\ y=-2,\ z=2$

Substituting (1, -2, 2) in (4) $26+4+4-34=0$

Hence the vertex of the cone is (1, –2, 2)

Ex. 3. *Show that the equation* $2y^2-8yz-4zx-8xy+6x-4y-2z+5=0$ *represents a cone whose vertex is* $\left(-\frac{7}{6},\frac{1}{3},\frac{5}{6}\right)$ *(Similiar (K. U. A.12)(S.V.M M18)*

Sol. Making the given equation homogeneous, we get

$$S(x,y,z,t)\equiv 2y^2-8yz-4zx-8xy+6xt-4yt-2zt+5t^2=0$$

$$\frac{\partial S}{\partial x}=-4z-8y+6t\,; \qquad \frac{\partial S}{\partial y}=4y-8z-8x-4t$$

$$\frac{\partial S}{\partial z}=-8y-4x-2t\,; \qquad \frac{\partial S}{\partial t}=6x-4y-2z+10t$$

Equating $t=1$ coordinates of the vertex satisfy the equations

$4y+2z-3=0$ .... (1) $\qquad 2x-y+2z+1=0$ .... (2)

$2x+4y+1=0$ .... (3) $\qquad 3x-2y-z+5=0$ .... (4)

Solving (1), (2) and (3) we get $x=-\frac{7}{6}, y=\frac{1}{3}, z=\frac{5}{6}$

Substituting in (4) : $3\left(-\frac{7}{6}\right)-2\left(\frac{1}{3}\right)-\frac{5}{6}+5=0 \Rightarrow -21-4-5+30=0$

Hence the vertex of the cone is $\left(-\frac{7}{6},\frac{1}{3},\frac{5}{6}\right)$

7. 14. Theorem. ***The cone*** $E(x,y,z)=0$ ***will have three mutually perpendicular generators*** $\Leftrightarrow a+b+c=0$ ***(A.U. M18)***

Proof. Given the equation of the cone

$$E(x,y,z)=ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

Let $\frac{x}{p}=\frac{y}{q}=\frac{z}{r}$ ...(1) be a generator of the cone,

$\therefore$ $E(p,q,r)=0 \Rightarrow ap^2+bq^2+cr^2+2fqr+2grp+2hpq=0$ ...(2)

The equation to the plane $\perp r$ to (1) and passing through the vertex is

$px+qy+rz=0$ ...(3)

Let this plane intersect the cone along two real generators and (l, m, n) be the d.c's of one of the generators.

$\therefore$ $al^2+bm^2+cn^2+2fmn+2gnl+2hlm=0$ ...(4) $pl+qm+rn=0$ ...(5)

Eliminating n between (4) and (5), we get

$$l^2(ar^2+cp^2-2grp)+2lm(cpq+hr^2-gqr-frp)+m^2(br^2+cq^2-2fqr)=0$$

$$\Rightarrow \frac{l^2}{m^2}(ar^2+cp^2-2grp)+2\frac{l}{m}(cpq+hr^2-gqr-frp)+(br^2+cq^2-2fqr)=0 \quad ...(6)$$

If (l_1, m_1, n_1) and (l_2, m_2, n_2) are the direction cosines of the two generators of intersection then $\frac{l_1}{m_1}, \frac{l_2}{m_2}$ are the roots of (6).

$$\therefore \quad \frac{l_1l_2}{m_1m_2}=\frac{br^2+cq^2-2fqr}{ar^2+cp^2-2grp} \Rightarrow \frac{l_1l_2}{br^2+cq^2-2fqr}=\frac{m_1m_2}{ar^2+cp^2-2grp}$$

$=\frac{n_1n_2}{aq^2+bp^2-2hpq}=k,$ by symmetry. $\therefore$ $l_1l_2+m_1m_2+n_1n_2$

$$=k[a(q^2+r^2)+b(r^2+p^2)+c(p^2+q^2)-2fqr-2grp-2hpq]$$

$$=k(a+b+c)(p^2+q^2+r^2) \quad ...(2)$$

The two generators of itersection of the plane (3) with the cone are at right angles.

$\Leftrightarrow$ $l_1l_2+m_1m_2+n_1n_2=0$

$\Leftrightarrow$ $a+b+c=0$ $[(\because (p,q,r)\neq(0,0,0)]$

Since plane (3) is perpendicular to generator (1), the two generators of intersection of the plane (3) with the cone are perpendicular to generator (1).

$\therefore$ These three generators are mutually perpendicular

$\Leftrightarrow$ Two generators of intersection are perpendicular $\Leftrightarrow$ $a+b+c=0$.

Note. 1. Above condition is satisfied whatever is the direction of the generator. From this we get if three mutually perpendicular lines are generators to the cone, then $a+b+c=0$

Note. 2. Let $F(x,y,z)\equiv ax^2+by^2+cz^2 2fyz+2gzx+2hxy+2hx+2vy+2wz+d=0$ be a cone. Shifting the origin to the vertex the transformed equation is

$E(x,y,z)=ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$.

the cone $F(x,y,z)=0$, has three mutually perpendicular generators

$\Leftrightarrow E(x,y,z)=0$ has three mutually perpendicular generators $\Leftrightarrow$ $a+b+c=0$

SOLVED PROBLEMS

Ex. 1. *Show that the two lines of intersection of the plane* $ax+by+cz=0$ *with the cone* $yz+zx+xy=0$ *will be perpendicular if* $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$. ***(N. U. AII)***

Sol. Given cone is $yz + zx + xy = 0$.

In this equation Coeff. of x^2 + coeff. of y^2 + coeff. of $z^2 = 0$

$\therefore$ The cone contains sets of three mutually perpendicular generators.

The plane $ax + by + cz = 0$ cuts the cone in perpendicular generators if its normal line through the vertex (0, 0, 0).

i.e., $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$ is a generator of the cone.

$\Rightarrow bc + ca + ab = 0 \Rightarrow \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = 0$

Ex. 2. *If the line* $x = \dfrac{1}{2}y = z$ *represents one of the three mutually perpendicular generators of the cone* $11yz + 6zx - 14xy = 0$, *find the equations of the other two.*

(S. K. U. A11)

Sol. The given cone is $11yz + 6zx - 14xy = 0$

The plane through the vertex of the cone and perpendicular to the generator.

$\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{1}$ ...(1) is $x + 2y + z = 0$ the other two generators perpendicular to (1) are the lines of intersection of $11yz + 6zx - 14xy = 0$ and $x + 2y + z = 0$.

Let l, m, n be the direction ratios of one of the common lines.

Then $11mn + 6nl - 14lm = 0$ ...(2) and $l + 2m + n = 0 \quad \Rightarrow n = -l - 2m$

Substituting in (2) $11m(-l - 2m) + 6l(-l - 2m) - 14lm = 0$

$\Rightarrow 6l^2 + 37lm + 22m^2 = 0 \Rightarrow (2l + 11m)(3l + 2m) = 0$

$\Rightarrow 2l + 11m = 0$ or $3l + 2m = 0$

(*i*) Solving $l + 2m + n = 0$ $\qquad 2l + 11m + 0 . n = 0$

We get $\dfrac{l}{-11} = \dfrac{m}{2} = \dfrac{n}{7}$

(*ii*) Solving $l + 2m + n = 0$

$3l + 2m + 0 . n = 0$ $\qquad \dfrac{l}{-2} = \dfrac{m}{3} = \dfrac{n}{-4}$

$\therefore$ The other two perpendicular generators are $\dfrac{x}{-11} = \dfrac{y}{2} = \dfrac{z}{7}$ and $\dfrac{x}{2} = \dfrac{y}{-3} = \dfrac{z}{4}$

Ex. 3. *Show that if a right circular cone has sets of three mutually perpendicular generators, its semivertical angle must be* $\tan^{-1}\sqrt{2}$. ***(A. N.U. A12, A.K.N.U M18)***

Sol. Let the origin be the vertex, l, m, n be direction cosines of the axis of the cone and α be its semivertical angle

Then the equation to cone is $(lx + my + nz)^2 = (l^2 + m^2 + n^2)(x^2 + y^2 + z^2)\cos^2\alpha$

$\because$ the cone contains three mutually perpendicular generators, then

i.e., coeff of x^2 + coeff of y^2 + coeff of $z^2 = 0$.

coeff of $x^2 = l^2 - (l^2 + m^2 + n^2)\cos^2\alpha$

coeff of $y^2 = m^2 - (l^2 + m^2 + n^2)\cos^2\alpha$

coeff of $z^2 = n^2 - (l^2 + m^2 + n^2)\cos^2\alpha$

Adding, we have by (1) $(l^2 + m^2 + n^2) - 3(l^2 + m^2 + n^2)\cos^2\alpha = 0$

$\Rightarrow 1 - 3\cos^2\alpha = 0 \Rightarrow \sec^2\alpha = 3$

$\Rightarrow 1 + \tan^2\alpha = 3 \Rightarrow \tan^2\alpha = 2 \Rightarrow \tan\alpha = \sqrt{2} \Rightarrow \alpha = \tan^{-1}\sqrt{2}$

Ex. 4. *Show that cone whose vertex is the origin and which passes through the curve of intersection of the surface* $2x^2 - y^2 + 2z^2 = 3d^2$ *any plane at a distance d, from the origin has three mutually perpendicular generators.*

Sol. Equation to any plane at a distance d from the origin is $lx + my + nz = d$ ... (1)

where l, m, n are the actual d.c's of normal to the plane .

Homogenising the equation of the sphere with that of the plane, we have

$$2x^2 - y^2 + 2z^2 = 3d^2\left(\frac{lx + my + nz}{d}\right)^2$$

Now co.eft. of x^2 + co. eft. of y^2 + co.eft. of z^2

$= (2 - 3l^2) - 1 - 3m^2 + (2 - 3n^2) = 3 - 3(l^2 + m^2 + n^2) = 3 - 3(1) = 0$

Hence plane (1) cuts the cone in three mutually perpendicular generators.

Ex. 5. *If the plane* $2x - y + cz = 0$ *cuts the cone* $yz + zx + xy = 0$ *in perpendicular lines find c.*

Sol. Given cone $yz + zx + xy = 0$ .... (1)

contains sets of three mutually perpendicular generators.

$2x - y + cz = 0$ cuts (1) in perpendicular lines $\Rightarrow$ the normal of the plane lies on it.

$\Rightarrow (2, -1, c)$ must satisfy the cone equation $\Rightarrow (-1)(c) + c(2) + (2)(-1) = 0 \Rightarrow c = 2$

Ex. 6. *Find the locus of the point from which three mutually perpendicular lines can be drawn to intersect the central conic* $ax^2 + by^2 = 1, z = 0$.

Sol. Let the point P be (x_1, y_1, z_1)

Any line through P is $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n} = k$ .... (1)

Any point on the line is $(x_1 + lk, y_1 + mk, z_1 + nk)$

the point lies on the base curve $ax^2 + by^2 = 1, z = 0$

$\Leftrightarrow a(x_1 + lk)^2 + b(y_1 + mk)^2 = 1, z_1 + nk = 0$

Eliminating k, we have $a\left(x_1 - \dfrac{lz_1}{n}\right)^2 + b\left(y_1 - \dfrac{mz_1}{n}\right)^2 = 1$

$\Rightarrow a(nx_1 - lz_1)^2 + b(ny_1 - mz_1) = n^2$

using (1), to the cone is $a\left[x_1(z - z_1) - z_1(x - x_1)\right]^2 + b\left[y_1(z - z_1) - z_1(y - y_1)\right]^2 = (z - z_1)^2$

This contains three mutually perpendicular generators

$\Leftrightarrow$ co.eft. of x^2 + co. eft. of y^2 + co.eft. of $z^2 = 0$

$\Rightarrow az_1^2 + bz_1^2 + ax_1^2 + by_1^2 - 1 = 0$. $\quad\therefore$ Locus of P is $a(x^2+z^2)+b(y^2+z^2)=1$

EXERCISE 7 (*f*)

1. Find the vertices of the following cones :

(*a*) $2x^2+2y^2+7z^2-10yz-10zx+2x+2y+26z-17=0$ **(*A.U. M18*)**

(*b*) $4x^2-y^2+2z^2+2xy-3yz+12x-11y+6z+4=0$ **(*K. U. M18*)**

(*c*) $x^2-2y^2+3z^2-4xy+5yz-6zx+8x-19y-2z-20=0$ **(*O. U. A11, S. K. D. M15*)**

2. If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represents one of the three mutually perpendicular generators of the cone $5yz-8zx-3xy=0$, find the other two.

3. If $\frac{x}{1}=\frac{y}{1}=\frac{z}{2}$ is one of the three mutually perpendicular generators of the cone $3yz-2zx-2xy=0$, find the other two.

4. Show that three mutually perpendicular tangent lines can be drawn to sphere $x^2+y^2+z^2=r^2$ from any point on the sphere $2(x^2+y^2+z^2)=3r^2$

[**Hint.** The enveloping cone will have three mutually perpendicular generators.]

5. Prove that the plane $lx+my+nz=0$, cuts the cone $(b-c)x^2+(c-a)y^2+(a-b)z^2+2fyz+2gzx+2hxy=0$ in perpendicular lines if $(b-c)l^2+(c-a)m^2+(a-b)n^2+2fmn+2gml+2hlm=0$

6. Show that the locus of a point from which three mutually perpendicular tangent lines can be drawn to the cone $ax^2+by^2+cz^2=1$ is $a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=a+b+c$.

7. Show that the locus of the point from which three mutually perpendicular lines be drawn to meet the curve $x^2+y^2=1, z=0$ is $x^2+y^2+2z^2=1$.

8. Show that the equation to the cone whose vertex is at the origin and which passes through the curve of intersection of the sphere $x^2+y^2+z^2=3a^2$, and any plane at a distance 'a' from the origin has mutually perpendicular generators.

ANSWERS

1. (*a*) (2, 2, 1) (*b*) $(-1,-2,-3)$ (*c*) $(1,-2,3)$

2. $\frac{x}{5}=\frac{y}{-4}=\frac{z}{1}$; $x=y=-z$ **3.** $\frac{x}{2}=\frac{y}{-4}=\frac{z}{1}$; $\frac{x}{3}=\frac{y}{1}=\frac{z}{-2}$

7. 15. INTERSECTION OF A LINE WITH A CONE

Let the equation to the cone S be

$$S(x,y,z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$$

Let the equation to a line be $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}\ (=r)$

Let P be a point on this line

$\therefore\ P = (lr + x_1, mr + y_1, nr + z_1)$ $\qquad\therefore\ P \in S \Leftrightarrow S(lr + x_1, mr + y_1, nr + z_1) = 0$

$\Leftrightarrow\ a(lr + x_1)^2 + b(mr + y_1)^2 + c(nr + z_1)^2 + 2f(mr + y_1)(nr + z_1) + 2g(nr + z_1)(lr + x_1)$
$+ 2h(lr + x_1)(mr + y_1) + 2u(lr + x_1) + 2v(mr + y_1) + 2w(nr + z_1) + d = 0.$

$\Leftrightarrow\ r^2(al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm) + 2r[l(ax_1 + hy_1 + gz_1 + u)$
$+ m(hx_1 + by_1 + fz_1 + v) + n(gx_1 + fy_1 + cz_1 + w)] + S(x_1, y_1, z_1) = 0.$

$\Leftrightarrow\ r^2E(l, m, n) + 2r[lU_1 + mV_1 + nW_1] + S_{11} = 0$

(*i*) This will be a quadratic eqn. in $r \Leftrightarrow E(l, m, n) \neq 0$

The equation will have two real and dictinctive roots.

$\Leftrightarrow\ (lU_1 + mV_1 + nW_1)^2 - E(l, m, n)\, S_{11} > 0$

Then there will be two real points of the line common with the cone. The line segment joining the two point is called the chord of the cone.

(*ii*) If $E(l, m, n) \neq 0$ and $(lU_1 + mV_1 + nW_1) - E(l, m, n)\, S_{11} < 0$ then there are no common points.

(*iii*) If $E(l, m, n) \neq 0$ and $(lU_1 + mV_1 + nW_1)^2 = E(l, m, n)\, S_{11}$ then the two roots of the equation are real and equal. Hence the line meets the curve in two coincident points. Then the line is called the tangent line at that common point.

The set of all the tangent lines to the cone at the common point is called the tangent plane of the cone at that point.

(*iv*) If $E(l, m, n) = lU_1 + mV_1 + nW_1 = S_{11} = 0$, then the line becomes a generator of the cone.

7. 16. TANGENT PLANE

Definition. *Let $S = 0$, be the cone and L be a tangent line to the cone at P on it. The locus of the line L is called the* **tangent plane** *to the cone at P.*

7. 17. Theorem. ***If $P(x_1, y_1, z_1)$ is a point on the cone S = 0, then the equation of the tangent plane to the cone at P is $S_1 = 0$.***

Proof. Given equation to the cone $S = S(x, y, z) = 0$

Let the equation to the line passing through $P(x_1, y_1, z_1)$ be

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}\ (= r) \qquad ...(1)$$

$P(x_1, y_1, z_1) \in S = 0 \Rightarrow S_{11} = 0$

The point $(lr + x_1, mr + y_1, nr + z_1)$ of the line (1) lies on $S = 0$

$\Leftrightarrow\ S(lr + x_1, mr + y_1, nr + z_1) = 0$

$\Leftrightarrow r^2E(l, m, n)^0 + 2r[lU_1 + mV_1 + nW_1] + S_{11} < 0$

The line is a tangent line to the cone

$$\Leftrightarrow\ (lU_1 + mV_1 + nW_1)^2 - E(l, m, n).S_{11} = 0 \Leftrightarrow\ lU_1 + mV_1 + nW_1 = 0 \qquad ...(2)$$

Eliminating l, m, n in (1) and (2), the locus of tangent line is

$(x - x_1)U_1 + (y - y_1)V_1 + (z - z_1)W_1 = 0$

i.e. $U_1x + V_1y + W_1z = U_1x_1 + V_1y_1 + W_1z_1$

i.e. $S_1 = S_{11}$ *i.e.* $S_1 = 0$ $[\because S_{11} = 0]$

$\therefore$ The equation to the tangent plane at $P(x_1, y_1, z_1)$ to the cone $S = 0$ is $S_1 = 0$.

Corollary. If the equation of the cone

$E(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

then the equation to the tangent plane at (x_1, y_1, z_1) on the cone is $U_1x + V_1y + W_1z = 0$

i.e. $x(ax_1 + hy_1 + gz_1) + y(hx_1 + by_1 + fz_1) + z[gx_1 + fy_1 + cz_1] = 0$

i.e. $axx_1 + byy_1 + czz_1 + f(y_1z + yz_1) + g(z_1x + zx_1) + h(x_1y + xy_1) = 0$

Note 1. The tangent plane at a point P to the cone is also the tangent plane at every point on the generator.

2. The tangent plane at a point P to the cone contains the generator through P.

3. The equation to the normal line to the tangent plane at $P(x_1, y_1, z_1)$ is

$$\frac{x - x_1}{U_1} = \frac{y - y_1}{V_1} = \frac{z - z_1}{W_1}$$

7. 18. Theorem. ***The necessary and sufficient condition for the plane*** $\pi = lx + my + nz = 0$ ***to be a tangent plane to the cone***

$$E(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \textit{ is } \rho = \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & o \end{vmatrix} = 0$$

(*i*) Necessary Condition

Let $P(x_1, y_1, z_1)$ be the point of contact of the given tangent plane π with the cone

$E(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

$\therefore$ The equation to the tangent plane is $U_1x + V_1y + W_1z = 0$

$\Rightarrow (ax_1 + hy_1 + gz_1)x + (hx_1 + by_1 + fz_1)y + (gx_1 + fy_1 + cz_1)z = 0$

Comparing with the given tangent plane π

i.e. $lx + my + nz = 0$ we have $\dfrac{U_1}{l} = \dfrac{V_1}{m} = \dfrac{W_1}{n}$ $(= -k$, where $k \neq 0)$

$$\Rightarrow \frac{ax_1 + hy_1 + gz_1}{l} = \frac{hx_1 + by_1 + fz_1}{m} = \frac{gx_1 + fy_1 + cz_1}{n} = -k$$

$\Rightarrow ax_1 + hy_1 + gz_1 + lk = 0;$ $hx_1 + by_1 + gz_1 + mk = 0;$ $gx_1 + fy_1 + cz_1 + nk = 0$

Also $lx_1 + my_1 + nz_1 = 0$. $(\because P \in \pi)$

The non-zero solution (x_1, y_1, z_1, k) satisfy the equations

$ax + hy + gz + lt = 0;$ $hx + by + fz + mt = 0$ ...(I)

$gx + fy + cz + nt = 0;$ $lx + my + nz = 0$.

$$\text{Hence } \rho = \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & o \end{vmatrix} = 0 \Rightarrow \rho = Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$$

where A, B, C, F, G, H are the cofactors of a, b, c, f, g, h in the determinant.

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

***(ii)* Sufficiency of the condition.** Given $\rho = 0$, to prove the plane π is a tangent plane to the cone $E(x, y, z) = 0$.

Proof. If $\rho = 0$, there exists a non-zero solution (x_1, y_1, z_1, k) to I.

If $k = 0$, then $U_1 = 0,\ V_1 = 0,\ W_1 = 0 \Rightarrow (x_1, y_1, z_1)$ is the vertex of the cone.

This contradicts the fact that $(x_1, y_1, z_1) \neq (0, 0, 0)$. Hence $k \neq 0$.

Corresponding to the non-zero solution (x_1, y_1, z_1, k) of the equations I, we have

$U_1 = -kl,\ V_1 = -km,\ W_1 = -kn$ ...(1) and $lx_1 + my_1 + nz_1 = 0$ ...(2)

(2) $\Rightarrow P \in \pi$

and $E(x_1, y_1, z_1) = U_1x_1 + V_1y_1 + W_1z_1 = -k(lx_1 + my_1 + nz_1) = 0.$

$\therefore$ (x_1, y_1, z_1) is a point on the cone.

$\therefore$ $P(x_1, y_1, z_1)$ is a common point of the plane π and the cone.

$\therefore$ Equation to the tangent plae at P to the cone is $U_1x + V_1y + W_1z = 0$

since $l : m : n = U_1 : V_1 : W_1$, $lx + my + nz = 0$ is the tangent plane $P(x_1, y_1, z_1)$ to the cone.

7. 19. RECIPROCAL CONE

Theorem. ***The locus of the lines perpendicular to the tangent planes of the cone $E(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ and passing through its vertex is the cone.***

$$\begin{vmatrix} a & b & g & x \\ h & b & f & y \\ g & f & c & z \\ x & y & z & 0 \end{vmatrix} = 0$$

Proof. Let $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ be a line perpendicular to a tangent plane of the cone and passing through the vertex (0, 0, 0).

The cone is $E(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ ...(1)

$\therefore$ $lx + my + nz = 0$ is the tangent plane to (1).

$\Leftrightarrow Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$

where A, B, C, F, G, H are the cofactors of a, b, c, f, g, h in the determinant.

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Hence the locus of the normal line is $Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$

i.e $$\begin{vmatrix} a & b & g & x \\ h & b & f & y \\ g & f & c & z \\ x & y & z & 0 \end{vmatrix} = 0$$

This equation represents a cone called the reciprocal cone of (1).

Corollary. The reciprocal cone of

$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$ ...(2) is the cone $E(x, y, z) = 0$

Proof. By the above theorem the reciprocal cone of (2) is

$A'x^2 + B'y^2 + C'z^2 + 2F'yz + 2G'zx + 2H'xy = 0$ ...(3)

where A', B', C', F', G', H' are the cofactors of A, B, C, F, G, H in determinent

$$\Delta = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} \qquad \therefore \quad A' = BC - F^2$$

$= (ca - g^2)(ab - h^2) - (gh - af)^2 = a(abc + 2fgh - af^2 - bg^2 - ch^2)$

$= a\,\Delta$, where $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

Similarly $B' = b\,\Delta$, $C' = c\,\Delta$, $F' = f\,\Delta$, $G' = g\,\Delta$, $H' = h\,\Delta$

Then the equation (3) reduces to $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

i.e. $E(x, y, z) = 0$

Thus cones (1) and (2) are Reciprocal cones to each other.

Note 1. A cone and its reciprocal cone will have the same vertex.

2. Corresponding to each tangent plane of a cone there exists a generator of the reciprocal cone which is perpendicular to the tangent plane and vice versa.

3. A cone $E(x, y, z) = 0$ has three mutually $\perp r$ tangent planes

$\Leftrightarrow$ the reciprocal cone of $E(x, y, z) = 0$ has three mutually $\perp r$ generators

$\Leftrightarrow (bc - f^2) - (ca - g^2) - (ab - h^2) = 0 \Leftrightarrow bc + ca + ab = f^2 + g^2 + h^2$

SOLVED PROBLEMS

Ex. 1. *The semi-vertical angle of a right circular cone having three mutually perpendicular (i) generators is* $\tan^{-1}\sqrt{2}$ ***(A. U. A11)***

(ii) tangent planes is $\tan^{-1}\dfrac{1}{\sqrt{2}}$. ***(N. U. A88, S. K. D. M 15)***

Sol. Let the equations to the right circular cone be $x^2 + y^2 = z^2\tan^2\alpha$

(*i*) If the cone contains three mutually perpendicular generators then $a + b + c = 0$

(*ii*) The given cone contains three mutually perpendicular tangent planes

$\Leftrightarrow$ its reciprocal cone contains three mutually $\perp r$ generators

$\therefore$ Equations to the reciprocal cone of (1) is $-\tan^2\alpha\, x^2 - \tan^2\alpha\, y^2 + 1.z^2 = 0$ ...(2)

Eqn. (2) will have three mutually perpendicular generators if

$-\tan^2\alpha - \tan^2\alpha + 1 = 0 \Rightarrow \alpha = \tan^{-1}\dfrac{1}{\sqrt{2}}$.

Ex. 2. *Show that the general equation to a cone which touches the three coordinate planes is* $\sqrt{ax}+\sqrt{by}+\sqrt{cz}=0$.

Sol. The general equation of the cone containing the three coordinate axes is

$ayz+bzx+cxy=0$ ...(1)

The reciprocal cone of (1) wil have the coordinate planes.

$\therefore$ The equation to the reciprocal cone of (1) is

$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$ where $A=-a^2$; $B=-b^2$; $C=-c^2$

$F=bc$; $G=ca$; $H=ab$

$\therefore$ The equation to the required cone is $-a^2x^2-b^2y^2-c^2z^2+2bcyz+2cazx+2abxy=0$

$\Rightarrow\quad (ax+by-cz)^2=4abxy \Rightarrow\quad ax+by-cz=\pm 2\sqrt{abxy}$

$\Rightarrow\quad ax+by\mp 2\sqrt{abxy}=cz \Rightarrow\quad (\sqrt{ax}\pm\sqrt{by})^2=cz$

$\Rightarrow\quad \sqrt{ax}\pm\sqrt{by}=\pm\sqrt{cz} \Rightarrow\quad \sqrt{ax}\pm\sqrt{by}\pm\sqrt{cz}=0$

Ex. 3. *Show that the reciprocal cone of* $ax^2+by^2+cz^2=0$ *is the cone* $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0$. **(A. U. M12, N. U. A11, A.U. A11, S.K.U. A11, K.U.A11)**

Sol. Given cone is $ax^2+by^2+cz^2=0$

$\therefore$ The equation to the reciprocal cone is

$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$ where $A=bc$, $B=ca$, $C=ab$

$F=0$, $G=0$, $H=0$

$\therefore$ The equation to the reciprocal cone is $bcx^2+cay^2+abz^2=0 \Rightarrow \frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0$

Ex. 4. *Find the equations of the tangent planes to the cone* $9x^2-4y^2+16z^2=0$, *which contains the line* $\frac{x}{32}=\frac{y}{72}=\frac{z}{27}$.

Sol. The given line $\frac{x}{32}=\frac{y}{72}=\frac{z}{27}$ is the line of intersection of the planes

$72x-32y=0$ and $27y-72z=0$

i.e. $9x-4y=0$ and $3y-8z=0$ ...(1)

$\therefore$ The plane passing through line (1) is $9x-4y+\lambda(3y-8z)=0$

i.e. $9x+y(3\lambda-4)-8\lambda z=0$ ...(2)

$\therefore$ The equation to the normal line of (2) is $\frac{x}{9}=\frac{y}{3\lambda-4}=\frac{z}{-8\lambda}$ ...(3)

Now plane (2) is a tangent plane to the cone $9x^2-4y^2+16z^2=0$ ...(4)

$\Leftrightarrow$ The normal line (3) is a generator of the reciprocal cone of the cone (4)

$\therefore$ The equation of the reciprocal cone of (4) is $\frac{x^2}{9}-\frac{y^2}{4}+\frac{z^2}{16}=0$ ...(5)

Since (3) is a generator of (5)

$\frac{9^2}{9}-\frac{(3\lambda-4)^2}{4}+\frac{(-8\lambda)^2}{16}=0$ *i.e.* $7\lambda^2+24\lambda+20=0 \Rightarrow \lambda=-2$ or $\frac{-10}{7}$

Hence the equations of tangent planes from (2) are

$$9x-10y+16z=0 \text{ and } 63x-58y+80z=0$$

Ex. 5. *Prove that the equation* $\sqrt{fx}\pm\sqrt{gy}\pm\sqrt{hz}=0$ *represents a cone that touches the coordinate planes and find its reciprocal cone.* **(S.V.U., A12, S. K. D. M15)**

Sol. The given equation is $\sqrt{fx}\pm\sqrt{gy}=\mp\sqrt{hz}$

$\Rightarrow fx+gy\pm 2\sqrt{fgxy}=hz$

$\Rightarrow (fx+gy-hz)^2=4fgxy$

$\Rightarrow f^2x^2+g^2y^2+h^2z^2-2ghyz-2hfzx-2fgxy=0$ ... (1)

This being a homogeneous equation of second degree, represents a quadric cone

The coordinate plane $x=0$ meets (1) in $g^2y^2+h^2z^2-2ghyz=0 \Rightarrow (gy-hz)^2=0$

$\Rightarrow$ which being a perfect square $\Rightarrow x=0$ touches it similarly we can show that $y=0, z=0$ also touch (1).

Again from the cone (1) : $'a'=f^2, 'b'=g^2, 'c'=h^2, 'f'=-gh, 'g'=-hf, 'h'=-fg$

$\therefore$ $\mathrm{A}=bc-f^2=g^2h^2-(-gh)^2=0$

Similarly $\mathrm{B}=0, \mathrm{C}=0, \mathrm{F}=gh-af=(-hf)(-fg)-f^2(-gh)=2f^2gh$

Similarly $\mathrm{G}=2g^2hf, \mathrm{H}=2h^2fg$

$\therefore$ Reciprocal cone is $\mathrm{A}x^2+\mathrm{B}y^2+\mathrm{C}z^2+2\mathrm{F}yz+2\mathrm{G}zx+2\mathrm{H}xy=0$

$\Rightarrow 2f^2ghyz+2g^2hfzx+2h^2fgxy=0 \Rightarrow fyz+gzx+hxy=0$

Ex. 6. *Find the condition that one plane* $ux+vy+wz=0$ *may touch the cone* $ax^2+by^2+cz^2=0$ **(A.K.N.U. M18)**

Sol. Equation to the normal to the given plane is $\dfrac{x}{u}=\dfrac{y}{v}=\dfrac{z}{w}$ .... (1)

Equation to the reciprocal cone of $ax^2+by^2+cz^2=0$ is $\dfrac{x^2}{a}+\dfrac{y^2}{b}+\dfrac{z^2}{c}=0$ .... (3)

Now the plane touches the cone (2)

$\Leftrightarrow$ the normal of the plane lies on cone (2)

$\Leftrightarrow \dfrac{u^2}{a}+\dfrac{v^2}{b}+\dfrac{w^2}{c}=0$ which is the required condition.

Ex. 7. *Find the equation of the cone which touches the three coordinate planes and the planes* $x+2y+3z=0, 2x+3y+4z=0$.

Sol. The equation to the cone touching the three axes can be taken as

$$\sqrt{fx}\pm\sqrt{gy}\pm\sqrt{hz}=0$$

Its reciprocal cone is $fyz+gzx+hxy=0$ .... (2)

The planes $x+2y+3z=0$ and $2x+3y+4z=0$

touch the cone (1) $\Leftrightarrow$ their normals lies on (2)

$\Rightarrow$ D.r.'s of the normal $(1,2,3)$ and $(2,3,4)$ satisfy (2)

(*i*) $f(2)(3)+g(3)(1)+h(1)(2)=0 \Rightarrow 6f+3g+2h=0$ .... (3)

(*ii*) $f(3)(4)+g(4)(2)+h(2)(3)=0 \Rightarrow 6f+4g+3h=0$ .... (4)

Solving (3) and (4) : $\frac{f}{9-8}=\frac{g}{12-18}=\frac{h}{24-18}$

Hence (1) becomes $\sqrt{x}+\sqrt{-6y}+\sqrt{6z}=0$

EXERCISE 7 (*g*)

1. Prove that the perpendiculars drawn from the origin to the tangent planes to the cone $3x^2+4y^2+5z^2+2yz+4zx+6xy=0$ lie on the cone $19x^2+11y^2+3z^2+6yz-10zx-26xy=0$.
2. Find the equation of the quadric cone which touches the three coordinate plaes and the planes $x+y+z=0$ and $2x-3y+z=0$.
3. Prove that the perpendiculars drawn from the origin to tangent planes to the $2x^2+3y^2+4z^2+2yz+4zx+6xy=0$ lie on the cone $11x^2+4y^2-3z^2+8yz-6zx-20xy=0$.
4. Find the condition that the plane $lx+my+nz=0$ may touch the cone $2x^2-3y^2+z^2=0$ and find the equation of the reciprocal cone.
5. Find the equation of the cone which touches the three coordinate plaes and the three mutually perpendicular planes $x-y+z=0$, $2x+3y+z=0$, $4x-y-5z=0$.
6. Find the plane which touches the cone $x^2+2y^2-3z^2+2yz-5zx+3xy=0$ along the generator whose direction ratios are $(1,1,1)$. ***(A.K.N.U. M18)***

ANSWERS

2. $64x^2+9y^2+25z^2-30yz+80zx+48xy=0$
4. $3l^2-2m^2+6n^2=0$, $3x^2-2y^2+6z^2=0$
5. $64x^2+9y^2+25z^2+30yz+80zx-48xy=0$
6. $x^2+36y^2+36z^2+72yz-12zx+12xy=0$

7. 20. INTERSECTION OF TWO CONES WITH A COMMON VERTEX

In general two cones with a common vertex intersect along four comman generators.

Let $S=0$ and $S'=0$ be two cones with origin as the comman vertex, then $S+\lambda S'=0$ represents the general equation of a cone whose vertex is at the origin and which passes through the four comman generators of the two cones.

Cor. Let λ be so chosen that $S'+\lambda S'=0$ becomes the product of two linear factors the two linear factors equated to zero represent the equations to a pair of planes through the common generators.

In that case the values of λ are the roots of the λ - cubic equation

$$\begin{vmatrix} a+\lambda a' & h+\lambda h' & g+\lambda g' \\ h+\lambda h' & b+b'\lambda & f+\lambda f' \\ g+\lambda g' & f+\lambda f' & c+\lambda c' \end{vmatrix}=0$$

The three values of λ give the three pairs of planes through the four comman generators.

SOLVED PROBLEMS

Ex. 1. *Find the equation of the cone which passes through the common generators of the cones* $2x^2-4y^2-z^2=0$ *and* $10xy-2yz+5zx=0$ *may the line with direction ratios* $(1,2,3)$.

Sol. Let the required cone be $2x^2-4y^2-z^2+\lambda(10xy-2yz+5zx)=0$ .... (1)

This is a quadric cone with vertex at the origin.

The line with d.r.'s $(1,2,3)$ lie on (1)

$\Leftrightarrow 2(1)^2-4(2)^2-(3)^2+\lambda[10(1)(2)-2(2)(3)+5(3)(1)]=0 \Rightarrow -23+23\lambda=0 \Rightarrow \lambda=1$

$\therefore$ Required cone is $2x^2-4y^2-z^2+10xy-2yz+5zx=0$

Ex. 2. *Find the condition that the lines of the section of the plane* $lx+my+nz=0$ *and the cones* $ax^2+by^2+cz^2=0$ *and* $fyz+gzx+hxy=0$ *should be concident.*

Sol. Any cone through the intersection of the two given cones is

$$ax^2+by^2+cz^2+\lambda(fyz+gzx+hxy)=0 \quad \text{.... (1)}$$

Given that the plane $lx+my+nz=0$ cuts (1) in coincident lines

$\Rightarrow$ for some value of λ (1) must represent a pair of planes.

Let $l_1x+m_1y+n_1z=0$ .... (2) be the other plane.

Then $ax^2+by^2+cz^2+\lambda(fyz+gzx+hxy)=(lx+my+nz)(l_1x+m_1y+n_1z)$

$\Rightarrow ll_1=a, mm_1=b, nn_1=c \quad \Rightarrow l_1=a/l,\ m_1=b/m,\ n_1=c/n$

Again $\lambda f=mn_1+m_1n=\dfrac{cm}{n}+\dfrac{bn}{m}=\dfrac{cm^2+bn^2}{mn}$

Similarly $\lambda g=\dfrac{an^2+cl^2}{nl}$ and $\lambda h=\dfrac{am^2+bl^2}{lm}$

$\Rightarrow \dfrac{cm^2+bn^2}{fmn}=\dfrac{an^2+cl^2}{gnl}=\dfrac{am^2+bl^2}{hlm}$ which is the required condition.

EXERCISE 7 (*h*)

1. Show that the equation of the cone through the intersection of the cones $x^2-2y^2+3z^2-4yz+5zx-6xy=0$ and $2x^2-3y^2+4z^2-5yz+6zx-10xy=0$ and the line with d.r's $(1,1,1)$ is $y^2-2z^2+3yz-4zx+2xy=0$.

2. Show that the plane $3x+2y-4z=0$ passes through a pair of common geerators $27x^2+20y^2-32z^2=0$ and $2yz+zx-4xy=0$. Show that the plane through other two generators is $9x+10y+8z=0$.

UNIT - V

8. **The Cylinder**

   Definition of a cylinder. Equation to the cylinder whose generators intersect a given conic and are parallel to a given line. Enveloping cylinder of a sphere. The right circular cylinder. Equation of the right circular cylinder with a given axis and radius.

9. **The Central Conicoid**

   The general equation of the second degree and the various surfaces represented by it; Shapes of some surfaces. Nature of Ellipsoid. Nature of Hyperboloid of one sheet.

8

THE CYLINDER

8.1. Definition. *Let P be a point and L be a line through P, with direction ratios l, m, n. If S is a surface such that* $P \in S \Rightarrow L \subset S$ *then S is called a cylinder.*

The line L is called a generator of the cylinder.

Note. Every generator of the cylinder has direction ratios *l, m, n.*

8. 2. ELLIPTIC CYLINDER

The surface represented by the equation

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is called the **Elliptic Cylinder.**

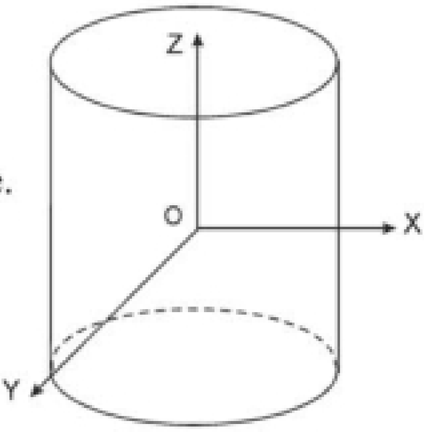


Fig. 6

The section of this cylinder with the plane $z = k$ is an ellipse.
Every such ellipse has the centre on the z-axis.
Hence z - axis is called the axis of the cylinder.
If $a = b$ then the cylinder is a **surface of revolution.**
The section with the plane $z = k$ is a circle.
Hence the surface is called the **right circular cylinder.**

8. 3 HYPERBOLIC CYLINDER

The surface represented by the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is called the **hyperbolic cylinder with Z-axis as the axis.**

The section common to the plane $z = k$ with the cylinder is a **hyperbola.**

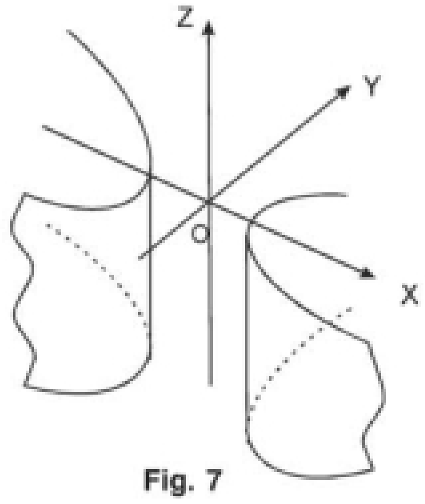


Fig. 7

8. 4 PARABOLIC CYLINDER

The surface represented by the equation
$y^2 = 2a^2x$ is called the parabolic cylinder.
The section of the plane $z = k$
with the cylinder is a parabola.
Similarly, $x^2 = 2b^2y, x^2 = 2c^2z$ etc,
are all parabolic cylinders.

8. 5 CYLINDER WITH BASE GUIDING CURVE

Definition. *Let P be a point and L be a line through P, with direction ratios l, m, n. C be a curve in the plane not parallel to L. If S is a surface such that* $P \in C \Rightarrow L \subset S$, *then S is the cylinder with C as the base guiding curve.*

L is called the generator of the cylinder.

Another Definition. *A cylinder is a surface generated by a straight line which is always parallel to a fixed line with the condition that it may intersect a given curve or touch a given surface.*

The given curve is called the guiding curve.

8.6. Theorem. ***The equation of the cylinder whose generators are parallel to the line*** $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ ***and the base curve*** $f(x,y)=0, z=0$ ***is*** $f\left(x-\frac{lz}{n}, y-\frac{mz}{n}\right)=0$ **(*O.U. A 88*)**

Proof. All generators are parallel to $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$

Let $P(x_1, y_1, z_1)$ be a point on the cylinder.

$\therefore$ The equation to a line L through P with direction ratios (l, m, n) is $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad (=r)$

Any point on this line is $(lr+x_1, mr+y_1, nr+z_1)$

This point lies on the curve $f(x,y)=0, z=0$

$\Leftrightarrow \; f(x_1+lr, y_1+mr)=0, \; z_1+nr=0 \Rightarrow r=\frac{-z_1}{n}$

$\Leftrightarrow f\left(x_1-\frac{lz_1}{n}, y_1-\frac{mz_1}{n}\right)=0$

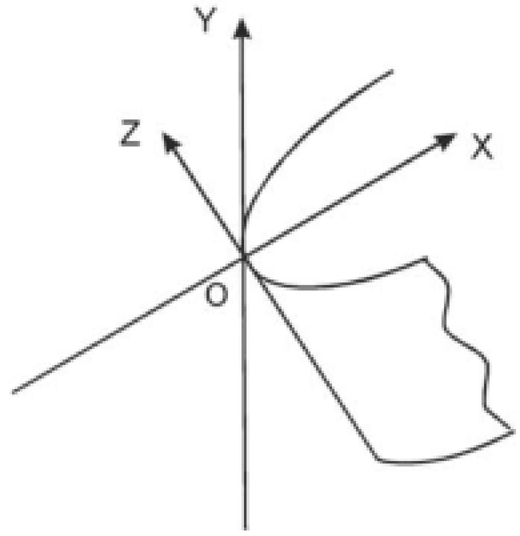


Fig. 8

Hence the locus of P is the cylinder whose equation is $f\left(x-\frac{lz}{n}, y-\frac{mz}{n}\right)=0$

8. 7. EQUATION OF A CYLINDER

Find the equation of the cylinder whose generators intersect the conic $ax^2+2hxy+by^2+2gx+2fy+c=0, z=0$ and are parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$. **(*O.U. M 15*)**

Let $P(\alpha, \beta, \gamma)$ be a point on the cylinder. Equation to the generator through P is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=k$$

Any point on the line is $(\alpha+lk, \beta+mk, \gamma+nk)$

This point P lies on the conic

$$\Leftrightarrow a(\alpha+lk)^2+2h(\alpha+lk)(\beta+mk)+b(\beta+mk)^2+2g(\alpha+lk)+2f(\beta+mk)+c=0$$

and $\gamma+nk=0 \Rightarrow k=-\frac{\gamma}{n}$

Eliminating k from (1)

$$a\left(\alpha-\frac{l\gamma}{n}\right)^2+2h\left(\alpha-\frac{l\gamma}{n}\right)\left(\beta-\frac{m\gamma}{n}\right)+b\left(\beta-\frac{m\gamma}{n}\right)^2+2g\left(\alpha-\frac{l\gamma}{n}\right)+2f\left(\beta-\frac{mk}{n}\right)+c=0$$

Hence the locus of the point P is

$$a\left(x-\frac{lz}{n}\right)^2+2h\left(x-\frac{lz}{n}\right)\left(y-\frac{mz}{n}\right)+b\left(y-\frac{mz}{n}\right)^2+2g\left(x-\frac{lz}{n}\right)+2f\left(y-\frac{mz}{n}\right)+c=0$$

$$\Rightarrow a\,(nx-lz)^2+2h\,(nx-lz)(ny-mz)+b\,(ny-mz)^2+2gn\,(nx-lz)+2\,fn\,(ny-mz)+cn^2=0$$

Cor. I. If generator is paralled to Z- axis substituting $l=0, m=0, n=1$ in the above equation, we get

$ax^2+2hxy+by^2+2gx+2fy+c=0$, or $f(x,y)=0$

This equation in three dimension represents a cylinder, whose axis is parallel to Z-axis. Where as in two dimension it represents a conic or a pair of straight lines under certain condition.

Similarly if the axis be parallel to Y- axis then the equation will be of the type $f\,(x,z)=0$ and $f(y,z)=0$ if the axis be parallel to X-axis.

Cor. II. Equation of the cylinder which intersects the curve $f(x,y,z)=0, \phi\,(x,y,z)=0$ and whose generators are parallel to Z - axis is obtained by eliminating Z between the two given equations.

In case the axis be parallel to Y - axis then eliminate y and if it be parallel to X - axis then eliminate x.

SOLVED PROBLEMS

Ex. 1. *Find the equation of the cylinder whose generators are parallel to* $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ *and which passes through the curve* $x^2+y^2=16, z=0$

(O. U. M15, S. V.U. M 15, A11, A. N. U. M15, S.U.M M18, A.U M18)

Sol. Given the d.rs of the generators as $(1,2,3)$

Let P (x_1, y_1, z_1) be a point on the cylinder .

$\therefore$ Equation to the generator be $\frac{x-x_1}{1}=\frac{y-y_1}{2}=\frac{z-z_1}{3} \quad (=r)$

Any point on the line is $(r+x_1, 2r+y_1, 3r+z_1)$.

This point lies on the curve $x^2+y^2=16,\ z=0$

$\Leftrightarrow \quad (x_1+r)^2+(y_1+2r)^2=16, 3r+z_1=0 \Rightarrow r=-\frac{z_1}{3} \Leftrightarrow \left(x_1-\frac{z_1}{3}\right)^2+\left(y_1-\frac{2z_1}{3}\right)^2=16$

$\therefore$ The locus of P is the cylinder $9x^2+9y^2+5z^2-6zx-12yz-144=0$.

Ex. 2. *Find the equation of the quadric cylinder whose generators are parallel to z- axis and guiding curve* $ax^2+by^2+cz^2=1, lx+my+nz=p$

Sol. The equation of the required cylinder is obtained by eliminating z between the equations $ax^2+by^2+cz^2=1$ and $lx+my+nz=p$

$\Rightarrow$ Substituting $z=\frac{p-lx-my}{n}$ in the other equation

$$ax^2+by^2+c\left(\frac{p-lx-my}{n}\right)^2=1$$

$\Rightarrow an^2x^2+bn^2y^2+c\,(p-lx-my)^2=n^2$ which is the required cone.

Ex. 3. *Find the equation of the cylinder, whose generators have direction cosines l, m, n and which passes through $x^2+z^2=a^2$ in ZOX - plane.* $(y=0)$ **(O. U. M15)**

Sol. Let $P(x_1, y_1, z_1)$ be a point on the generator

$\therefore$ Equation to the generator is $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=k$

Any point on this is (x_1+lk, y_1+mk, z_1+nk)

The point lies on $x^2+z^2=a^2, y=0$

$\Leftrightarrow (x_1+lk)^2+(z_1+nk)^2=a^2, y_1+mk=0 \Rightarrow k=-\frac{y_1}{m}$

Eliminating k, $\left(x_1-\frac{ly_1}{m}\right)^2+\left(z_1-\frac{ny_1}{m}\right)^2=a^2$

$\therefore$ Locus of P is the cylinder $(mx-ly)^2+(mz-ny)^2=a^2m^2$

Ex. 4. *Find the equation of the cylinder whose generators are parallel to the line $y=mx, z=nx$ and which intersect the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0$.*

Sol. Given line is $\frac{x}{1}=\frac{y}{m}=\frac{z}{n}$ ... (1)

Let $P(x_1, y_1, z_1)$ be a point on the generator

$\therefore$ Equation to the generator is $\frac{x-x_1}{1}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=k$

Any point on the generator is (x_1+k, y_1+mk, z_1+nk)

This meets the curve $b^2x^2+a^2y^2=a^2b^2, z=0$

$\Leftrightarrow b^2(x_1+k)^2+a^2(y_1+mk)^2=a^2b^2, z_1+nk=0$

Eliminating k, $b^2\left(x_1-\frac{z_1}{n}\right)+a^2\left(y_1-\frac{mz_1}{n}\right)^2=a^2b^2$

$\therefore$ Locus of P is the cylinder $b^2(nx-z)^2+a^2(ny-mz)^2=a^2b^2n^2$

EXERCISE 8 (*a*)

1. Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{1}=\frac{-y}{2}=\frac{z}{3}$ and whose base curve is $x^2+2y^2=1, z=3$

   (O. U. A10, A12, N. U. A10, S.K.U M18, K.U M18, A.U M18, V.S.P.U M18)

2. Find the equation of the cylinder whose generators have the direction ratios $(1,-2,3)$ and whose guiding curve is $x^2+2y^2=1, z=0$.

3. Find the equation of the cylinder whose generators intersect the curve $ax^2 + by^2 = 2z$, $lx + my + nz = p$ and are parallel to Z- axis. ***(O.U. M15)***

4. The cylinder has the guiding curve $x^2 + y^2 = z, x + y + z = 1$ and its generators are parallel to Z - axis. Find its equation. ***(A.U. A11)***

ANSWERS

1. $3x^2 + 6y^2 + 3z^2 + 8yz - 2zx + 6x - 24y - 18z + 24 = 0$

2. $3x^2 + 6y^2 + 3z^2 - 2zx + 8yz - 3 = 0$ **3.** $n(ax^2 + by^2) + 2lx + 2my - 2p = 0$

4. $x^2 + y^2 + x + y - 1 = 0$

8. 8. THE RIGHT CIRCULAR CYLINDER

Definition. *Let (l, m, n) be the direction numbers of the normal to the plane π containing a circle C. Let L be a normal line to the plane π and passing through a point P.*

If S is the surface such that $P \in C \Rightarrow L \subset S$ then S called the right circular cylinder. The normal through the centre of the circle is called the axis of the cylinder and the radius of the circle is called the radius of the cylinder.

Another Definition. *A right circular cylinder is a surface generated by a line which intersects a fixed circle and is perpendicular to its plane.*

The fixed circle is called the guiding circle. The normal to the plane of this circle through its centre is called the axis of the cylinder.

Every plane perpendicular to its axis intersects the cylinder along a circle. All thse circles have the same radius which is also called the radius of the cylinder.

8. 9. Theorem. ***The equation to the right circular cylinder with radius r and axis the line*** $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$ ***is***

$$[(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 - r^2](l^2 + m^2 + n^2) = [l(x-\alpha) + m(y-\beta) + n(z-\gamma)]^2$$

Proof. Let $P(x_1, y_1, z_1)$ be a point on the cylinder. $A(\alpha, \beta, \gamma)$ is the given point on the axis whose d.c's are $\dfrac{l}{\sqrt{(l^2+m^2+n^2)}}, \dfrac{m}{\sqrt{(l^2+m^2+n^2)}}, \dfrac{n}{\sqrt{(l^2+m^2+n^2)}}$

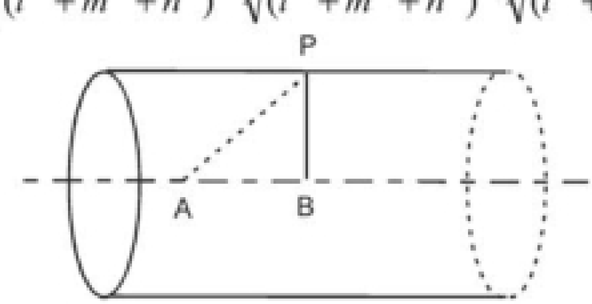


Fig. 9

If B is the projection of P on the axis, then AB = Proj. of AP on the axis

$$= \frac{l(x_1-\alpha) + m(y_1-\beta) + n(z_1-\gamma)}{\sqrt{(l^2+m^2+n^2)}} \quad \because \quad AP^2 = AB^2 + PB^2 \Rightarrow AP^2 - PB^2 = AB^2$$

$$\Rightarrow (x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2-r^2=\left[\frac{l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)}{\sqrt{l^2+m^2+n^2}}\right]^2$$

$\therefore$ The locus of P is the right circular cylinder given by the equation

$$[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-r^2](l^2+m^2+n^2)=[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2$$

Corollary. If Z axis is the axis of the cylinder, then the equation to the axis is

$$\frac{x-0}{0}=\frac{y-0}{0}=\frac{z-0}{1}$$

$\therefore$ The equation to the right circular cylinder with z axis as its axis and radius r is
$(x^2+y^2+z^2-r^2)(1)=z^2 \Rightarrow x^2+y^2=r^2$

SOLVED PROBLEMS

Ex. 1. *Find the equation to the right circular cylinder whose guiding circle is* $x^2+y^2+z^2=9,\ x-y+z=3$ ***(S.V. U. M15, A.N.U. M15, 06, A11, A.U M18, K.U M18, S.U.M M18)***

Sol. Given sphere is $x^2+y^2+z^2=9$

$\therefore$ Centre $=(0,0,0)$, radius $=3$. Given plane is $x-y+z=3$ .... (1)

$\therefore$ The equation to the normal of the plane through O $\frac{x}{1}=\frac{y}{-1}=\frac{z}{1}$

Let $P(x_1, y_1, z_1)$ be a point on the cylinder. A is the projection B on axis. ***(N.U.A2006)***

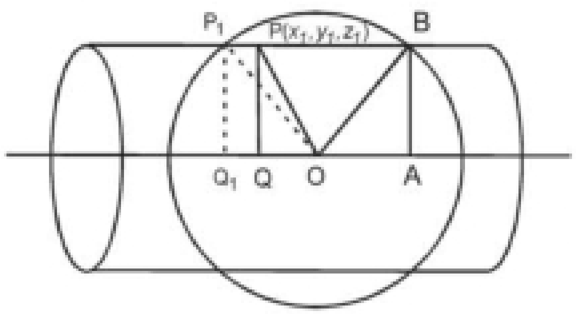


Fig. 10

OA $=\perp r$ distance of O from plane (1) $=\left|\frac{-3}{\sqrt{(1+1+1)}}\right|=\sqrt{3}$

$AB^2=$ (radius of the circle)$^2=OB^2-OA^2=3^2-3=6$

$\therefore$ Equation of the right circular cylinder is

$$[(x-0)^2+(y-0)^2+(z-0)^2-6]\ [1+1+1]=[1(x-0)-1(y-0)+1(z-0)]^2$$

$$\Rightarrow [3(x^2+y^2+z^2)-6]=[x-y+z]^2 \Rightarrow x^2+y^2+z^2+xy+yz-zx-9=0$$

Ex. 2. *Find the equation of a right circular cylinder of radius 2 whose axis passes through the point* $(1,2,3)$ *and has direction ratios* $(2,-3,6)$. **(K. U. A11, S. V. U. A12)**

Sol. Equation of the axis will be $\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z-3}{6}$

radius of the cylinder $r=2$

Equation to the required cylinder is

$$[(x-1)^2+(y-2)^2+(z-3)^2-(2)^2][(2)^2+(-3)^2+(6)^2]=[2(x-1)-3(y-2)+6(z-3)]^2$$

$\Rightarrow (x^2+y^2+z^2-2x-4y-6z+10)(49)=(2x-3y+6z-14)^2$

$\Rightarrow 45x^2+40y^2+13z^2+36yz-24zx+12xy-42x-280y-126z+294=0$

Ex. 3. *Find the equation of the right circular cylinder whose axis is* $\frac{x-2}{2}=\frac{y-1}{1}=\frac{z}{3}$ *and which passes through* $(0,0,3)$.

Sol. Let r be the radius of the cylinder.

$\therefore$ Equation to the cylinder is

$$[(x-2)^2+(y-1)^2+z^2-r^2][(2)^2+(1)^2+3^2]=[2(x-2)+1(y-1)+3z]^2 \quad \dots (1)$$

This cylinder passes through $(0,0,3)$

$\Leftrightarrow [4+1+9-r^2](14)=(-4-1+9)^2 \Rightarrow r^2=90/7$

$\therefore$ Required cylinder is from (1) : $\left[(x-2)^2+(y-1)^2+z^2-\frac{90}{7}\right](14)=(2x+y+3z-5)^2$

$\Rightarrow 14(x^2+y^2+z^2-4x-2y)-180=(2x+y+3z-5)^2$

$\Rightarrow 10x^2+13y^2+5z^2-4xy-6yz-12zx-36x-18y+30z-135=0$

Ex. 4. *Find the equation of the right circular cylinder whose axis is* $x-2=z, y=0$ *and passes through the point* $(3,0,0)$.

Sol. Equation to the axis is $\frac{x-2}{1}=\frac{y}{0}=\frac{z}{1}$. Let r be the radius of the cylinder.

Equation to the cylinder is $[(x-2)^2+y^2+z^2-r^2](1+0+1)=[(x-2)+0.y+1.z]^2$

This passes through $(3,0,0)$ $\Leftrightarrow (1-r^2)(2)=1 \Rightarrow r^2=\frac{1}{2}$

$\therefore$ Required cylinder is $\left[(x-2)^2+y^2+z^2-\frac{1}{2}\right](2)=(x-2+z)^2$

$\Rightarrow x^2+2y^2+z^2-2zx-4x+4z+3=0$

EXERCISE 8 (*b*)

1. Find the equation to the right circular cylinder whos axis is $\frac{x-1}{2}=\frac{y}{3}=\frac{z-3}{1}$ and of radius 2. ***(A. N. U. M15, O. U A12, S.V.U. M18)***
2. Find the equation to the right circular cylinder whose axis $x=2y=-z$ and having the radius 4.
3. Find the equation to the right cicular cylinder whose guiding curve is the circle through the points $(1,0,0),(0,1,0)$ and $(0,0,1)$.
4. Find the equation of the right circular cylinder of radius 2 whose axis is the line $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$.

5. Find the equation of the right circular cylinder of radius 5 and whose axis is the line $\frac{x}{2}=\frac{y}{3}=\frac{z}{6}$. ***(A.U. M18)***

6. Find the right circular cylinder whose arc section is the circle $x^2+y^2+z^2-x-y-z=0$, $x+y+z=1$.

7. Find the equation to the right circular cylinder of radius 3 and having for its axis the line $\frac{x-1}{2}=\frac{y-3}{2}=\frac{z-5}{-1}$.

ANSWERS

1. $10x^2+5y^2+13z^2-12xy-6yz-4zx-8x+30y-74z+59=0$

2. $5x^2+8y^2+5z^2+4yz+8zx-4xy-144=0$

3. $x^2+y^2+z^2-yz-zx-xy-1=0$

4. $5x^2+8y^2+5z^2-4xy-4yz-8zx+22x-16y-14z-10=0$

5. $45x^2+40y^2+13z^2-12xy-36yz-24zx-1225=0$

6. $x^2+y^2+z^2-yz-zx-xy=1$

7. $5x^2+5y^2+8z^2+4yz+4zx-8xy-6x-42y-96z+225=0$

8. 10. ENVELOPING CYLINDER

Definition. *The set of parallel tangent lines with direction ratios (l, m, n) to a surface form a cylinder called the enveloping cylinder in the direction of (l, m, n).*

8.11. Theorem. ***The equation to the enveloping cylinder of the surface $x^2+y^2+z^2=a^2$, in the direction of (l, m, n) is***

$$(lx+my+nz)^2=(l^2+m^2+n^2)(x^2+y^2+z^2-a^2)$$

Proof. Let $P(x_1, y_1, z_1)$ be a point on the tangent line of the given surface

$\therefore$ The equation to the line through P with d.r's (l, m, n) be

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}\,(=r) \qquad \text{.... (1)}$$

The point $(lr+x_1, mr+y_1, nr+z_1)$ of the line lies on the surface

$$x^2+y^2+z^2-a^2=0 \qquad \text{..... (2)}$$

$\Leftrightarrow\ (lr+x_1)^2+(mr+y_1)^2+(nr+z_1)^2-a^2=0$

$\Leftrightarrow\ r^2(l^2+m^2+n^2)+2r\,(lx_1+my_1+nz_1)+(x_1^2+y_1^2+z_1^2-a^2)=0$

The line (1) is a tangent line to (2)

$\Leftrightarrow\ (lx_1+my_1+nz_1)^2-(l^2+m^2+n^2)(x_1^2+y_1^2+z_1^2-a^2)=0$

$\therefore$ The locus of P is the enveloping cylinder

$(lx+my+nz)^2=(l^2+m^2+n^2)(x^2+y^2+z^2-a^2)$

SOLVED PROBLEMS

Ex. 1. *Find the equation of the enveloping cylinder of the sphere* $x^2+y^2+z^2-2x+4y-1=0$, *having its generators parallel to the line* $x=y=z$.

(N.U. A.06, A12, A. U. A12, A.U M18, S.V.U M18, K.U M18)

Sol. Given sphere is $x^2+y^2+z^2-2x+4y-1=0$. Given line is $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$ .... (1)

Let $P(x_1, y_1, z_1)$ be a point on the tangent line of the sphere.

$\therefore$ The equation to a line through P and parallel to line (1)

$$\frac{x-x_1}{1}=\frac{y-y_1}{1}=\frac{z-z_1}{1}(=r) \quad .... (2)$$

Any point on (2) is $(r+x_1, r+y_1, r+z_1)$

This lies on the sphere $\Leftrightarrow (r+x_1)^2+(r+y_1)^2+(r+z_1)^2-2(r+x_1)+4(r+y_1)-1=0$

$\Leftrightarrow 3r^2+2r(x_1+y_1+z_1+1)+x_1^2+y_1^2+z_1^2-2x_1+4y_1-1=0$

The line (2) is a tangent line $\Leftrightarrow 4(x_1+y_1+z_1+1)^2-4(3)(x_1^2+y_1^2+z_1^2-2x_1+4y_1-1)=0$

$\therefore$ The locus of P is the enveloping cylinder

$x^2+y^2+z^2-xy-yz-zx-4x+5y-z-2=0$

Ex. 2. *Find the equation of a right circular cylinder which envelopes a sphere with centre* (a,b,c) *and radius* r *and has the generators having d.r.'s* (l,m,n).

Sol. Equation to the axis of the cylinder passes through (a,b,c).

Hence equation to the axis is $\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$

Equation to the required cylinder is

$$[(x-a)^2+(y-b)^2+(z-c)^2-r^2](l^2+m^2+n^2)=[l(x-a)+m(y-b)+n(z-c)]^2$$

EXERCISE 8 (*c*)

1. Find the equation of the enveloping cylinder of the sphere $x^2+y^2+z^2=25$, whose generators are parallel to $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$. **(K. U. A11, S.K.U M18)**
2. Find the enveloping cylinder of the sphere $x^2+y^2+z^2+2x-4y-1=0$ having its generators parallel to $x=y=z$.
3. Find the equation of the enveloping cylinder of the conicoid $ax^2+by^2+cz^2=1$, whose generators are parallel to $x=y=z$. **(A.U M18)**

ANSWERS

1. $13x^2+10y^2+5z^2-4xy-6zx-12yz-350=0$

2. $x^2+y^2+z^2-yz-zx-xy+4x-5y+z-2=0$ **3.** $\sum(b+c)x^2-2\sum abxy-(a+b+c)=0$

9

THE CENTRAL CONICOID

9.1. Definition. *Let* $S \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$ *define a locus in space. If the same locus cannot be defined by a first degree equation, then the surface S is called a* **conicoid** *or a* **quadric**.

In the previous chapters, we have seen that a line and a quadric have two points in common. The name conicoid is due to the fact that every plane section of a quadric is a conic.

The above general equation can, by proper transformation of coordinate axes, be reduced to any one of the following forms.

I (*i*) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ *Ellipsoid*

(*ii*) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$ *Hyperboloid of one sheet*

(*iii*) $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$ *Hyperboloid of two sheets*

II (*i*) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=2z$ *Elliptic paraboloid*

(*ii*) $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=2z$ *Hyperbolic paraboloid*

9. 2 THE ELLIPSOID

(A.U. A11)

The surface S defined by the equation

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ is called an *ellipsoid.*

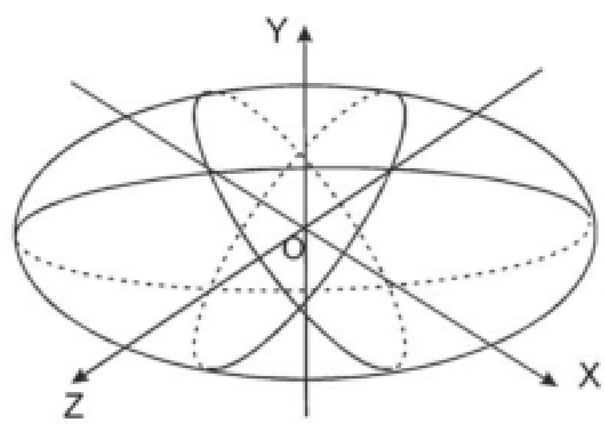


Fig. 11

(*i*) $(x, y, z) \in S \Rightarrow (-x, -y, -z) \in S$

$\therefore$ The origin is the mid point of all chords joining two such points. Such chords are called *diameters* and the origin is called the centre of the conicoid.

(*ii*) $(x, y, z) \in S \Rightarrow (x, y, -z) \in S$

$\therefore$ The surface is symmetrical about XY plane.

Similarly, it is also symmetrical bout YZ plane and ZX plane. These coordinate plaes are called the *principal planes* and their common lines of intersection which are the coordinate axes are called *principal* axes of the conicoid.

(*iii*) $A(a, 0, 0)$, $A'(-a, 0, 0)$ are the common points of the conicoid with the X-axis. Similarly $B(0, b, 0)$, $B'(0, -b, 0)$ and $C(0, 0, c)$, $C'(0, 0, -c)$ are the common points with Y and Z axes.

The length AA', BB', CC' are called the *principal diameters* of the ellipsoid.

(*iv*) The plane section common to the XY plane and the ellipsoid is given by the equations $z = 0$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

This equation represents an ellipsoid in XY plane. Similarly, the sections of YZ and ZX planes with the ellipsoid, are ellipses.

(*v*) The equation of the surface is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ *i.e.* $\frac{y^2}{a^2} + \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2}$

when $x < -a$ and $x > a$ is $x^2 > a^2$ *i.e.* $\frac{x^2}{a^2} > 1$ *i.e.* $1 - \frac{x^2}{a^2} < 0$ $\therefore \frac{y^2}{b^2} + \frac{z^2}{c^2} < 0$

$\therefore$ The surface does not exist when $x < -a$ and $x > a$. Similarly the surface does not exist when $y < -b, y > b$ and $z < -c, z > c$.

$\therefore$ The surface is bounded by the rectangular parallelopiped formed by the planes $x = \pm a, y = \pm b$ and $z = \pm c$.

$\therefore$ The ellipsoid is a bounded surface.

9. 3. THE HYPERBOLOID OF ONE SHEET

The surface S defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ *is called a hyperboloid of one sheet.*

(*i*) The origin bisects all chords of the conicoid passing through it and therefore is the centre of the conicoid.

(*ii*) The surface is symmetrical about three coordinate planes. So they form the principal planes and the axes form the principal axes of the hyperboloid.

Z axis has no common point with the surface.

(*iii*) The section of the plane $z = k$ with the conicoid is the ellipse whose equations are $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{a^2}, z = k$.

For all real values of k this will be an ellipse and for $k = 0$, it is called the principal ellipse.

(*iv*) The plane $x = k$ has the heperboloid as the common section with the conicoid given by $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 - \frac{k^2}{a^2}, x = k$.

Similar is the case with the plane $y = \lambda$.

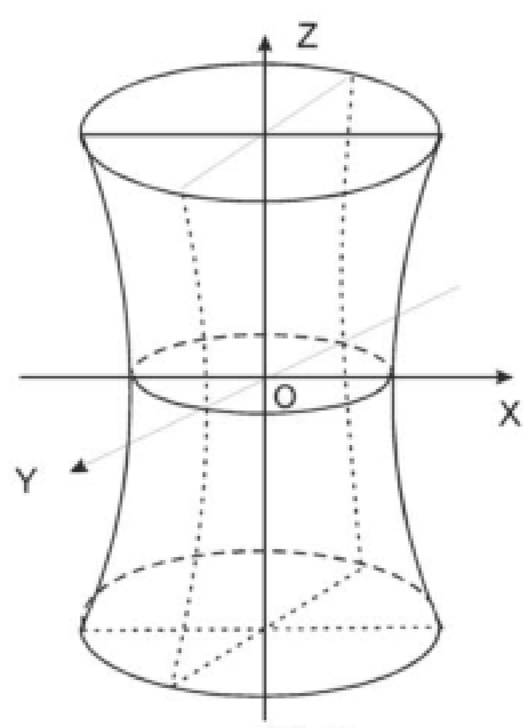


Fig. 12

Note. (*i*) The surface is not a bounded set.

(*ii*) The equation $\frac{x^2}{2}-\frac{y^2}{3}-\frac{z^2}{4}=-1$ *i.e.* $\frac{-x^2}{2}+\frac{y^2}{3}-\frac{z^2}{4}=1$ represents the hyperboloid of one sheet.

9. 4. THE HYPERBOLOID OF TWO SHEETS

The surface defined by the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ is called the hyperboloid of two sheets.

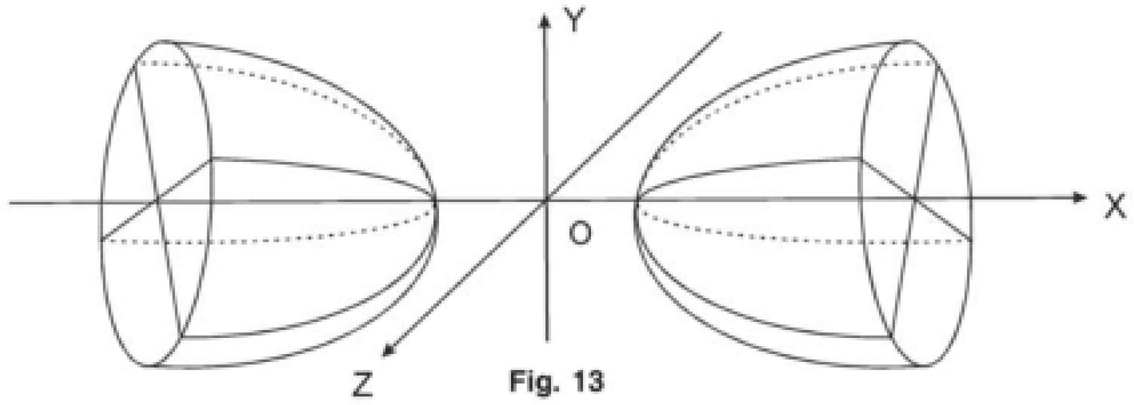


Fig. 13

(*i*) The chords of the coincoid through the origin are bisected at the origin. Therefore, origin is called the centre.

(*ii*) The surface is symmetrical about the coordinate planes. Therefore, the coordinate planes are the principal planes and the coordinate axes are the principal planes.

(*iii*) Only X-axis has common points $(a, 0, 0)$ and $(-a, 0, 0)$ with the hyperboloid; Y and Z axes do not have points in common with the hyperboloid.

(*iv*) The plane sections of the hyperboloid with XY and ZX planes are hyperbolas while the plane section with the YZ plane is a null set.

Note 1. A hyperboloid is not a bounded surface.

2. The equation $\frac{x^2}{3}+\frac{y^2}{4}-\frac{z^2}{6}=-1$ *i.e* $\frac{-x^2}{3}-\frac{y^2}{4}+\frac{z^2}{6}=1$ represents hyperboloid of two sheets.

9. 5. CENTRAL CONICOID

The surfaces cosidered above are represented in general by the equation

$ax^2+by^2+cz^2=1$ where $abc \neq 0$

The surface is

(*i*) an ellipsoid if a, b, c are all positive.

(*ii*) hyperboloid of one sheet if two of them are positive and one negative.

(*iii*) hyperboloid of two sheets if two of them are negative and one positive.

In each case the origin is the centre of the surface, the coordinate planes are the principal planes and the coordinate axes are the principal axes.

Note. These three surfaces have a centre and three principal planes and are, therefore, called central conicoids.

9. 6. NOTATION

The following notation is adopted in this chapter

$S = ax^2 + by^2 + cz^2 - 1, \qquad S_1 = axx_1 + byy_1 + czz_1 - 1, \quad S_{11} = ax_1^2 + by_1^2 + cz_1^2 - 1$

9. 7. COMMON POINTS OF A LINE WITH A CONICOID

Let the equation to the conicoid S be $S(x, y, z) = ax^2 + by^2 + cz^2 - 1 = 0$

Let the equation to the line L be $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \; (= r)$

If P is a point on the line, then $P = (lr + x_1, mr + y_1, nr + z_1)$

Now, $P \in S \Rightarrow S(lr + x_1, mr + y_1, nr + z_1) = 0$

$\Rightarrow a(lr + x_1)^2 + b(mr + y_1)^2 + c(nr + z_1)^2 - 1 = 0$

$\Rightarrow r^2(al^2 + bm^2 + cn^2) + 2r(alx_1 + bmy_1 + cnz_1) + S_{11} = 0$

Let $\Delta \equiv (alx_1 + bmy_1 + cnz_1)^2 - (al^2 + bm^2 + cn^2)S_{11}$

(*i*) This will be a quadratic equation in r

$\Rightarrow al^2 + bm^2 + cn^2 \neq 0$

and will have two real roots $\Rightarrow \Delta > 0$.

In this case there will be two points of the line common with the conicoid. The line segment joining the two points is the chord of the conicoid.

(*ii*) In the above quadratic equation if $\Delta = 0$, then the two roots are real and equal. Hence the line and the conicoid have only one common point. Then the line is called the tangent line at that common point.

(*iii*) Again in the quadratic equation if $\Delta < 0$, then there are no common points of the line with the conicoid.

(*iv*) If $al^2 + bm^2 + cn^2 = alx_1 + bmy_1 + cnz_1 = S_{11} = 0$, then all the points of the line lie on the conicoid. In this case the line will be a generator and the conicoid will be the hyperboloid of one sheet.

9. 8. TANGENT PLANES

Definition. *Let L be a tangent line at a point P on the conicoid S. The locus of L is called the* **tangent plane** *of S at P.*

9. 9. Theorem. ***Let $P(x_1, y_1, z_1)$ be a point on the conicoid $S = 0$. The equation to the tangent plane at P to the conicoid is $S_1 = 0$.***

Proof. Let the equation to the conicoid be $S \equiv ax^2 + by^2 + cz^2 - 1 = 0$

Let the equation to a line passing through $P(x_1, y_1, z_1)$ be

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \; (= r) \qquad \ldots(1)$$

$P(x_1, y_1, z_1) \in \text{conicoid} \Rightarrow S_{11} \equiv ax_1^2 + by_1^2 + cz_1^2 - 1 = 0$

The point $(lr+x_1, mr+y_1, nr+z_1)$ of the line lies on $S=0$

$\Rightarrow a(lr+x_1)^2+b(mr+y_1)^2+c(nr+z_1)^2-1=0$

$\Rightarrow r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)+S_{11}=0$

The line (*i*) is a tangent line to the conicoid

$\Leftrightarrow (alx_1+bmy_1+cnz_1)^2-(al^2+bm^2+cn^2)S_{11}=0$

$\Leftrightarrow alx_1+bmy_1+cnz_1=0 \qquad (\because S_{11}=0)$

Hence the locus of the tangent line is the plane

$ax_1(x-x_1)+by_1(y-y_1)+cz_1(z-z_1)=0$

i.e. $axx_1+byy_1+czz_1=ax_1^2+by_1^2+cz_1^2=1$

Hence the equation to the tangent plane at $P(x_1, y_1, z_1)$ to the conicoid

$axx_1+byy_1+czz_1-1=0$ *i.e.* $S_1=0$.

Note 1. The equation to the tangent plane at (x_1, y_1, z_1) on the ellipsoid

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1$

2. The equation to the normal to the tangent plane at $P(x_1, y_1, z_1)$ on the conicoid is

$$\frac{x-x_1}{ax_1}=\frac{y-y_1}{by_1}=\frac{z-z_1}{cz_1}$$

9.10. Theorem. ***The necessary and sufficient condition that the plane $lx+my+nz=p$ may be a tangent plane to the conicoid $ax^2+by^2+cz^2=1$ $(abc\neq 0)$ is $p^2=\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}$.*** **(O. U. A11, N.U.A.06)**

Proof. (*i*) Necessary Condition

Let the given plane $lx+my+nz=p$ ...(1)

be the tangent plane at $P(x_1, y_1, z_1)$ on the conicoid $ax^2+by^2+cz^2=1$

But the equation of the plane at P to the conicoid is $axx_1+byy_1+czz_1=1$ ...(2)

(1) and (2) represent the same plane.

$$\Rightarrow \frac{l}{ax_1}=\frac{m}{by_1}=\frac{n}{cz_1}=\frac{p}{1} \Rightarrow x_1=\frac{l}{ap},\ y_1=\frac{m}{bp},\ z_1=\frac{n}{cp} \qquad (\because p\neq 0)$$

$\because$ P lies on the conicoid $\Rightarrow ax_1^2+by_1^2+cz_1^2=1$

$$\Rightarrow a\left(\frac{l}{ap}\right)^2+b\left(\frac{m}{bp}\right)^2+c\left(\frac{n}{cp}\right)^2=1 \Rightarrow \frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2>0$$

The point of contact is $\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$

(*ii*) Sufficiency of the condition

Let $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2>0$

$\therefore\ a\left(\frac{l}{ap}\right)^2+b\left(\frac{m}{bp}\right)^2+c\left(\frac{n}{cp}\right)^2=1 \Rightarrow \left(\frac{l}{ap},\frac{m}{bp},\frac{n}{cp}\right)$ is a point on the conicoid.

Now the tangent plane at this point to the conicoid is

$$a\left(\frac{l}{ap}\right)x+b\left(\frac{m}{bp}\right)y+c\left(\frac{n}{cp}\right)z=1 \Rightarrow lx+my+nz=1$$

Note 1. The equation of the tangent plane to the conicoid $ax^2+by^2+cz^2=1$ can be taken in the form

$$lx+my+nz=\pm\sqrt{\left[\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right]}$$

(*i*) If $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}<0$, then the conicoid has no tangent planes.

(*ii*) If l, m, n are the d.c's of the normal to the tangent plane then the distance of that plane from the origin $=\sqrt{\left[\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right]}$

2. The plane $lx+my+nz=p$ is a tangent plane to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

$\Leftrightarrow p^2=a^2l^2+b^2m^2+c^2n^2$

Equation of the tangent plane to the ellipsoid can be taken in the form

$$lx+my+nz=\pm\sqrt{\left[a^2l^2+b^2m^2+c^2n^2\right]}$$

9. 11. DIRECTOR SPHERE

Definition. *let the conicoid S have a set of three mutually perpendicular tangent planes and P be their common point. The locus of P is a sphere called the* **Director Sphere** *of S.*

9. 12. Theorem. ***Let the conicoid $ax^2+by^2+cz^2=1$, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}>0$, have a set of three mutually perpendicular tangent planes and P be their common point. The locus of P is the sphere $x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.***

Proof. Let a set of three mutually perpendicular tangent planes to the conicoid be

$$l_1x+m_1y+n_1z=p_1;\quad l_2x+m_2y+n_2z=p_2;\quad l_3x+m_3y+n_3z=p_3$$

where $p_i=\sqrt{\left(\frac{l_i^2}{a}+\frac{m_i^2}{b}+\frac{n_i^2}{c}\right)},\quad i=1,2,3.$

Let (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) be taken to be the direction cosines of the normals of the respective planes.

Then, from earlier chapters we know that (l_1, l_2, l_3), (m_1, m_2, m_3) and (n_1, n_2, n_3) form the direction cosines of the three mutually perpendicular directions.

$\therefore$ We have $l_1^2+l_2^2+l_3^2=m_1^2+m_2^2+m_3^2=n_1^2+n_2^2+n_3^2=1$

and $l_1m_1+l_2m_2+l_3m_3=m_1n_1+m_2n_2+m_3n_3=n_1l_1+n_2l_2+n_3l_3=0$

Let $P(x_1, y_1, z_1)$ be the common point of the planes, then

$l_1x_1+m_1y_1+n_1z_1=p_1;\quad l_2x_2+m_2y_2+n_2z_2=p_2;\quad l_3x_3+m_3y_3+n_3z_3=p_3$

Squaring and adding these three equations, we get

$$x_1^2+y_1^2+z_1^2=p_1^2+p_2^2+p_3^2=\frac{\sum l_i^2}{a}+\frac{\sum m_i^2}{b}+\frac{\sum n_i^2}{c}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

$\therefore$ The locus of P is the sphere $x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

Note 1. The sphere $x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ is called the Director Sphere of the conicoid.

2. The Director Sphere of the ellipsoid

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $x^2+y^2+z^2=a^2+b^2+c^2$. ***(A.U. A12)***

SOLVED PROBLEMS

Ex. 1. *Find the equations of tangent planes to $2x^2-6y^2+3z^2=5$, which pass through the line $x+9y-3z=0=3x-3y+6z-5$.* ***(A.U. A12, N. U. A06)***

Sol. Given conicoid is $\frac{2}{5}x^2-\frac{6}{5}y^2+\frac{3}{5}z^2=1$ ...(1)

Any plane through the given line is $3x-3y+6z-5+\lambda(x+9y-3z)=0$

$\Rightarrow x(3+\lambda)+y(9\lambda-3)+z(6-3\lambda)=5$. This touches the conicoid.

$$\Leftrightarrow (3+\lambda)^2\frac{5}{2}+(9\lambda-3)^2\left(\frac{-5}{6}\right)+(6-3\lambda)^2\left(\frac{5}{3}\right)=5^2$$

$\Leftrightarrow \lambda^2=1 \quad \Leftrightarrow \lambda=\pm 1$

$\therefore$ The tangent planes are $4x+6y+3z=5$ and $2x-12y+9z=5$

Ex. 2. *Prove that the locus of the foot of the perpendicular drawn from the centre of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ to any of its tangent planes is $a^2x^2+b^2y^2+c^2z^2=(x^2+y^2+z^2)^2$*

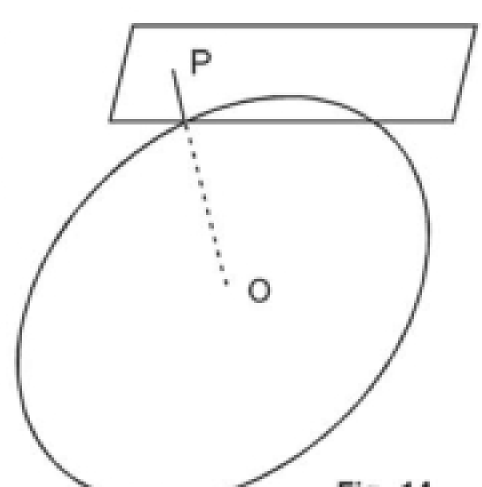


Fig. 14

Sol. Let $P(x_1, y_1, z_1)$ be the foot of the perpendicular from O to the tangent plane π.

$\therefore$ D.r.'s of the normal OP to the plane are x_1, y_1, z_1.

$\therefore$ The equation to the plane through P and $\perp$ to OP is

$x_1(x-x_1)+y_1(y-y_1)+z_1(z-z_1)=0$

$\Rightarrow xx_1 + yy_1 + zz_1 = x_1^2 + y_1^2 + z_1^2$ ...(1)

This touches the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

$\Leftrightarrow a^2x_1^2 + b^2y_1^2 + c^2z_1^2 = (x_1^2+y_1^2+z_1^2)^2$

$\therefore$ The locus of P is $a^2x^2+b^2y^2+c^2z^2=(x^2+y^2+z^2)^2$

Ex. 3. *Any three mutually perpendicular lines drawn through a fixed point C meet the conicoid* $ax^2+by^2+cz^2=1$ *in* $P_1, P_2; Q_1, Q_2; R_1, R_2$ *respectively. Prove that* $\frac{1}{CP_1.CP_2}+\frac{1}{CQ_1.CQ_2}+\frac{1}{CR_1.CR_2}=$ *constant.*

Sol. Let $C=(\alpha,\beta,\gamma)$

The equation to any line through C be $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$

Let (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) be direction cosines of the three perpendicular lines through C.

The equation to the line through C and with d.c's (l_1, m_1, n_1) is

$\frac{x-\alpha}{l_1}=\frac{y-\beta}{m_1}=\frac{z-\gamma}{n_1}(=r)$ ...(1)

Any point on the line is $(l_1r+\alpha, m_1r+\beta, n_1r+\gamma)$. If this point lies on the given conicoid $ax^2+by^2+cz^2=1$ then, $a(l_1r+\alpha)^2+b(m_1r+\beta)^2+c(n_1r+\gamma)^2=1$

$\Leftrightarrow r^2(al_1^2+bm_1^2+cn_1^2)+2r(a\alpha l_1+b\beta m_1+n\gamma n_1)+(a\alpha^2+b\beta^2+c\gamma^2-1)=0$

The line meets the conicoid in P_1 and P_2.

$\therefore$ The roots of this quadratic equation are the values CP_1 and CP_2.

$$\therefore \quad CP_1.CP_2=\frac{a\alpha^2+b\beta^2+c\gamma^2-1}{al_1^2+bm_1^2+cn_1^2}\Rightarrow\frac{1}{CP_1.CP_2}=\frac{al_1^2+bm_1^2+cn_1^2}{a\alpha^2+b\beta^2+c\gamma^2-1}$$

Similarly, the lines CQ_1Q_2, CR_1R_2 meet the conicoids where

$$\frac{1}{CQ_1.CQ_2}=\frac{al_2^2+bm_2^2+cn_2^2}{a\alpha^2+b\beta^2+c\gamma^2-1};\quad \frac{1}{CR_1.CR_2}=\frac{al_3^2+bm_3^2+cn_3^2}{a\alpha^2+b\beta^2+c\gamma^2-1}$$

Hence $\frac{1}{CP_1.CP_2}+\frac{1}{CQ_1.CQ_2}+\frac{1}{CR_1.CR_2}$

$$=\frac{a}{\Delta}(l_1^2+l_2^2+l_3^2)+\frac{b}{\Delta}(m_1^2+m_2^2+m_3^2)+\frac{c}{\Delta}(n_1^2+n_2^2+n_3^2)$$

where $\Delta=a\alpha^2+b\beta^2+c\gamma^2-1$

$\because$ $CP_1P_2, CQ_1Q_2, CR_1R_2$ are three mutually $\perp r$ lines

$\Rightarrow l_1^2+l_2^2+l_3^2=m_1^2+m_2^2+m_3^2=n_1^2+n_2^2+n_3^2=1$

$$\therefore \quad \frac{1}{CP_1.CP_2}+\frac{1}{CQ_1.CQ_2}+\frac{1}{CR_1.CR_2}=\frac{a+b+c}{a\alpha^2+b\beta^2+c\gamma^2-1}=a \text{ constant.}$$

EXERCISE 9 (*a*)

1. Find the equations to the tangent planes to the conicoid $3x^2-6y^2+9z^2+17=0$, parallel to the plane $x+4y-2z=8$.
2. Find the equations to the tangent planes which pass through the line
   (*a*) $7x-6y+9=0,\ z=3$ and touching $7x^2-3y^2-z^2+21=0$.
   (*b*) $7x+10y-30=0,\ 5y-3z=0$ and touching $7x^2+5y^2+3z^2=60$
   (*c*) $\frac{x}{3}=\frac{y-3}{-3}=\frac{z}{1}$ and touching $\frac{x^2}{6}+\frac{y^2}{3}+\frac{z^2}{2}=1$
3. Show that the plane $3x+12y-6z-17=0$ touches the conicoid $3x^2-6y^2+9z^2+17=0$. Find the point of contact. ***(N. U. A12, A.U. A11, A12)***
4. A tangent plane to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ meets the co-ordinate axes in A, B and C. Find the locus of the centroid of *(i)* the triangle ABC *(ii)* the tetrahedron OABC.
5. If $2r$ is the distance between two parallel tangent planes to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, prove that the line through the origin perpendicular to the planes lies on the cone $x^2(a^2-r^2)+y^2(b^2-r^2)+z^2(c^2-r^2)=0$.
6. Tangent planes are drawn in the conicoid $ax^2+by^2+cz^2=1$ through the point (α,β,γ). Prove that the perpendiculars to them from the origin generate the cone
   $(\alpha x+\beta y+\gamma z)^2=\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}$
7. A tangent plane to the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ makes equal angles with the coordinate planes. Show that it forms with them a tetrahedron of volume $\frac{1}{6}(a^2+b^2+c^2)^{3/2}$

ANSWERS

1. $3x+12y-6z=\pm 17$ **2.** (*a*) $7x-6y-4z+21=0,\ 14x-12y-z+21=0$
(*b*) $14x+5y+9z=60,\ 7x+5y+3z=30$ (*c*) $x+y=3,\ x+2y+3z=6$.

3. $(-1,2,\frac{2}{3})$ **4.** (*i*) $\frac{a^2}{x^2}+\frac{b^2}{y^2}+\frac{c^2}{z^2}=9$ (*ii*) $\frac{a^2}{x^2}+\frac{b^2}{y^2}+\frac{c^2}{z^2}=16$

9. 13. ENVELOPING CONE

Definition. *$S=0$ is a conicoid and P is a point $\notin S$. The set of tangent lines, if they exist, to the conicoid and passing through P form a cone with vertex at P. The cone is called the* **enveloping cone** *of the point P with respect to the conicoid.*

Theorem. ***$P(x_1, y_1, z_1)$ is not a point on the conicoid $S=0$. The equation to the enveloping cone of P w.r.t. to conicoid $S=0$ is $S_1^2=SS_{11}$.***

Proof. Let the equation to the conicoid be $ax^2+by^2+cz^2=1$

$P\notin S \Rightarrow S_{11}\neq 0$. Let a line L passing through P be $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}\ (=r)$

The point $(lr + x_1, mr + y_1, nr + z_1)$ of the line lies on $S = 0$.

$\Leftrightarrow a\,(lr + x_1)^2 + b\,(mr + y_1)^2 + c\,(nr + z_1)^2 = 1$

$\Leftrightarrow r^2(al^2 + bm^2 + cn^2) + 2r(alx_1 + bmy_1 + cnz_1) + S_{11} = 0$

The line L is a tangent line to the conicoid.

$\Leftrightarrow (alx_1 + bmy_1 + cnz_1)^2 = S_{11}(al^2 + bm^2 + cn^2)$

(*i*) If $al^2 + bm^2 + cn^2 = 0$. Then $alx_1 + bmy_1 + cnz_1 = 0$

$\Rightarrow$ The line L is an asymptotic line to the conicoid.

(*ii*) If $al^2 + bm^2 + cn^2 \neq 0$, then L is a tangent line

$\Leftrightarrow (alx_1 + bmy_1 + cnz_1)^2 = S_{11}(al^2 + bm^2 + cn^2) \Rightarrow$ L lies on the curve

$[ax_1(x - x_1) + by_1(y - y_1) + cz_1(z - z_1)]^2 = S_{11}[a(x - x_1)^2 + b(y - y_1)^2 + c(z - z_1)^2]$

$\Rightarrow (S_1 - S_{11})^2 = S_{11}[S + S_{11} - 2S_1] \Rightarrow S_1^2 = SS_{11}$

This is a homogeneous equation of second degree representing a cone.

SOLVED PROBLEMS

Ex. 1. *A point P moves so that the section of the enveloping cone of the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ *with P as vertex by the plane is a circle. Show that P lies on one of the conics* $\frac{y^2}{b^2 - a^2} + \frac{z^2}{c^2} = 1$.

Sol. Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$

Equation to enveloping cone is $S_1^2 = S.S_{11}$

$\therefore$ Enveloping cone with vertex $P(x_1, y_1, z_1)$ is

$$\left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} - 1\right)^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right)$$

plane $z = 0$ cuts it along

$$\left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right)$$

This will be a circle $\Leftrightarrow$ co.eft. of $x^2 =$ co.eft. of y^2 and co.eft. of $xy = 0$.

$$\Rightarrow \frac{x_1^2}{a^4} - \frac{1}{a^2}\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right) = \frac{y_1^2}{b^4} - \frac{1}{b^2}\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right) \text{ and } \frac{x_1 y_1}{a^2 b^2} = 0$$

$$\Rightarrow \frac{1}{a^2}\left(\frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right) = \frac{1}{b^2}\left(\frac{x_1^2}{a^2} + \frac{z_1^2}{c^2} - 1\right) \text{ and } x_1 y_1 = 0$$

$$\therefore \text{ Locus of P is } \frac{1}{a^2}\left(\frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) = \frac{1}{b^2}\left(\frac{x^2}{a^2} + \frac{z^2}{b^2} - 1\right),\ xy = 0$$

(*i*) If $x = 0$, then

$$\frac{1}{a^2}\left(\frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) = \frac{1}{b^2}\left(\frac{z^2}{c^2} - 1\right) \Rightarrow \frac{y^2}{a^2 b^2} + \frac{z^2}{c^2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{1}{a^2} - \frac{1}{b^2}$$

$$\Rightarrow \frac{y^2}{a^2b^2}+\frac{z^2(b^2-a^2)}{a^2b^2c^2}=\frac{b^2-a^2}{a^2b^2} \Rightarrow \frac{y^2}{b^2-a^2}+\frac{z^2}{c^2}=1; x=0$$

(*ii*) For $y=0$, we get $\frac{x^2}{a^2-b^2}+\frac{z^2}{c^2}=1$

Ex. 2. *Find a section of the enveloping cone of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, *whose vertex is P by the plane* $z=0$ *is a rectangular hyperbola show that the locus of P is* $\frac{x^2+y^2}{a^2+b^2}+\frac{z^2}{c^2}=1$

Sol. Let $P(x_1, y_1, z_1)$ be the vertex.

Equation to the enveloping cone of the ellipsoid is $SS_{11}=S_1^2$

$$\Rightarrow \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^1}{c^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}-1\right)^2$$

Its section by the plane $z=0$ is $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$

This represents a rectangular hyperbola

$\Leftrightarrow$ co.eft. of $x^2 +$ co.eft. of $y^2=0$

$$\Rightarrow \frac{1}{a^2}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)-\frac{x_1^2}{a^4}+\frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)-\frac{y_1^2}{b^4}=0$$

$$\Rightarrow \frac{1}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)+\frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)=0$$

$$\Rightarrow \frac{x_1^2+y_1^2}{a^2b^2}+\left(\frac{z_1^2}{c^2}-1\right)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=0 \Rightarrow \frac{x_1^2+y_1^2}{a^2+b^2}+\frac{z_1^2}{c^2}=1$$

$\therefore$ Locus of P is $\frac{x^2+y^2}{a^2+b^2}+\frac{z^2}{c^2}=1$

9. 14. ENVELOPING CYLINDER

Definition. *The locus of tangent lines to the conicoid* $S=0$ *and with the direction numbers* $(l, m, n), E(l, m, n) \neq 0$ *is a cylinder, called the* **enveloping cylinder** *of the conicoid in the direction of* (l, m, n).

9.15. Theorem. ***The equation to the enveloping cylinder of the conicoid*** $ax^2+by^2+cz^2=1$ ***in the direction of*** (l,m,n) ***when*** $al^2+bm^2+cn^2 \neq 0$ ***is***

$$(alx+bmy+cnz)^2=(al^2+bm^2+cn^2)(ax^2+by^2+cz^2-1)$$

Proof. Let $P(x_1, y_1, z_1)$ be a point on the tangent line.

$\therefore$ The equation to the line with d.r's (l, m, n) and containing P is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \; (=r) \qquad ...(1)$$

The point $(lr + x_1, mr + y_1, nr + z_1)$ of the line lies on the conicoid

$S \equiv ax^2 + by^2 + cz^2 - 1 = 0$ ...(2)

$\Leftrightarrow\ a(lr + x_1)^2 + b(mr + y_1)^2 + c(nr + z_1)^2 = 1$

$\Leftrightarrow\ r^2(al^2 + bm^2 + cn^2) + 2r(alx_1 + bmy_1 + cnz_1) + S_{11} = 0$

the line (1) is a tangent line to (2) $\Leftrightarrow\ (alx_1 + bmy_1 + cnz_1)^2 = (al^2 + bm^2 + cn^2)S_{11}$

$\therefore$ The locus of P is the cylinder

$(alx + bmy + cnz)^2 = (al^2 + bm^2 + cn^2)S = (al^2 + bm^2 + cn^2)(ax^2 + by^2 + cz^2 - 1)$

SOLVED PROBLEMS

Ex. 1. *Prove that the enveloping cylinder of the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ *whose generator are parallel to the lines* $\frac{x}{0} = \frac{y}{\pm\sqrt{a^2 - b^2}} = \frac{z}{c}$ *meet the plane* $z = 0$ *in circles.*

Sol. Let (x_1, y_1, z_1) be any point on the cylinder

$\therefore$ Equation to the generator be $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r^2$

Any point on the line $(x_1 + lr, y_1 + mr, z_1 + nr)$

This lies on the conic $\Leftrightarrow \frac{(x_1 + lr)^2}{a^2} + \frac{(y_1 + mr)^2}{b^2} + \frac{(z_1 + nr)^2}{c^2} = 1$

$$\Rightarrow r^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) + 2r\left(\frac{lx_1}{a^2} + \frac{my_1}{b^2} + \frac{nz_1}{c^2}\right) + \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right) = 0$$

The line touches the conic

$$\Rightarrow 4\left(\frac{lx_1}{a^2} + \frac{my_1}{b^2} + \frac{nz_1}{c^2}\right)^2 - 4\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right) = 0$$

Locus of P is the cylinder

$$\left(\frac{lx}{a^2} + \frac{my}{b^2} + \frac{nz}{c^2}\right)^2 = \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

In this put $l = 0, m = \pm\sqrt{a^2 - b^2}$ and $n = 0$

$$\Rightarrow \left(\frac{y\sqrt{a^2 - b^2}}{b^2} - \frac{z}{c}\right)^2 = \left(0 + \frac{a^2 - b^2}{b^2} + 1\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

This meets the plane $z = 0$ in the conic

$$\left[\frac{y\sqrt{a^2 - b^2}}{b^2}\right]^2 = \left(\frac{a^2}{b^2}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

$$\frac{y^2(a^2 - b^2)}{b^4} = \frac{a^2}{b^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \Rightarrow \frac{y^2(a^2 - b^2)}{a^2b^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$$

$$\Rightarrow \frac{y^2}{b^2} - \frac{y^2}{a^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \Rightarrow x^2 + y^2 = a^2$$ which is a circle.

EXERCISE 9 (*b*)

1. Show that the locus of the points from which three mutually perpendicular tangent lines can be drawn to the conicoid $ax^2+by^2+cz^2=1$ is the conicoid
$a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=a+b+c$ ***(A.U. A12)***

2. Prove that the enveloping cylinder of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, whose generators are parallel to the line $\frac{x}{0}=\frac{y}{\sqrt{a^2-b^2}}=\frac{z}{c}$ meets the plane $z=0$ in a circle.
(K. U. M15, A.U. A11)

Objective Type Questions

I. ***Fill in the blanks***

1. Centre of the sphere $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$ is ____ .
2. Equation of the normal to the sphere $x^2+y^2+z^2-2x-4y+6z-3=0$ at the point $(1, 1, 1)$ on it is ____ .
3. Equation of sphere passing through (0, 0, 0), (1, 0, 0), (0, 1, 0) (0, 0, 1) is ____ .
4. Radius of the sphere $x^2+y^2+z^2=ax+by+cz$ is ____ .
5. The pole of the plane $lx+my+nz=p$ with respect to the sphere $x^2+y^2+z^2=a^2$ is ____.
6. The radius of the sphere $x^2+y^2+z^2-4x+2y-6z+5=0$ is ____ .
7. Centre of the sphere $2x^2+2y^2+2z^2=x+y+z$ is ____ .
8. The direction cosines of the line joining $(1, 1, 0)$; $(0, 0, -1)$ are ____ .
9. The centre and radius of the sphere $x^2+y^2+z^2-2x+4y-6z=11$ are given by ____ .
10. The radius of the sphere $x^2+y^2+z^2-4x+6y-2z+5=0$ is ____ .
11. The direction cosines of the line joining the points (1, 1, 1) and $(2, 4, -7)$ are ____ .
12. The section of a sphere by a plane through its centre is called ____ .
13. The distance of the point (x, y, z) from $y-$axis is ____ .
14. The condition that the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ is normal to the plane $ax+by+cz=d$ is ____ .
15. The line joining the points (1, 2, 3), (4, 5, 7) and the line joining the points $(-4, 3, -6)$ and $(2, 9, 2)$ are ____ to each other.
16. The direction cosines of the normal to the plane $5x-y+3z=27$ are ____ .
17. The condition for the two spheres $x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$ and $x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$ to cut orthogonally is ____ .
18. The equation of the sphere drawn on the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) as diameter is ____ .
19. The equation of the sphere passing through the points $(0, 0, 0)$ $(a, 0, 0)(0, b, 0)$ and $(0, 0, c)$ is ____ .
20. The equation of the plane through the intersection of the planes $y+z=4$ and $z+x=5$ and passing through the point (0, 1, 3) is ____ .
21. The equation of the sphere with centre at $(2, -3, 4)$ and radius 5 is ____ .
22. The direction cosines of the normal to the plane $x-2y+2z=1$ are ____ .
23. The equation of the tangent plane to $3x^2-4y^2=2z$ at $(2, -1, 4)$ is ____ .
24. The radius of the sphere $x^2+y^2+z^2+6x-8y+4z+29=0$ is ____ .

II. Strike out the incorrect answer :

1. $(ax+by+cz+d)+\lambda\,(a_1x+b_1y+c_1z+d_1)=0$ is [a plane, a line]

2. $x^2+y^2+z^2=a^2;\, x+y+z=a$, together represents [a circle, a sphere]

3. Every plane section of a sphere is [a circle, a line]

4. The centre of the sphere $2x^2+2y^2+2z^2-4x-4y-4z+1=0$ is [(1, 1, 1)], [(2, 2, 2)]

5. Sphere $x^2+y^2+z^2+a^2=0$ is [real, not real]

6. The sphere $x^2+y^2+z^2=25$ and $x^2+y^2+z^2-24x-40y-18z+225=0$ touch [internally, externally].

7. Equation of the sphere which touches the sphere $x^2+y^2+z^2=3$ at $(1,1,-1)$ and passes through the origin is $[x^2+y^2+z^2-x-y+z=0,\, 2(x^2+y^2+z^2)-x-y-z=0]$

8. The sphere $x^2+y^2+z^2-2a(x+y+z)+2a^2=0$ [touches ; does not tough] the co-ordinate planes.

9. The spheres $x^2+y^2+z^2-24x-40y-18z+225=0$ and $x^2+y^2+z^2=2025$ touch [internally, externally]

10. The radical plane of the two spheres $x^2+y^2+z^2+3x+y-z+6=0$ and $x^2+y^2+z^2+6x+4y+2z=8$ is $[3x+3y+3z-14=0,\, 9x+5y+z-2=0]$.

11. The distance of the plane $x+y+z=1$ from the origin is $\left[1;\dfrac{1}{\sqrt{3}}\right]$.

12. The straight line $\dfrac{x-2}{5}=\dfrac{y-1}{6}=\dfrac{z+1}{4}$ [lies, does not lie] on the plane $2x+3y-7z=5$.

13. The distance between the planes $x+2y+2z+3=0$ and $x+2y+2z+7=0$ is $[4, 4/3]$.

14. The centre of the sphere $x^2+y^2+z^2-2x+4y-6z+8=0$ is $[(1,-2,3);(-1,2,-3)]$.

15. The plane $3x-4y=0$ [passes ; does not pass] through the $z-$axis.

16. The spheres $x^2+y^2+z^2=64$ and $x^2+y^2+z^2-12x+4y-6z+48=0$ touch [internally ; externally].

17. The centre of the circle $x^2+y^2+z^2+12x-12y-16z+111=0;\, 2x+2y+z=17$ is $[(-4,8,9);(-1,2,3)]$.

18. Equation of a sphere touching the three co-ordinate planes is $[x^2+y^2+z^2\pm 2ax\pm 2ay\pm 2az+2a^2=0;\, x^2+y^2+z^2-ax-by-cz=0]$.

19. Equation of the sphere on the join of the points $(-2,2,-2)$ and $(2,-1,4)$ as diameter is $[x^2+y^2+z^2-5x-3y-11z+8=0;\, x^2+y^2+z^2-y-2z-14=0]$

20. The line $2x-1=y+3=4-z$ [intersects ; does not intersect] the sphere $x^2+y^2+z^2-6x+8y-4z+4=0$.

III. Put a ✓ mark on the correct answer :

1. Equation of the $x-$axis is

(a) $x=0$ (b) $y+z=0$ (c) $y=0, z=0$.

2. The equation of the plane passing through $(2,-1,3)$ and parallel to the plane $3x-4y+7z=0$ is

(a) $4x-3y+7z=32$ (b) $3x-4y+7z=23$ (c) $3x-4y+7z=31$

3. $ax+by+cz=0$ is parallel to

(a) $x=0$ (b) $by=cz$ (c) none of (a) and (b)

4. $x^2+y^2=9-z^2$ is a

(a) sphere (b) a pair of planes (c) none of (a) and (b)

5. The interior of the sphere $x^2+y^2+z^2=12$ is

(a) (4, 0, 0) (b) (1, 1, 2) (c) (1, 2, 3)

6. $by+cz+d=0$ is perpendicular to

(a) $by=cz$ (b) $x=0$ (c) $by+cz=0$

7. The radius of the sphere $x^2+y^2+z^2-ax-by-cz=0$

(a) $\dfrac{a+b+c}{4}$ (b) $\dfrac{\sqrt{a}}{2}+\dfrac{\sqrt{b}}{2}+\dfrac{\sqrt{c}}{2}$ (c) $\dfrac{\sqrt{a^2+b^2+c^2}}{2}$

8. If a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of the lines which are parallel, then

(a) $a_1=a_2, b_1=b_2, c_1=c_2$ (b) $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$ (c) $a_1a_2+b_1b_2+c_1c_2=0$

9. The direction cosines of a line equally inclines to the axes are

(a) 1, 1, 1 (b) $\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$ (c) $\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}$

10. The angle between the lines, whose direction ratios are 1, 1, 2 and $\sqrt{3}-1, -\sqrt{3}-1, 1$, is

(a) 30° (b) 90° (c) 60°

11. The equation of the xoy plane, is

(a) $z=0$ (b) $x=0$ (c) $y=0$

12. The angle between the planes $2x+y+z=6$, $x-y+2z=3$, is

(a) $\dfrac{\pi}{3}$ (b) $\dfrac{\pi}{2}$ (c) $\dfrac{\pi}{4}$

13. The straight lines $\dfrac{x-2}{3}=\dfrac{y-5}{4}=\dfrac{z-7}{2}$ and $\dfrac{x-4}{-2}=\dfrac{y-5}{3}=\dfrac{z-9}{-2}$, represent

(a) parallel lines (b) different lines (c) perpendicular lines

14. The direction cosines of the line joining the points $(4,3,-5)$ and $(-2,1,-8)$ are

(a) $2, 4, -13$ (b) $6, 2, 3$ (c) $\dfrac{6}{7}, \dfrac{2}{7}, \dfrac{3}{7}$

15. The direction cosines of the normal to the plane $2x-3y+6z=7$, are

(a) $\dfrac{1}{3}, \dfrac{2}{3}, \dfrac{7}{3}$ (b) $\dfrac{2}{7}, \dfrac{-3}{7}, \dfrac{6}{7}$ (c) $2, -3, 6$

16. The angle between the planes $3x-4y+5z=0$, $2x-y-2z=5$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{6}$

17. The line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{0}$ is perpendicular to

(a) $x-$axis (b) $y-$axis (c) $z-$axis

18. The line $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ is

(a) parallel to (b) perpendicular to (c) lying in the plane $2x+y-2z=3$.

19. The foot of the perpendicular from $(3,-1,11)$ to the line $\frac{x}{2}=\frac{y-2}{3}=\frac{z-3}{4}$, is

(a) (0, 2, 3) (b) (2, 3, 4) (c) (2, 5, 7)

20. $x(x-a)+y(y-b)+z(z-c)=0$ is

(a) a pairs of planes (b) sphere (c) plane.

ANSWERS

I. 1. $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$ **2.** $x-1=0,\frac{y-1}{1}=\frac{z-1}{-4}$ **3.** $x^2+y^2+z^2-x-y-r=0$

4. $\frac{\sqrt{a^2+b^2+c^2}}{2}$ **5.** $\left(\frac{a^2l}{p},\frac{a^2m}{p},\frac{a^2n}{p}\right)$ **6.** 3 **7.** $\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$ **8.** $\pm\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$

9. $(1,-2,3), 5$ **10.** 3 **11.** $\pm\left(\frac{-1}{\sqrt{74}},\frac{-3}{\sqrt{74}},\frac{8}{\sqrt{74}}\right)$ **12.** Great circle

13. $\sqrt{x^2+z^2}$ **14.** $\frac{a}{l}=\frac{b}{m}=\frac{c}{n}$ **15.** parallel **16.** $\pm\left(\frac{5}{\sqrt{35}},\frac{-1}{\sqrt{35}},\frac{3}{\sqrt{35}}\right)$

17. $2(u_1u_2+v_1v_2+w_1w_2)=d_1+d_2$

18. $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$ **19.** $x^2+y^2+z^2-ax-by-cz=0$

20. $y+z-4=0$ **21.** $x^2+y^2+z^2-4x+6y-8z+4=0$ **22.** $\pm\left(\frac{1}{3},\frac{-2}{3},\frac{2}{3}\right)$

23. $6x+4y-z-4=0$ **24.** 0

II. **1.** a line **2.** a sphere **3.** a line **4.** $(2,2,2)$ **5.** real

6. internally **7.** $2(x^2+y^2+z^2)-x-y-z=0$ **8.** does not touch

9. externally **10.** $9x+5y+z-2=0$

11. 1 **12.** lies **13.** 4 **14.** $(-1,2,-3)$ **15.** does not pass

16. externally **17.** $(-1,2,3)$ **18.** $x^2+y^2+z^2-ax-by-cz=0$

19. $x^2+y^2+z^2-y-2z-14=0$ **20.** does not intersect

III.

1.	c	**2.**	c	**3.**	c	**4.**	a	**5.**	b	**6.**	b	**7.**	c
8.	b	**9.**	b	**10.**	b	**11.**	a	**12.**	a	**13.**	c	**14.**	c
15.	b	**16.**	b	**17.**	c	**18.**	b	**19.**	c	**20.**	a		

Problems for Practicals

Bisectors of angles between two planes

1. Find the equation to the plane bisecting the acute angle between the planes $3x-2y-6z=0, -2x+y-2z-2=0$
2. Find the equation to the plane bisecting the acute angle between the planes $5x+12y-13z=0, 3x+4y-5z+1=0$
3. Find the equation to the plane bisecting the acute angle between the planes $x+2y+2z-19=0, 4x-3y+12z+3=0$
4. Find the equation to the plane bisecting the acute angle between the planes $3x-6y+2z+5=0, 4x-12y+3z-3=0$
5. Find the equation to the plane bisecting the obtuse angle between the planes $2x-y+2z+2=0, 3x-2y-6z=0$
6. Find the equation to the plane bisecting the obtuse angle between the planes $3x+4y-5z+1=0, 5x+12y-13z=0$
7. Find the equation to the plane bisecting the obtuse angle between the planes $x+2y-2z-19=0, 4x-3y+12z+3=0$
8. Find the equation to the plane bisecting the obtuse angle between the planes $3x+4y-5z+1=0, 5x+12y-13z=0$
9. Find the equation to the plane bisecting the angle between the planes $x+2y+2z=9, 4x-3y+12z+13=0$ and does not contain the origin. ***(N. U. A11)***
10. Find the equation to the plane bisecting the angle between the planes $4x-3y+12z+13=0, x+2y+2z=9$ and containing the origin.

ANSWERS

1.	$23x-13y+32z+20=0$	**2.**	$14x-8y+13=0$
3.	$x+35y-10z=256$	**4.**	$25x+17y+62z-238=0$
5.	$5x-y-4z+8=0$	**6.**	$64x+112y-130z+13=0$
7.	$25x+17y+62z-238=0$	**8.**	$14x-8y+13=0$
9.	$x+35y-10z-156=0$	**10.**	$25x+17y+62z-78=0$

The length and equations of the line of S.D. between two straight lines

1. Find the length of the S. D. between the lines $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}, \frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$

2. Find the equations to the line of S.D. between the lines
$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}, \frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$

3. Find the length of the S. D. between the lines $\frac{x}{1}=\frac{y}{2}=\frac{z}{1}$ and
$x+y+2z-3=0=2x+3y+3z-4$

4. Find the length of the S. D. between the lines $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-1}{2};\frac{x-4}{4}=\frac{y-5}{5}=\frac{z-2}{3}$

5. Find the equations to the line of S. D. between the lines
$\frac{x-4}{4}=\frac{y-5}{5}=\frac{z-2}{3};\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-1}{2}$

6. Find the length of the S. D. between the lines $3x-2y-z+3=0=2x+3y-5z-6=0$ and
$\frac{x-2}{2}=\frac{y+1}{3}=\frac{z}{4}$

7. Find the equations of the S. D. between the lines $\frac{x+3}{-4}=\frac{y-6}{3}=\frac{z}{2}; \frac{x+2}{-4}=\frac{y}{1}=\frac{z-7}{1}$
(N. U. A11)

8. Find the length of the S. D. between the lines

$\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7};\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$

9. Find the equations to the line of the S.D. between the lines

$\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7};\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$

10. Find the length of the S.D. between the lines

$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4};\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ what is your conclusion ?

11. Find the length of the S.D. between any to opposite edges of the tetrahedron formed by the lines $y+z=0, z+x=0, x+y=0, x+y+z=a$

12. Find the coordinates of the points where the S. D. between the lines

$\frac{x-12}{-9}=\frac{y-1}{4}=\frac{z-5}{2};\frac{x-23}{-6}=\frac{y-19}{-4}=\frac{z-25}{3}$ meets them.

13. Find the length of the S. D. between the lines $2x+y-z=0=x-y+2z$;

$x+2y-3z-4=0=2x-3y+4z-5$ **(K. U. A12)**

14. Find the equations to the line of the S. D. between the lines
$5x-2y-3z+6=0=x-3y+2z-3;\ \frac{x}{4}=\frac{y+1}{3}=\frac{z-2}{2}$

15. Show that the equation to the plane containing the line $\frac{y}{b}+\frac{z}{c}=1, x=0$ and parallel to the line $\frac{x}{a}-\frac{z}{c}=1, y=0$ is $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}=-1$ and if $2d$ is the S.D. prove that $\frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$

16. Prove that the S.D. between a diagonal and an edge not meeting in a cube of side a is $a/\sqrt{2}$.

ANSWERS

1. $3\sqrt{30}$ **2.** $4x-5y-17z+79=0=22x-5y+19z-83$ **3.** $13/\sqrt{66}$

4. $1/\sqrt{6}$ **5.** $2x-7y+11z+6=0=2x-10y+14z-6$ **6.** $97/13\sqrt{6}$

7. $\frac{x-1}{1}=\frac{y-3}{-4}=\frac{z+2}{8}$ **8.** 14 **9.** $9x-4y-z-14=0, 117x+4y-41z-490=0$

10. 0, The lines are coplanar **11.** $\frac{2a}{\sqrt{6}}$ **12.** $(3,5,7),(11,11,31)$ **13.** $\frac{2\sqrt{14}}{7}$

14. $7x-2y-11z+20=0, 13x-13z+24=0$

Sphere through a given circle

1. The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes in the points A, B, C. Find the equations of the circumcircle of the triangle ABC.

2. If the circumcircle of the triangle ABC is given by the equations

$x^2+y^2+z^2-ax-by-cz=0,\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ find the centre of the circle.

3. Find the equation to the sphere which passes through the point (a,b,c) and the circle $x^2+y^2=r^2, z=0$

4. Find the equation to the sphere having the circle $x^2+y^2+z^2+10y-4z-8=0$, $x+y+z-3=0$ as the great circle.

5. Find the equations of the circle lying on the sphere $x^2+y^2+z^2-2x+4y-6z+3=0$ and having its centre at $(2,3,-4)$

6. Find the equation to the sphere on which the following circles lie

$x^2+y^2+z^2-y+2z=0=x-y+z-2=0\,;x^2+y^2+z^2+x-3y+z-5=0=2x-y+4z-1$

7. Find the radius of the circle $x^2+y^2+z^2-x+z-2=0=x+2y-z-4$

8. Find the equation to the sphere which has the circle

$x^2+y^2+z^2-x+z-2=0=x+2y-z-4$ as one of its great circles.

9. Find the equation to the sphere having the circle

$x^2+y^2+z^2+10y-4z-8=0=x+y+z-3$ as one of its great circles.

10. Find the equation to the sphere which passes through the circle $x^2+y^2=4, z=0$ and is cut by the plane $x+2y+2z=0$ in a circle of radius 3 units. ***(A. U. A12)***

11. Show that the four points $(-8,5,2),(-5,2,2),(-7,6,6),(-4,3,6)$ are concyclic.

12. Find the centre and radius of the circle $x^2+y^2+z^2=25, 2x+3y+2z=9$

13. Find the centre of the circle $x^2+y^2+z^2+2x-2y-4z-19=0$; $x+2y+2z+7=0$

14. Find the equation to the sphere passing through the circle $x^2+y^2+z^2+2x+3y+6=0=x-2y+4z-9$ and through the centre of the sphere $x^2+y^2+z^2-2x+4y-6z+5=0$. ***(K. U. A12)***

15. Find the equation to the sphere having the extremities of a diameter as $(2,-1,4)$ and $(-2,2,-2)$. If the plane $2x+y-z-3=0$ cuts the sphere, find the area of the circle formed.

16. Show that the centres of all sections of the sphere $x^2+y^2+z^2=a^2$ by planes through the point (α,β,γ) lie on the sphere $x(x-\alpha)+y(y-\beta)+z(z-\gamma)=0$

17. The spheres $x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$;
$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$ are orthogonal
$\Leftrightarrow 2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2$

18. Show that every sphere through the circle $x^2+y^2-2ax+r^2=0, z=0$ intersects orthogonally every sphere through the circle $x^2+z^2=r^2, y=0$

ANSWERS

1. $x^2+y^2+z^2-ax-by-cz=0$; $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

2. $\left(\frac{a}{2}+\frac{r}{a},\frac{b}{2}+\frac{r}{b},\frac{c}{2}+\frac{r}{c}\right)$ where $r=-\frac{1}{2(a^{-2}+b^{-2}+c^{-2})}$

3. $(x^2+y^2+z^2-r^2)c=z(a^2+b^2+c^2-r^2)$ **4.** $x^2+y^2+z^2-4x+6y-8z+4=0$

5. $x^2+y^2+z^2-2x+4y-6z+3=0;\ x+5y-7z-45=0$

6. $x^2+y^2+z^2+3x-4y+5z-6=0$ **7.** $r=1$

8. $x^2+y^2+z^2-2x-2y+2z+2=0$ **9.** $x^2+y^2+z^2-4x+6y-8z+4=0$

10. $x^2+y^2+z^2\pm 6z-4=0$ **14.** $x^2+y^2+z^2+7y-8z+24=0$

Angle of intersection of two spheres

1. If θ is the angle between two intersecting spheres whose radii are r_1 and r_2 and whose distance between the centres is d then prove that $\cos\theta=\frac{r_1^2+r_2^2-d^2}{2r_1r_2}$

2. If r_1,r_2 are the radii of two orthogonal spheres, find the radius of the common circle formed.

3. If two spheres $x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$,
$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$ are intersecting orthogonally, find the radius of the common circle formed.

4. Find the value of k if the spheres
$x^2+y^2+z^2+6y+2z+8=0,\ x^2+y^2+z^2+6x+8y+4z+k=0$ are orthogonal.

5. If the spheres $x^2+y^2+z^2-ax-by-cz+\lambda\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1\right)=0$ and
$x^2+y^2+z^2-2ax-2by-2cz=0$ are orthogonal, find the value of λ.

6. Find the equation to the sphere passing through the points $(0,3,0),(-2,-1,-4)$ and cutting the spheres $x^2+y^2+z^2+x-3z-2=0\ x^2+y^2+z^2+(1/2)x+(3/2)y+2=0$, orthogonally.

7. Find the angle between the two intersecting spheres
$x^2+y^2+z^2-2x-4y-z-11=0, x^2+y^2+z^2+2x-y+12z+5=0$

8. Find the radical line of the spheres $x^2+y^2+z^2+4y=0$,
$x^2+y^2+z^2+3x-2y+8z+6=0$; $x^2+y^2+z^2+2x+2y+2z+2=0$

9. Find the radical line of the spheres $x^2+y^2+z^2+2x+2y+2z+2=0$;
$x^2+y^2+z^2+4x+4y+4z+4=0$; $x^2+y^2+z^2+x+6y-4z-2=0$

10. Find the equation to the sphere cutting the sphere $x^2+y^2+z^2-4x+6y+4=0$ orthogonally and touching the plane $3x+2y-z+2=0$ at $(1,-2,1)$

11. Find the radical centre of the spheres
$x^2+y^2+z^2+4y=0$; $x^2+y^2+z^2+2x+2y+2z+2=0$;
$x^2+y^2+z^2+3x-2y+8z+6=0$; $x^2+y^2+z^2-x+4y-6z-2=0$

12. Find the equation to the sphere passing through the circle
$x^2+y^2+z^2-2x+3y-4z+6=0=3x-4y+5z-15$ and cutting the sphere
$x^2+y^2+z^2+2x+4y-6z+11=0$ orthogonally.

ANSWERS

2. $\dfrac{r_1r_2}{\sqrt{r_1^2+r_2^2}}$

3. $\dfrac{\sqrt{u_1^2+r_1^2+w_1^2-d_1}\sqrt{u_2^2+r_2^2+w_2^2-d_2}}{\sqrt{(u_1-u_2)^2+(v_1-v_2)^2+(w_1-w_2)^2}}$

4. $k=20$

5. $-\dfrac{(a^2+b^2+c^2)}{2}$

6. $x^2+y^2+z^2+2x-2y+4z-3=0$

7. $90°$

8. $x-y+z+1=0=3x-6y+8z+6$

9. $\dfrac{x}{2}=\dfrac{y-1}{5}=\dfrac{z}{3}$

10. $x^2+y^2+z^2+7x+10y-5z+12=0$

11. $\left(-\dfrac{1}{5},\dfrac{1}{2},\dfrac{-3}{10}\right)$

12. $5(x^2+y^2+z^2)-13x+19y-25z+45=0$

The Cone

1. Find the equation of the cone generated by the lines passing through (α, β, γ) and whose direction ratios satisfy $al^2 + bm^2 + cn^2 = 0$.
2. Find the angle between the lines of intersection of the plane and the cone given by
   (*a*) $2x + y - z = 0;\ 4x^2 - y^2 + 3z^2 = 0$ ***(N.U. A11)*** (*b*) $x + y + z = 0;\ x^2 + 2y^2 - z^2 = 0$
   (*c*) $6x - 10y - 7z = 0;\ 108x^2 - 20y^2 - 7z^2 = 0$
3. Find the equation of the quadric cone which passes through the coordinate axes and the three lines $\frac{1}{2}x = y = -z;\ x = \frac{1}{3}y = \frac{1}{5}z;\ \frac{1}{8}x = 1 - \frac{y}{11} = \frac{1}{5}z$
4. Find the vertices of the cone (*a*) $2y^2 - 8yz - 4zx - 8xy + 6x - 4y - 2z + 5 = 0$
   (*b*) $2x^2 + 2y^2 + 7z^2 - 10yz - 10zx + 2x + 2y + 26z - 17 = 0$
5. Prove that the equation to the cone through the axes and the lines of intersection of the cone $11x^2 - 5y^2 + z^2 = 0$ and the plane $7x - 5y + z = 0$ is $14yz - 30zx + 3xy = 0$
6. Show that $33x^2 + 13y^2 - 95z^2 - 144yz - 96zx - 48xy = 0$ represents a right circular cone whose axis is $3x = 2y = z$. Find the semi vertical angle.
7. Find the equation of the right circular cone passing through the axes. Find its semi vertical angle and the equation of its axis.
8. A straight line $x = y = z$ rotates round the straight line $x = -y = -z$ about the origin, to generate a right circular cone. Find the equation of that cone.
9. Find the equation of the quadric cone which touches the three coordinate planes and the planes $x + y + z = 0$ and $2x - 3y + z = 0$.
   (**Hint :** Required cone is the reciprocal cone to the cone which passes through the normals through the origin to the given planes)
10. Find the cone whose semi vertical angle is $45°$ and which has its vertex at the origin and its axis along the line $x = -2y = z$. Show that the plane $z = 0$ cuts the cone in two lines inclined at an angle $\mathrm{Cos}^{-1}\left(\frac{4}{5}\right)$

ANSWERS

1. $a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2 = 0$ **2.** (*a*) $\mathrm{Cos}^{-1}\left(\frac{9}{\sqrt{105}}\right)$ (*b*) $\mathrm{Cos}^{-1}\left(\frac{2}{\sqrt{7}}\right)$
(*c*) $\mathrm{Cos}^{-1}\left(\frac{16}{7\sqrt{5}}\right)$ **3.** $16yz - 33zx - 25xy = 0$ **4.** (*a*) $(-7/6, 1/3, 5/6)$ (*b*) $(2,2,1)$
6. $60°$ **7.** $yz + zx + xy = 0;\ \mathrm{Cos}^{-1}\left(\frac{1}{\sqrt{3}}\right);\ x = y = z$
8. $x^2 + y^2 + z^2 + 3yz - 3zx - 3xy = 0$ **9.** $64x^2 + 9y^2 + 25z^2 - 30yz + 80zx + 48xy = 0$
10. $x^2 + 7y^2 + z^2 + 8yz - 16zx + 8xy = 0$

Cylinder

1. A cylinder cuts the plane $z=0$ in the curve $x^2+\frac{y^2}{4}=\frac{1}{4}$ and has its axis parallel to $3x=-6y=2z$. Find its equation.
2. Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{1}=\frac{-y}{2}=\frac{z}{3}$ and whose guiding curve is the ellipse $x^2+2y^2=1, z=3$.
3. Find the equation of the surface generated by the straight line which is parallel to the line $y=mx, z=nx$ and intersects the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$

   [**Hint :** The line $y=mx, z=nx$ can be put as $\frac{x}{1}=\frac{y}{m}=\frac{z}{n}$ etc.]
4. Find the equation to the right circular cylinder whose guiding curve is the circle $x^2+y^2+z^2=9, x-y+z=3$ ***(A.U. A11, N. U. A11, K. U. A11)***
5. Find the equation of the right circular cylinder of radius 2 and whose axis is the line $\frac{x-1}{2}=y-2=\frac{z-3}{2}$
6. Find the right circular cylinder whose axis is $x=2y=-z$ and radius 4.
7. Find the equation of the curve loping cylinder of the sphere $x^2+y^2+z^2-2x+4y=1$, having its generators parallel to the line $x=y=z$.

ANSWERS

2. $3x^2+6y^2+3z^2+8yz-2zx+6x-24y-18z+24=0$

3. $\frac{(nx-z)^2}{a^2}+\frac{(ny-mz)^2}{b^2}=n^2$ **4.** $x^2+y^2+z^2+xy+yz-zx-9=0$

5. $5x^2+8y^2+5z^2-4yz-8zx-4xy+22x-16y-14z-10=0$

6. $5x^2+8y^2+5z^2+4yz+8zx-4xy-144=0$

7. $x^2+y^2+z^2-xy-yz-zx-4x+5y-z-2=0$

Conicoid

1. Show that the length of the perpendicular p from the origin as the tangent plane at the point (x_1, y_1, z_1) of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is given by $\frac{1}{p^2}=\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}$
2. Show that the plane $3x+12y-6z-17=0$ touches the conicoid $3x^2-6y^2+9z^2+17=0$
3. Find the equation to the tangent plane to the surface
   (*a*) $4x^2-5y^2+7z^2+13=0$ and parallel to the plane $4x+20y-21z=0$
   (*b*) $x^2-2y^2+3z^2=2$ and parallel to the plane $x-2y+3z=0$.

4. Find the equation of the tangent planes to $2x^2-6y^2+3z^2=5$ and which pass through the line $x+9y-3z=0=3x-3y+6z-5$.

5. Find the locus of the feet of the perpendiculars from the origin to the tangent planes to the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which cut off from the axes, intercepts the sum of whose reciprocals is equal to $(1/k)$.

6. Tangent plans are drawn to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, through the point (α,β,γ) prove that the perpendiculars to them through the origin generate the cone

$(\alpha x+\beta y+\gamma z)^2=a^2x^2+b^2y^2+c^2z^2$

7. Find the equations to the two tangent planes passing through the line $7x+10y=30$, $5y-3z=0$ and thouch the conicoid $7x^2+5y^2+3z^2=60$.

8. Show that the plane $x+2y+3z=2$ touches the conicoid $x^2-2y^2+3z^2=2$ and find the point of contact.

9. Find the envoloping cone of the sphere $x^2+y^2+z^2-2x+4z=2$ with vertix at $(1,1,1)$

10. Prove that the enveloping cylinder of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ whose generators are parallel to the lines $\frac{x}{0}=\frac{y}{\sqrt{a^2+b^2}}=\frac{z}{c}$ meet the plane $z=0$ in circles. ***(A.U. A11)***

11. Find the enveloping cylinder of the sphere $x^2+y^2+z^2-2x+4y-1=0$ having its generators parallel to the line $x=y=z$.

12. Prove that the plane $2x-4y-z+3=0$ touches the paraboloid $x^2-2y^2=3z$ and find its point of contact.

ANSWERS

3. (a) $4x+20y-21z=\pm 13$ (b) $x-2y+3z\pm 2=0$

4. $4x+6y+3z-5=0,\ 2x-12y+9z-5=0$ **7.** $14x+5y+9z=60,\ 7x+5y+3z=30$

8. $(1,-1,1)$ **9.** $4x^2+3y^2-6yz-8x-16z-4=0$

11. $x^2+y^2+z^2-yz-zx-xy-4x+5y-z-2=0$ **12.** $(3,3,-3)$

KEY TO
A TEXTBOOK OF
B.Sc. MATHEMATICS
Vol. I
(2nd Semester)

UNIT - I

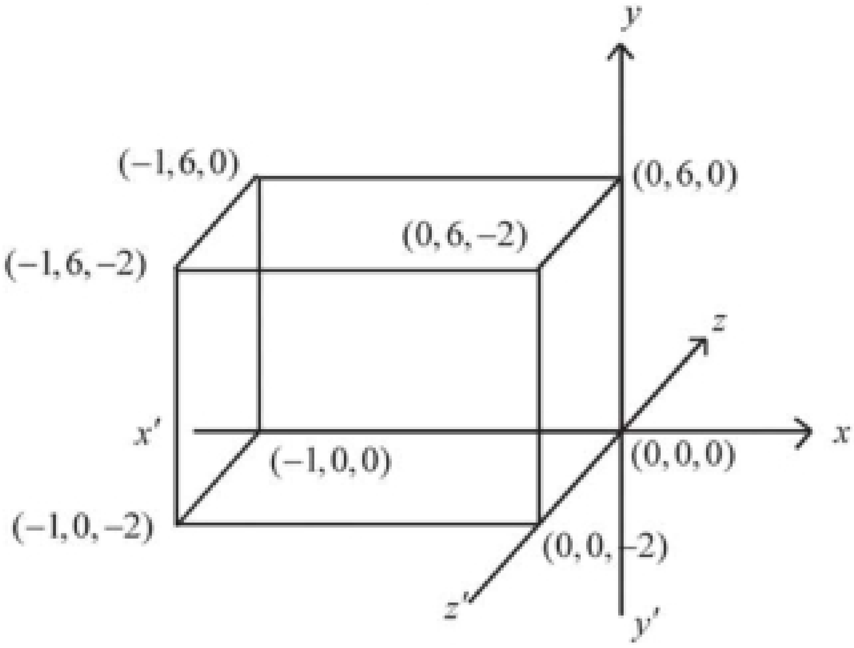
Coordinates

Exercise 2 (*a*)

1. Since any prpendicular to XY plane is parallel to Z-axis and since z-coordinate of any point in XY plane is zero, the foot of the perpendicular from the point $(-1,6,-2)$ to XY plane is the point $(-1,6,0)$.

Similarly the foot of $(-1,6,-2)$ in YZ plane is $(0,6,-2)$ and in ZX plane is $(-1,0,-2)$.

2.



3. (*i*) Given equation of the locus is $x=2$. This is a plane parallel to yz plane.

(*ii*) Given equation of the locus is $y=-3$. This is a plane parallel to zx plane.

4. (*i*) $x=0$ represents yz plane.

(*v*) $x=0$, $z=0$ are the equations of yz plane and xy plane.
They intersect in the line y-axis. The locus is y-axis.

(*vi*) $x=3$, $y=2$ are the equations of the plane parallel to yz plane and xz plane.
They intersect in a line parallel to z-axis. Thus the locus is a line parallel to z-axis.

5. (*i*) Given equation is $x^2+y^2=25, z=0$ ... (1)

The equation $x^2+y^2=25$ represents a circle in XY plane.

But from (1), $z=0$ for every point on the circle.

Hence (1) represents a circle in R^3 space.

(*ii*) Given equation is $y^2=4ax, z=0$ .... (1)

The equation $y^2=4ax$ represents a parabola in xy plane.

But from (i), $z=0$ for every point. Hence (1) represents a parabola.

(*vi*) Given equation is $4x^2+9y^2=36$ *i.e.,* $\frac{x^2}{9}+\frac{y^2}{4}=1$ ... (1)

In XY plane, the equation (1) represents an ellipse.

Any point (x, y, z) where (x, y) satisfies (1) and $z \in R$ belong to the locus represented by (1).

Hence for a fixed (x, y) satisfying (1), we have a line parallel to Z-axis and for all (x, y) satisfying (1), we have a set of lines parallel to Z-axis.

Thus the locus of the equation $4x^2 + 9y^2 = 36$ in 3 D-space is a system of lines parallel to Z-axis, called a cylinder.

6. Let $P = (2,3,5)$ and $Q = (5,7,10)$.

Through P and Q draw planes parallel to YZ plane meeting X-axis in A and B.

Hence $A = (2,0,0)$ and $B = (5,0,0)$.

$\therefore$ One edge of the rectangular parallelopiped with PQ as diagonal and having faces parallel to the coordinate planes is $|5-2| = 3$.

Similarly lengths of the other edges are $|7-3| = 4$, $|10-5| = 5$.

EXERCISE 2 (*b*)

1. $A = (-1,3,5)$ and $B = (4,-12,-20)$ $\Rightarrow \overline{OA} = (-1,3,5)$ and $\overline{OB} = (4,-12,-20)$

$\Rightarrow \overline{OA} = -4\,\overline{OB}$ $\Rightarrow$ O, A, B are collinear.

Alieter : $OA = \sqrt{1^2 + 3^2 + 5^2} = \sqrt{35}, OB = \sqrt{16 + 144 + 400} = 4\sqrt{35}$

$AB = \sqrt{25 + 225 + 625} = 5\sqrt{35}$ · $AB = AO + OB = 5\sqrt{35}$

A, O, B are collinear.

(*i*) Let $(A; OB) = \lambda_1 : \lambda_2$ $\therefore\ \lambda_2 \overline{OA} = \lambda_1 \overline{AB}$

$\Rightarrow \lambda_2(-1,3,5) = \lambda_1(5,-15,-25) \Rightarrow -\lambda_2 = 5\lambda_1 \Rightarrow \lambda_1 : \lambda_2 = -1 : 5 \Rightarrow (A; OB) = -1 : 5$

2. $(A = (2,-3,2\sqrt{3}) \Rightarrow \overline{OA} = (2,-3,2\sqrt{3})$

(*i*) Unit vector on $\overrightarrow{OA} = \dfrac{\overline{OA}}{|\overline{OA}|} = \dfrac{1}{\sqrt{(4+9+12)}} = (2,-3,2\sqrt{3})$

(*ii*) Unit vector on $\overrightarrow{OA} = \dfrac{\overline{OA}}{|\overline{OA}|} = \left(\dfrac{-2}{5}, \dfrac{+3}{5}, \dfrac{-2\sqrt{3}}{5}\right)$

4. (*iii*) $A = (2,3,5)$, $B = (-1,5,-1)$, $C = (4,-3,2)$

$\Rightarrow \overline{AB} = (-3,2,-6)$ $\overline{BC} = (5,-8,3)$, $\overline{CA} = (-2,6,3)$

$\Rightarrow |\overline{AB}|^2 = 9 + 4 + 36 = 49$, $|\overline{BC}|^2 = 25 + 64 + 9 = 98$, $|\overline{CA}|^2 = 4 + 36 + 9 = 49$

$\Rightarrow AB = CA, AB^2 + CA^2 = BC^2 \Rightarrow \Delta ABC$ is right angled isosceles.

5. Let $A = (a,b,c)$, $B = (b,c,a)$, $C = (c,a,b)$

$\Rightarrow \overline{AB} = (4,2,-6)$ and $\overline{AC} = (6,3,-9)$

$\Rightarrow 3\,\overline{AB} = 2\,\overline{AC} \Rightarrow \overline{AB} = \frac{2}{3}\overline{AC} \Rightarrow$ A, B, C are collinear.

OR : Area of $\Delta ABC = \frac{1}{2}|\overline{AB}\times\overline{AC}| = \frac{1}{2}$ modulus of $\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 4 & 2 & -6 \\ 6 & 3 & -9 \end{vmatrix} = \frac{1}{2}|\overline{0i} - \overline{0j} + \overline{0k}| = 0$

$\Rightarrow$ A, B, C are collinear.

7. (*i*) $AB = BC = CD = DA$ and $AC \neq BD\,(\overline{AB}.\overline{AD} \neq 0) \Rightarrow$ $ABCD$ is a rhombus.

(*ii*) $AB = BC = CD = DA$ and $AC = BD$ (or $\overline{AB}.\overline{AD} = 0$) $\Rightarrow$ $ABCD$ is a square.

8. Suppose $A = (x, y, z)$ is the centre of the sphere.

Let $P_1 = (a,0,0)$, $P_2 = (0,b,0)$, $P_3 = (0,0,c)$ and O = (0, 0, 0) be the points on it.

Then we have $P_1A = P_2A = P_3A = OA$

$$\Rightarrow (x-a)^2 + (y-0)^2 + (z-0)^2 \quad \text{... (1)}$$
$$= (x-0)^2 + (y-b)^2 + (z-0)^2 \quad \text{... (2)}$$
$$= (x-0)^2 + (y-0)^2 + (z-c)^2 \quad \text{... (3)}$$
$$= (x-0)^2 + (y-0)^2 + (z-0)^2 \quad \text{... (4)}$$

From (1) and (2) we get, $-2ax + a^2 = -2by + b^2$

$$-2ax + 2by = b^2 - a^2 \quad \text{... (5)}$$

From (2) and (3) $-2by + 2cz = c^2 - b^2$ ... (6)

From (3) and (4) $-2cz + c^2 = 0 \Rightarrow z = \frac{c}{2}$ ... (7)

Substituting in (6) we get $y = b/2$ and $x = \frac{c}{2}$

$\therefore$ The centre $= \left(\frac{a}{2}, b/2, c/2\right)$ radius $= \sqrt{a^2 + b^2 + c^2}\,/2$

9. Let the given points be $O = (0,0,0)$, $A = (2,0,0)$, $B = (0,4,0)$, $C = (0,0,6)$.

Do as in Problem.6. $a = 2, b = 4, c = 6$. Centre = (1, 2, 3) and

radius $= \frac{\sqrt{a^2 + b^2 + c^2}}{2} = \frac{\sqrt{4+16+36}}{2} = \sqrt{14}$

10. $PA + PB = 2k \Rightarrow PA = 2k - PB \Rightarrow PA^2 = 4k^2 + PB^2 - 4k\,.\,PB$

$\Rightarrow PA^2 - PB^2 - 4k^2 = -4k\,.\,PB$

$\Rightarrow (x-a)^2 + y^2 + z^2 - (x+a)^2 - y^2 - z^2 - 4k^2 = -4k\,.\,PB$

$\Rightarrow ax + k^2 = k\,.\,PB \Rightarrow a^2x^2 + 2ax\,k^2 + k^4 = k^2\,.\,PB^2$

$\Rightarrow \frac{a^2}{k^2}x^2 + k^2 + 2ax = (x+a)^2 + y^2 + z^2$

$$\Rightarrow \left(1-\frac{a^2}{k^2}\right)x^2+y^2+z^2=k^2-a^2$$

$\therefore$ Equation to the locus of P is $\left(1-\frac{a^2}{K^2}\right)x^2+y^2+z^2=K^2-a^2$.

11. Proceed as in Ex. 10.

EXERCISE 2 (*c*)

3. Given points are $P=(3,2,-4)$, $Q=(5,4,-6)$, $R=(9,8,-10)$

Then $PQ=\sqrt{(5-3)^2+(4-2)^2+(-6+4)^2}=\sqrt{4+4+4}=2\sqrt{3}$

$QR=\sqrt{(5-9)^2+(4-8)^2+(-6+10)^2}=\sqrt{16+16+16}=4\sqrt{3}$

$PR=\sqrt{(3-9)^2+(2-8)^2+(-4+10)^2}=6\sqrt{3}$

$\therefore\ PQ+QR=PR \Rightarrow P,Q,R$ are collinear.

Q divides PR in the ratio $PQ=QR=1:2$

4. Given $A=(2,3,4)$, $B=(3,-2,2)$ and $C=(6,-17,-4)$. Let $(C\,;A,B)=\lambda_1:\lambda_2$

$$\therefore \left(\frac{3\lambda_1+2\lambda_2}{\lambda_1+\lambda_2},\frac{-2\lambda_1+3\lambda_2}{\lambda_1+\lambda_2}\frac{2\lambda_1+4\lambda_2}{\lambda_1+\lambda_2}\right)$$

$$\therefore \frac{3\lambda_1+2\lambda_2}{\lambda_1+\lambda_2}=6 \quad \Rightarrow 3\lambda_1=-4\lambda_2 \quad \Rightarrow \lambda_1:\lambda_2=-4:3$$

5. Given $A=(-2,3,4)$ and $B=(1,2,3)$.

Let $(P\,;A,B)=\lambda_1:\lambda_2$. P lies on XZ plane. y-coordinate of $P=0$.

$$\Rightarrow \frac{2\lambda_1+3\lambda_2}{\lambda_1+\lambda_2}=0 \quad \Rightarrow 2\lambda_1=-3\lambda_2 \quad \Rightarrow \lambda_1:\lambda_2=-3:2$$

$$\Rightarrow P=\left(\frac{-3-4}{-3+2},0,\frac{-9+8}{-3+2}\right)=(7,0,1)$$

6. (*i*) Given $A=(1,2,3)$, $B=(2,10,1)$.

Let Q divide AB in the ratio $1:m$ then $Q=\left(\frac{2l+m}{l+m},\frac{10l+2m}{l+m},\frac{l+3m}{l+m}\right)$

given x-coordinate of Q is $-1 \Rightarrow \frac{2l+m}{l+m}=-1$

$\Rightarrow 2l+m=-l-m \quad \Rightarrow 3l=-2m \Rightarrow \frac{l}{m}=\frac{-2}{3}$. $\quad \therefore\ l=m=-2:3$.

y-coordinate of $Q=\frac{-20+6}{-2+3}=-14$. $\quad z$-coordinate of $Q=\frac{-2+9}{-2+3}=7$

7. (*ii*) Given $A=(1,1,1)$ and $B=(-2,4,1)$ Let $C=(x,y,z)$

Centroid of $\Delta\ ABC=(0,0,0)$

$$\Rightarrow \left(\frac{1-2+x}{3}, \frac{1+4+y}{3}, \frac{1+1+z}{3}\right) = (0,0,0)$$

$\Rightarrow x = 1, y = -5, z = -2 \quad \Rightarrow C = (1, -5, -2)$

8. Given $A = (5,-1,-1)$, $B = (-1,5,-1)$, $C = (-1,-1,5)$ and $D = (-3,-3,-3)$

$\overline{AB} = (-6,6,0) \Rightarrow AB = |\overline{AB}| = \sqrt{(36+36+0)} = 6\sqrt{2}$, etc.

Show that $AB = AC = AD = BC = BD = CD$. $\therefore$ $ABCD$ is a regualr tetrahedron.

9. Given $A = (3,-4,11), B = (1,4,-5), C = (17,-18,-3), D = (7,-6,1)$

$(P\,;A,B) = 1:\lambda_1$ and $(Q\,;C,D) = 1:\lambda_2$,

$$\Rightarrow P = \left(\frac{3\lambda_1+1}{\lambda_1+1}, \frac{-4\lambda_1+4}{\lambda_1+1}, \frac{11\lambda_1+5}{\lambda_1+1}\right)^2 \text{ and } Q = \left(\frac{17\lambda_2+7}{\lambda_2+1}, \frac{-18\lambda_2-6}{\lambda_2+1}, \frac{-3\lambda_2+1}{\lambda_2+1}\right)$$

$$P = Q \Rightarrow \frac{3\lambda_1+1}{\lambda_1+1} = \frac{17\lambda_2+7}{\lambda_2+1}, \text{etc.}$$

$\Rightarrow 3\lambda_1\lambda_2 + \lambda_2 + 3\lambda_1 + 1 = 17\lambda_1\lambda_2 + 7\lambda_1 + 17\lambda_2 + 7$, etc.

$\Rightarrow 14\lambda_1\lambda_2 + 4\lambda_1 + 16\lambda_2 = -6$ ... (1)

$14\lambda_1\lambda_2 + 2\lambda_1 + 22\lambda_2 = -10$ ... (2) $14\lambda_1\lambda_2 + 10\lambda_1 - 2\lambda_2 = -6$ ... (3)

$\Rightarrow 2\lambda_1 - 6\lambda_2 = 4$ from (1) and (2) $\Rightarrow \lambda_1 - 3\lambda_2 = 2$ ... (4)

$\Rightarrow 14\lambda_2\,(3\lambda_2+2) + 4(3\lambda_2+2)4 + 16\lambda_2 = -6$ from (1)

$\Rightarrow 3\lambda_2{}^2 + 4\lambda_2 + 1 = 0 \quad \Rightarrow (\lambda_2+1)(3\lambda_2+1) = 0 \quad \Rightarrow \lambda_2 = -\frac{1}{3} \; (\because \lambda_2 \neq -1)$

$\therefore$ From (4), $\lambda_1 + 1 = 2 \Rightarrow \lambda_1 = 1$ $\qquad \therefore P = (2,0,3) = Q$

(OR): Let $P = (2,0,3)$.

Given $A = (3,-4,11)$, $B = (1,4,-5)$, $C = (17,-18,-3)$ and $D = (7,-6,1)$.

$\therefore$ $\overline{PA} = (1,-4,8)$ $\qquad PA = \sqrt{1^2+4^2+8^2} = \sqrt{81}$

$\overline{PB} = (-1,4,-8)$ $\qquad PB = \sqrt{1^2+4^2+8^2} = \sqrt{81}$

$\overline{PC} = (15,-18,-6)$ $\qquad PC = \sqrt{15^2+18^2+6^2} = \sqrt{585} = 3\sqrt{65}$

$\overline{PD} = (5,-6,-2)$ $\qquad PD = \sqrt{5^2+6^2+2^2} = \sqrt{65}$

A, P, B are collinear. *Also C, P, D* are collinear.

$\therefore$ P is the common point to the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

10. Let $D = (x,y,z)$. Diagonals $\overline{AC}, \overline{BD}$ intersect each other.

$$\therefore \left(\frac{x+1}{2}, \frac{y+4}{2}, \frac{z+7}{2}\right) = \left(\frac{4+2}{2}, \frac{7+1}{2}, \frac{19-3}{2}\right) \Rightarrow (x,y,z) = (5,4,9)$$

Exercise 2 (*d*)

1. (*i*) $O=(0,0,0)$, $P=(6,2,3) \Rightarrow$ d.rs. $\overrightarrow{OP}$ are $6-0, 2-0, 3-0$

$\Rightarrow$ d.rs. $\overrightarrow{OP}$ are 6, 2, 3

$\Rightarrow$ d.cs. $\overrightarrow{OP}$ are $\dfrac{6}{\sqrt{(6^2+2^2+3^2)}}, \dfrac{2}{\sqrt{6^2+2^2+3^2}}, \dfrac{3}{\sqrt{6^2+2^2+3^2}}$

$\Rightarrow$ d.cs. $\overrightarrow{OP}$ are $\dfrac{6}{7}, \dfrac{2}{7}, \dfrac{3}{7}$

Note. d.cs. $\overleftrightarrow{OP}$ are $\dfrac{6}{7}, \dfrac{2}{7}, \dfrac{3}{7}$ (or) $-\dfrac{6}{7}, \dfrac{-2}{7}, \dfrac{-3}{7}$

2. Proceed as in Ex. 1 (*i*)

3. (*i*) Let $\overrightarrow{OP}$ be the ray making angles α, β, γ with $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$.

$\therefore$ d.cs. of $\overrightarrow{OP}$ are $\cos\alpha, \cos\beta, \cos\gamma$. $\quad \therefore \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

$\Rightarrow 1-\sin^2\alpha+1-\sin^2\beta+1-\sin^2\gamma=1 \Rightarrow \sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$

(*ii*) Let $(\overrightarrow{OP}, \overrightarrow{OX})=\alpha$, $(\overrightarrow{OP}, \overrightarrow{OY})=\alpha$ and $(\overrightarrow{OP}, \overrightarrow{OZ})=\alpha$

$\therefore$ $\cos\alpha$, $\cos\alpha$, $\cos\alpha$ are the d.cs. of $\overrightarrow{OP}$

$\Rightarrow \cos^2\alpha+\cos^2\beta+\cos^2\alpha=1 \quad \Rightarrow \cos\alpha = \pm 1/\sqrt{3}$

$\therefore$ d.cs. of the line $\overleftrightarrow{OP}$ (making equal angles with coordinate axes) are $\pm 1/\sqrt{3}$, $\pm 1/\sqrt{3}$, $\pm 1/\sqrt{3}$.

Noting that $+1/\sqrt{3}$, $+1/\sqrt{3}$, $+1/\sqrt{3}$; $-1/\sqrt{3}$, $-1/\sqrt{3}$, $-1/\sqrt{3}$ are two sets of d.cs. of the same line, we can have four lines making equal angles with the coordinate axes.

4. (*i*) If *A, B, C, D* are the given points. d.rs. of $\overleftrightarrow{AB}$ = d.rs. of $\overleftrightarrow{CD} \Rightarrow \overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

(*ii*) If *A, B, C* are given points, d.rs. of $\overrightarrow{AB}$ = d.rs. of $\overrightarrow{AC} \Rightarrow$ *A, B, C* are collinear.

(*iii*) Let $A=(2,3,4), B=(0,1,2), C=(2,0,4), D=(7,-4,3)$

D.r.s of $\overline{AB}$ are $(-2,-2,-2)$ and D.r.s of $\overline{CD}$ are $(5,-4,-1)$

Since $(-2)(5)+(-2)(-4)+(-2)(-1)=0$. $\overline{AB} \perp \overline{CD}$.

5. d.cs. of L_1 are l_1, m_1, n_1, and d.cs. of L_2 are l_2, m_2, n_2.

If $L \perp L_1, L \perp L_2$ and if d.cs. of L are l, m, n, then

$ll_1+mm_1+nn_1=0$, $ll_2+mm_2+nn_2=0$ *i.e.,* $\dfrac{l}{m_1n_2-m_2n_1}=\dfrac{m}{n_1l_2-n_2l_1}=\dfrac{n}{l_1m_2-l_2m_1}$

If $(L_1, L_2)=\theta$, then $\sin\theta=\sqrt{\left[\sum(m_1n_2-m_2n_1)^2\right]}$

$L_1 \perp L_2 \Rightarrow \theta=\dfrac{\pi}{2} \Rightarrow \sum(m_1n_2-m_2n_1)^2=1$

$\therefore$ d.cs. of L are $\dfrac{m_1n_2-m_2n_1}{\sqrt{\left[\sum(m_1n_2-m_2n_1)^2\right]}}$, ..., ...

i.e., $m_1n_2-m_2n_1$, $n_1l_2-n_2l_1$, $l_1m_2-l_2m_1$ are the DR's of L.

6. $\overline{OP} = (-1, 2, 3), \overline{OQ} = (-3, 4, 5)$.

If d.rs. of a normal to the plane containing $\overrightarrow{OP}, \overrightarrow{OQ}$ are (l, m, n)

We have $l(-1) + m(2) + n(3) = 0$, $\quad l(-3) + m(4) + n(5) = 0$

Solving , we get the d.r.'s as $(1, 2, -1)$

7. d.cs. of x-axis are (1, 0, 0). Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$

$\therefore \overline{PQ} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$

Projection of $\overline{PQ}$ on x-axis $= 12 = \sum l(x_2 - x_1)$

$\Rightarrow 1(x_2 - x_1) + 0(y_2 - y_1) + 0(z_2 - z_1) = 12 \quad \Rightarrow x_2 - x_1 = 12$

Similarly $y_2 - y_1 = 3$ and $z_2 - z_1 = 4$

$\therefore PQ = |\overline{PQ}| = \sqrt{\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]} = 13$

8. (*i*) The direction ratios of two lines are given by $\quad l + 5m + 3n = 0 \quad \ldots (1)$

$7l^2 + 5m^2 - 3n^2 = 0 \ \cdots (2) \qquad$ From (1) $l = -(5m + 3n)$

Substituting in (2) we get, $7(-5m - 3n)^2 + 5m^2 - 3n^2 = 0$

$\Rightarrow 7(25m^2 + 9n^2 + 30mn) + 5m^2 - 3n^2 = 0 \Rightarrow 180m^2 + 60n^2 + 210mn = 0$

Dividing with n^2 we get, $\Rightarrow 6\left(\frac{m}{n}\right)^2 + 7\left(\frac{m}{n}\right) + 2 = 0$

$\Rightarrow 6p^2 + 7p + 2 = 0 \quad \Rightarrow (3p + 2)(2p + 1) = 0 \Rightarrow p = \frac{-2}{3}$ and $p = \frac{1}{2}$.

Case I. Let $p = \frac{m}{n} = \frac{-2}{3} \quad \Rightarrow \frac{m}{2} = \frac{n}{-3}$

We get $l = -5m - 3n = -5\left(\frac{-2n}{3}\right) - 3n = \frac{10n - 9n}{3} = \frac{n}{3}$

$\therefore \frac{l}{-1} = \frac{m}{2} = \frac{n}{3}$ which gives the d.r.s. $(-1, 2, 3)$

Case II. Let $p = \frac{m}{n} = \frac{-1}{2} \Rightarrow \frac{m}{1} = \frac{n}{-2}$

We get $l = -5n - 3n = -5\left(\frac{-n}{2}\right) - 3n = \frac{5n - 6n}{2} = \frac{-n}{2}$

$\frac{l}{1} = \frac{m}{1} = \frac{n}{-2} \Rightarrow$ d.r's are $(1, 1, -2)$

(*iii*) Direction ratios are given by $l + m - n = 0 \quad \ldots (1) \quad 6ln - 12lm + mn = 0 \quad \ldots (2)$

From (1) $l = -m + n \qquad \ldots (3)$ Substituting in (2) we get

$6(-m + n)n - 12(-m + n)m + mn = 0$

$\Rightarrow -6mn + 6n^2 + 12m^2 - 12mn + mn = 0 \Rightarrow 6n^2 + 12m^2 - 17mn = 0$

Dividing with n^2, we get, $12\left(\frac{m}{n}\right)^2 - 17\left(\frac{m}{n}\right) + 6 = 0$

$12p^2 - 17p + 6 = 0 \left\{\text{where } p = \frac{m}{n}\right\} \Rightarrow (3p-2)(4p-3) = 0 \Rightarrow p = \frac{2}{3}$ and $p = \frac{3}{4}$

Case I. Let $p = \frac{m}{n} = \frac{2}{3} \Rightarrow \frac{m}{2} = \frac{n}{3}$

Substituting in (3) $l = -m + n = \frac{-2n}{3} + n = \frac{-2n+3n}{3} = \frac{n}{3}$

Then $\frac{l}{1} = \frac{m}{2} = \frac{n}{3}$ d.r's are (1, 2, 3). Dcs are $\left(\pm\frac{1}{\sqrt{14}}, \pm\frac{2}{\sqrt{14}}, \pm\frac{3}{\sqrt{14}}\right)$

Case II : Let $p = \frac{m}{n} = \frac{3}{4} \Rightarrow \frac{m}{3} = \frac{n}{4}$

Substituting in (3) we get $l = \frac{-3n}{4} + n = \frac{-3n+4n}{4} = \frac{n}{4}$

$\frac{l}{1} = \frac{m}{3} = \frac{n}{4}$ d.rs. are (1, 3, 4). Dcs are $\left(\frac{\pm 1}{\sqrt{26}}, \frac{\pm 3}{\sqrt{26}}, \frac{\pm 4}{\sqrt{26}}\right)$

9. The direction cosines of the given lines satisfy the equations :

$l + m + n = 0$ ... (1) $2mn + 3nl - 5lm = 0$ ... (2)

Substituting $n = -(l+m)$ from (1) into (2) we get,

$-2m(l+m) - 3l(l+m) - 5lm = 0 \Rightarrow 3l^2 + 10lm + 2m^2 = 0$

$\Rightarrow 3\left(\frac{l}{m}\right)^2 + 10\left(\frac{l}{m}\right) + 2 = 0$ (Dividing by m^2)

Let its roots are $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$. Then, $\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{2}{3} \Rightarrow \frac{l_1 l_2}{2} = \frac{m_1 m_2}{3} = K$ (say) ... (3)

Again substituting $l = -(m+n)$ from (1) into (2) we get

$2mn - 3n(m+n) + 5m(m+n) = 0 \Rightarrow 5m^2 + 4mn - 3n^2 = 0$

$\Rightarrow 5\left(\frac{m}{n}\right)^2 + 4\left(\frac{m}{n}\right) - 3 = 0$. Dividing by n^2

Let its roots be $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$. Then $\frac{m_1}{n_1} \cdot \frac{m_2}{n_2} = \frac{-3}{5} \Rightarrow \frac{m_1 m_2}{3} = \frac{n_1 n_2}{-5} = K$ ... (4)

From (3) and (4) we get, $\frac{l_1 l_2}{2} = \frac{m_1 m_2}{3} = \frac{n_1 n_2}{-5} = K$

$\therefore l_1 l_2 = 2K$, $m_1 m_2 = 3K$ and $n_1 n_2 = -5K$

$\therefore l_1 l_2 + m_1 m_2 + n_1 n_2 = 2K + 3K - 5K = 0$

Hence the given lines are $\perp$ to each other.

10. (*i*) Drs are given by the equations $3l + m + 5n = 0$ ... (1) $6mn - 2nl + 5lm = 0$ ... (2)

From (1) $m = -3l - 5n$

We have, $6n(-3l - 5n) - 2nl + 5l(-3l - 5n) = 0$; $-18nl - 30n^2 - 2nl - 15l^2 - 25ln = 0$

Dividing with $l^2 \Rightarrow 30\dfrac{n^2}{l^2} + 45\dfrac{n}{l} + 15 = 0$

$2p^2 + 3p + 1 = 0$ where $p = \dfrac{n}{l} \Rightarrow (2p+1)(p+1) = 0 \Rightarrow p = -1, p = -\dfrac{1}{2}$

Case I : $p = \dfrac{n}{l} = -1 \Rightarrow \dfrac{l}{1} = \dfrac{n}{-1}$

from (1) $m = -3l + 5l = 2l \quad \Rightarrow \dfrac{m}{2} = \dfrac{l}{1}$. $\qquad \therefore \dfrac{l}{1} = \dfrac{m}{2} = \dfrac{n}{-1}$ drs $= (1, 2, -1)$

Case II : $p = \dfrac{n}{l} = \dfrac{-1}{2} \Rightarrow \dfrac{n}{1} = \dfrac{l}{-2}$ $\qquad$ from (3) $m = -3l - 5\left(\dfrac{-l}{2}\right) = \dfrac{-6l + 5l}{2} = \dfrac{-l}{2}$

$\dfrac{l}{-2} = \dfrac{m}{1} = \dfrac{n}{1}$, drs $= (-2, 1, 1)$

θ is (given) the angle between the lines

$$\Rightarrow \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{-2+2-1}{\sqrt{6}\,.\,\sqrt{6}} = \frac{1}{6}$$

11. Let l_1, m_1, n_1 and l_2, m_2, n_2 be d.rs. of L_1, L_2.

d.rs. of the two rays L_1, L_2 are determined by

$al + bm + cn = 0$ ... (1) $\quad fmn + gnl + hlm = 0$ ... (2)

From (1) and (2) : $fmn - gn\left(\dfrac{bm + cn}{a}\right) - hm\left(\dfrac{bm + cn}{a}\right) = 0$

$\Rightarrow afmn - bgmn - chmn - cgn^2 - bhm^2 = 0$

$$\Rightarrow bh\left(\frac{m}{n}\right)^2 + (ch + bg - af)\frac{m}{n} + cg = 0 \qquad \text{... (3)}$$

$\dfrac{m_1}{n_1}, \dfrac{m_2}{n_2}$ are two roots of the equation $\Rightarrow \dfrac{m_1m_2}{n_1n_2} = \dfrac{cg}{bh} = \dfrac{g/b}{h/c} \quad \Rightarrow \dfrac{m_1m_2}{g/b} = \dfrac{n_1n_2}{h/c}$

By symmetry, we can have $\dfrac{l_1l_2}{f/a} = \dfrac{m_1m_2}{g/b} = \dfrac{n_1n_2}{h/c}$

(*i*) $L_1 \perp L_2 \Rightarrow l_1l_2 + m_1m_2 + n_1n_2 = 0 \Rightarrow \dfrac{f}{a} + \dfrac{g}{b} + \dfrac{h}{c} = 0$

(*ii*) $L_1 \parallel L_2 \Rightarrow \dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2} \Rightarrow \dfrac{m_1}{n_1} = \dfrac{m_2}{n_2}$

$\Rightarrow$ roots of (3) are equal $\Rightarrow$ discriminant of (3) is zero

$\Rightarrow (ch + bg + af)^2 = 4bhcg$

$\Rightarrow c^2h^2 + b^2g^2 + a^2f^2 + 2bcgh - 2abfg - 2cafh = 4bcgh$

$\Rightarrow (af + bg - ch)^2 = 4abfg$

$\Rightarrow af + bg - ch = \pm 2\sqrt{af\ bg}$ $\quad\Rightarrow af + bg \pm 2\sqrt{af\ bg} = ch$

$\Rightarrow (\sqrt{af} + \sqrt{bg})^2 = (\sqrt{ch})^2$ $\quad\Rightarrow \sqrt{af} \pm \sqrt{bg} = \pm\sqrt{ch}$

$\Rightarrow \sqrt{af} \pm \sqrt{bg} \pm \sqrt{ch} = 0$ $\quad\Rightarrow \sqrt{af} + \sqrt{bg} + \sqrt{ch} = 0$

12. d.rs. of concurrent rays $\overrightarrow{OP}, \overrightarrow{OQ}, \overrightarrow{OR}$ are $1, -1, 1$; $2, -3, 0$; $1, 0, 3$ consider

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \\ 1 & 0 & 3 \end{vmatrix} = 1(-9) + 1(6) + 1(3) = 0,$$

$\Rightarrow$ rays $\overrightarrow{OP}, \overrightarrow{OQ}, \overrightarrow{OR}$ are coplanar.

13. Let l, m, n be d.rs. of the line L

Let L_1, L_2, L_3 be the lines whose respective d.rs. are $(2, 1, 5)$; $(4, -2, 2)$; $(-6, 4, -1)$

$L \perp L_1 \Rightarrow 2l + m + 5n = 0$ ... (1) $\quad L \perp L_2 \Rightarrow 4l - 2m + 2n = 0$ ... (2)

$L \perp L_3 \Rightarrow -6l + 4m - n = 0$ ... (3)

From (1), (2) : $\dfrac{l}{2+10} = \dfrac{m}{20-4} = \dfrac{n}{-4-4} \Rightarrow \dfrac{l}{3} = \dfrac{m}{4} = \dfrac{n}{-2}$

But (3) is satisfied by the proportional values of l, m, n

$\therefore L \perp L_1, L \perp L_2$ and $L \perp L_3$

since L_1, L_2, L_3 are concurrent and since each is perpendicular to L, the lines L_1, L_2, L_3 are coplanar.

14. L_1, L_2, L_3 are three concurrent lines with d.rs. $l_1, m_1, n_1; l_2, m_2, n_2$;

$l_1 + \lambda l_2, m_1 + \lambda m_2, n_1 + \lambda n_2$

$$\text{Now} \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_1 + \lambda l_2 & m_1 + \lambda m_2 & n_1 + \lambda n_2 \end{vmatrix} = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_1 & m_1 & n_1 \end{vmatrix} + \lambda \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$\therefore$ The concurrent lines L_1, L_2, L_3 are coplanar

15. Projection of $\overline{\text{OP}}$ on the line having d.cs. $\left(\dfrac{2}{7}, \dfrac{-3}{7}, \dfrac{6}{7}\right)$ is $\sum l(x_2 - x_1)$

$$= 5\left(\frac{2}{7}\right) + 2\left(\frac{-3}{7}\right) + 4\left(\frac{6}{7}\right) = 4.$$

16. $\overline{AB} = (-3, 0, -3)$, $\overline{BC} = (4, 2, -4)$, $\overline{CA} = (-1, -2, 7)$

$\therefore \cos A = \dfrac{\overline{AB} \cdot \overline{AC}}{|\overline{AB}||\overline{AC}|} = \dfrac{(-3,0,-3) \cdot (1,2,-7)}{\sqrt{18} \cdot \sqrt{54}} = \dfrac{-3+21}{18\sqrt{3}} = \dfrac{1}{\sqrt{3}} \quad \therefore A = \cos^{-1}\dfrac{1}{\sqrt{3}}$

Also $\cos B = \dfrac{\overline{BA} \cdot \overline{BC}}{|\overline{BA}||\overline{BC}|} = \dfrac{(3,0,3) \cdot (4,2,-4)}{\sqrt{18} \cdot \sqrt{36}} = 0 \Rightarrow B = \dfrac{\pi}{2}$

Again $\cos C = \dfrac{\overline{CA} \cdot \overline{CB}}{|\overline{CA}||\overline{CB}|} = \dfrac{(-1,-2,7) \cdot (-4,-2,4)}{\sqrt{54} \cdot \sqrt{36}} = \dfrac{36}{18\sqrt{6}} = \sqrt{2/3} \Rightarrow C = \cos^{-1}\sqrt{2/3}$

Area of $\Delta ABC = \dfrac{1}{2}|\overline{AB} \times \overline{AC}| = \dfrac{1}{2}|(6, -24, -6)| = \dfrac{1}{2}\sqrt{(36+16\times 36+36} = 9\sqrt{2}$ sq. units

17. Proceed as in W. Ex. 11 by taking $a = b = c$

18. Vide Fig. 37 in W. Ex 11. Here $b = c = a$ (say)

d.cs. of the diagonals $\overrightarrow{OP}, \overrightarrow{AN}, \overrightarrow{BM}, \overrightarrow{CL}$ are respectively

$\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right), \left(-\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right), \left(\dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right), \left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{-1}{\sqrt{3}}\right)$

Let l, m, n be d.cs. of L.

Let L make angles $\alpha, \beta, \gamma, \delta$ respectively with $\overrightarrow{OP}, \overrightarrow{AN}, \overrightarrow{BM}, \overrightarrow{CL}$.

$\therefore \cos\alpha = \dfrac{1}{\sqrt{3}}l + \dfrac{1}{\sqrt{3}}m + \dfrac{1}{\sqrt{3}}n = \dfrac{l+m+n}{\sqrt{3}}$

Similarly $\cos\beta = \dfrac{-l+m+n}{\sqrt{3}}, \cos\gamma \dfrac{l-m+n}{\sqrt{3}}$ and $\cos\delta = \dfrac{l+m-n}{\sqrt{3}}$

$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

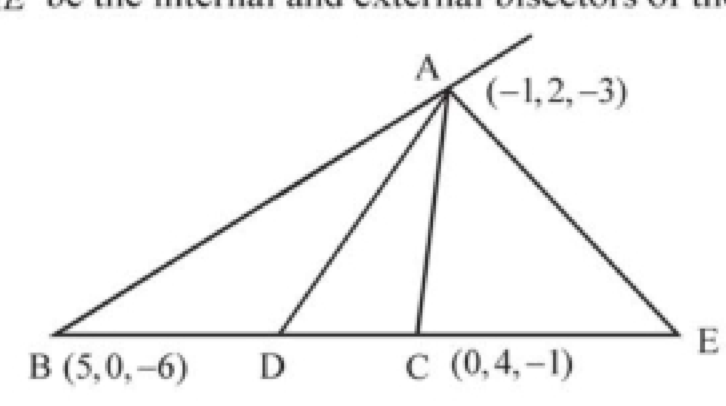
$$= \frac{(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2}{3} = \frac{4}{3}$$

19. (*i*) Show that AB is parallel and equal to CD.

(*ii*) Show that $AB = BC = CD = DA$ and $AB \perp BC$.

(*iii*) Show that $AB = BC = CD = DA$ and $AB \perp BC$.

20. Let $\overrightarrow{AD}, \overrightarrow{AE}$ be the internal and external bisectors of the angle $(\overrightarrow{AB}, \overrightarrow{AC})$.



$\therefore\ (D\,;B,C)=AB:AC$ and $(E\,;B,C)=AB\,;-AC$

Now $\overline{AB}=(6,-2,-3)$ and $\overline{AC}=(1,2,2)$

$AB=|\overline{AB}|=\sqrt{(36+4+9)}=7\,;\quad AC=|\overline{AC}|=\sqrt{(1+4+4)}=3$

$\Rightarrow(D\,;B,C)=7:3$ and $(E\,;B,C)=7:-3$

Hence $D=\left(\frac{15}{10},\frac{28}{10},\frac{-25}{10}\right)$ and $E=\left(\frac{-25}{4},\frac{28}{4},\frac{11}{4}\right)$

$\therefore$ d.rs. of $\overrightarrow{AD}$ are $\left(\frac{25}{10},\frac{8}{10},\frac{5}{10}\right)$ *i.e.,* (25, 8, 5)

and d.rs. of $\overrightarrow{AE}$ are $\left(\frac{-11}{4},\frac{20}{4},\frac{23}{4}\right)$ *i.e.,* $(-11,20,23)$

21. Given points are $A=(3,-1,11)$, $B=(0,2,3)$, $C=(4,8,11)$.

Let $(D, B, C)=1:\lambda$. Then $D=\left(\frac{4}{1+\lambda},\frac{8+2\lambda}{1+\lambda},\frac{11+3\lambda}{1+\lambda}\right)$ then

$\overline{AD}=\left(\frac{4}{1+\lambda}-3,\frac{8+2\lambda}{1+\lambda}+1,\frac{11+3\lambda}{1+\lambda}-11\right)=\left(\frac{1-3\lambda}{1+\lambda},\frac{9+3\lambda}{1+\lambda},\frac{-8\lambda}{1+\lambda}\right)$, $\overline{BC}=(4,6,8)$

Since $\overline{AD}\perp\overline{BC}$ we have, $4\left(\frac{1-3\lambda}{1+\lambda}\right)+\frac{6\,(9+3\lambda)}{1+\lambda}+\frac{-64\lambda}{1+\lambda}=0$

$4-12\lambda+54+18\lambda-64\lambda=0\quad\Rightarrow-58\lambda+58=0\quad\Rightarrow\lambda=1$

Substituting in D. $\therefore\ D=\left(\frac{4}{2},\frac{10}{2},\frac{14}{2}\right)=(2,5,7)$

22. $O=(0,0,0)$. Let $A=(x_1,y_1,z_1)$, $B=(x_2,y_2,z_2)$ and $C=(x_3,y_3,z_3)$

$\overline{OA}\perp\overline{BC}\Rightarrow x_1(x_3-x_2)+y_1(y_3-y_2)+z_1(z_3-z_2)=0$ ... (1)

$\overline{OB}\perp\overline{AC}\Rightarrow x_2(x_3-x_1)+y_2(y_3-y_1)+z_2(z_3-z_1)=0$ ... (2)

(2) $-$ (1): $x_3(x_2-x_1)+y_3(y_2-y_1)+z_3(z_2-z_1)=0\quad\Rightarrow OC\perp AB$.

The Plane

Exercise 3 (*a*)

1. x-intercept of the plane $2x-3y+4z=12$ is obtained by putting $y=0$ and $z=0$ in its equation. $\therefore$ x-intercept of the plane = 6.
Similarly y-intercept of the plane $=-4$ and z-intercept of the plane = 3.

2. Given plane is $2x-2y+z=5$ ... (1) d.rs. of its normal are $2,-2,1$

Dividing (1) by $\sqrt{(4+4+1)}=3$, we get $\frac{2}{3}x+\left(\frac{-2}{3}\right)y+\frac{1}{3}z=\frac{5}{3}$ ... (2)

Equation (2) or equation (3) is called noraml form equation of the plane (1).
For (2), d.cs. of its normal are $\frac{2}{3},\frac{-2}{3},\frac{1}{3}$ and distance of the origin from the plane $=\frac{5}{3}$.

3. (*i*) Since L, M are respectively the feet of the perpendiculars from $P\,(a,b,c)$ to YZ and ZX planes, $L=(0,b,c)$ and $M=(a,0,c)$. Also $O=(0,0,0)$.

$\therefore$ Equation to the plane $\overleftrightarrow{OLM}$ is $\begin{vmatrix} x-0 & y-0 & z-0 \\ 0-0 & b-0 & c-0 \\ a-0 & 0-0 & c-0 \end{vmatrix}=0$ *i.e.,* $bcx+cay-abz=0$

(*ii*) Proceed as in 3 (*i*).

4. $O\,(0,0,0)$. Let $P=(2,-3,4)$ d.rs. of $\overleftrightarrow{OP}$ are $2,-3,4$

$\therefore$ Equation to the plane through P and perpendicular to $\overleftrightarrow{OP}$ is
$2\,(x-2)-3\,(y+3)+4\,(z-4)=0$ *i.e.,* $2x-3y+4z=29$.

5. $O\,(0,0,0)$, $A=(a,b,c)$ are the given points. d.r.s. of $\overleftrightarrow{OA}$ are a,b,c.

d.cs. of $\overleftrightarrow{OA}$ are $\frac{\pm a}{\sqrt{a^2+b^2+c^2}},\frac{\pm b}{\sqrt{a^2+b^2+c^2}},\frac{\pm c}{\sqrt{a^2+b^2+c^2}}$

$\therefore$ Equation to the plane through A and at right angles of $\overleftrightarrow{OA}$ is
$a\,(x-a)+b\,(y-b)+c\,(z-c)=0$, *i.e.,* $ax+by+cz-a^2-b^2-c^2=0$

6. Let $P=(4,0,1)$. Given plane is $4x+3y-12z+8=0$ ... (1)
Required plane passes through P and is parallel to (1)
Any plane parallel to (1) is $4x-3y-12z=K$ ... (2)
K being an unknown constant
If (2) is the required plane, then $16+0-12=K \Rightarrow K=4$
$\therefore$ Equation to the required plane is $4x+3y-12z=4$.

7. Since the planes $x-2y+Kz=0$ ... (1) $2x+5y-z=0$ ... (2)
are at right angles, $1.2+(-2)\,5+K+(-1)=0$ *i.e.,* $K=-8$
Let l,m,n be d.rs. of the normal to the plane perpendicular to (1)&(2)

$\therefore\ 1.l+(-2)\,m+(-8)\,n=0$ *i.e.,* $l-2m-8n=0$ ... (3)

and $2.l+5\,m+(-1)\,n=0$ *i.e.,* $2l+5m-n=0$ ... (4)

From (3) and (4) : $\dfrac{l}{42}=\dfrac{m}{-15}=\dfrac{n}{9}$

$\therefore$ Equation to the plane through $(1,-1,-1)$ and

perpendicular to (1) and (2) is $42\,(x-1)-15\,(y+1)+9\,(z+1)=0$

i.e., $42x-15y+9z=48$ *i.e.,* $14x-5y+3z=16$

8. (*i*) Let $P=(2,2,1)$ and $Q=(9,3,6)$. Let the required plane pass through P and Q and be perpendicular to $2x+6y+6z=9$ ... (1)

Let l, m, n be d.cs. of the required plane.

Since normal to the required plane is perpendicular to $\overrightarrow{PQ}$,

$(9-2)\,l+(3-2)\,m+(6-1)\,n=0 \Rightarrow 7\,l+m+5\,n=0$ ... (2)

Since the required plane is perpendicular to (1), their normals are perpendicular.

$\therefore\ 2\,l+6\,m+6\,n=0$ ... (3)

From (2), (3) : $\dfrac{l}{-24}=\dfrac{m}{-32}=\dfrac{n}{40}\Rightarrow\dfrac{l}{3}=\dfrac{m}{4}=\dfrac{n}{-5}$

$\therefore$ Equation to the required plane is $3\,(x-2)+4\,(y-2)-5\,(z-1)=0$

i.e., $3x+4y-5z=9$.

(*ii*) Given points are $P\,(1,-2,4)$; $Q\,(3,-4,5)$ Dr's of PQ are $(3-1,-4+2,5-4)=(2,-2,1)$.

To the *xy*-plane *z*-axis is the normal whose dcs. are (0, 0, 1)

Let l,m,n are the dc's of the required plane. Then we have

$2l-2m+n=0$ ... (1) $n-0$ ... (2)

Sub (2) in (1) $\Rightarrow 2l-2m=0 \Rightarrow l=m$. $\therefore\ \dfrac{l}{1}=\dfrac{m}{1}=\dfrac{n}{0}$

So drs of the required plane are (1, 1, 0), since this passes through $(1,-2,4)$ equation of the plane is $1\,(x-1)+1\,(y+2)+0\,(3-4)=0$, $x-1+y+2=0$, $x+y+1=0$

(*iii*) Do as in (*i*)

9. Let the equation of the plane through the point $(-1,3,2)$ be

$l\,(x+1)+m\,(y-3)+n\,(z-2)=0$ ... (1)

Given planes are $x+2y+2z=5$ ... (2)

and $3x+3y+2z=8$ ... (3)

(1) $\perp$ (2) $\Rightarrow l.1+m.2+n.2=0 \Rightarrow l+2m+2n=0$ ... (4)

(1) $\perp$ (3) $\Rightarrow l.3+m.3+n.2=0 \Rightarrow 3l+3m+2n=0$ ... (5)

From (4) and (5) : $\frac{l}{4-6}=\frac{m}{6-2}=\frac{n}{3-6} \Rightarrow \frac{l}{2}=\frac{m}{-4}=\frac{n}{3}$

$\therefore$ Equation to the required plane is

$$2(x+1)-4(y-3)+3(z-2)=0 \Rightarrow 2x-4y+3z+8=0$$

10. Let l, m, n be d.rs. of the normal to the required plane. Since the required plane is parallel to x-axis, normal to the required plane is perpendicular to the x-axis.

$\therefore \; l.1+m.0+n.0=0 \Rightarrow l=0$. For the rest proceed as in Ex. 8.

11. Equation to the required plane is

$$(3-2)(x-2)+(4+1)(y-3)+(-1-5)(z-1)=0 \Rightarrow x+5y-6z+19=0$$

12. Given points are $(-1,6,2),(1,2,3)$ and $(-2,3,4)$

Equation to the plane passing through $(-1,6,2)$ and perpendicular to the join of the points $(1,2,3),(-2,3,4)$ is

$$(1+2)(x+1)+(2-3)(y-6)+(3-4)(z-2)=0$$

$$3x+3-y+6-z+2=0,\; 3x-y-z+11=0$$

13. (*i*) Let $P=(2,0,6)$ and $Q=(-6,2,4)$

Required plane passes through the mid-point of PQ and is perpendicular to PQ. Mid-point of $P=(-2,1,5)$

Equation to the required plane is $-8(x+2)+2(y-1)-2+(z-5)=0$, $4x+y+z+4=0$

14. (*i*) Given points are $(2,2,-1),(3,4,2),(7,0,6)$. Equation of the plane passing through three points is $\begin{vmatrix} x-2 & y-2 & z+1 \\ 3-2 & 4-2 & 2+1 \\ 7-2 & 0-2 & 6+1 \end{vmatrix}=0$

$\Rightarrow (x-2)(14+6)-(y-2)(7-15)+(z+1)(-2-10)=0$

$\Rightarrow 20x-40+8y-16-12z-12=0 \Rightarrow 20x+8y-12z-68=0 \Rightarrow 5x+2y-3z-17=0$

(*ii*) Given points are (1, 1, 1), $(1,-1,1),(-7,-3,-5)$. Equation of the plane passing through three points is $\begin{vmatrix} x-1 & y-1 & z-1 \\ 1-1 & -1-1 & 1-1 \\ -7-1 & -3-1 & -5-1 \end{vmatrix}=0$

$\Rightarrow (x-1)(12-0)-(y-1)(0-0)+(z-1)(0-16)=0$

$\Rightarrow -12x-12y-16z=0 \Rightarrow 12x-16z+4=0 \;\; \Rightarrow 3x-4z+1=0$

The above equation is parallel to y-axis.

15. (*i*) Equation to the plane containing the points

$$\begin{vmatrix} (x+6) & y-3 & z-2 \\ -7 & 14 & -3 \\ 9 & -5 & 2 \end{vmatrix} = 0 \Rightarrow x - y - 7z + 23 = 0.$$

If (5, 7, 3) is a point on the above plane $5 - 7 - 21 + 23 = 0$ satisfied.

$\Rightarrow$ the given points are coplanar.

(*ii*) Proceed as in Ex. 15 (*i*)

Since d.rs. of y-axis are 0, 1, 0 and since $3 \cdot 0 + 0 \cdot 1 + (-4) \cdot 0 = 0$, the normal to the plane (1) is perpendicular to y-axis *i.e.*, the plane (1) is parallel to y-axis.

16. Equation to the plane ABC is

$$\begin{vmatrix} x-3 & y-2 & z+5 \\ -3-3 & 8-2 & -5+5 \\ -3-3 & 2-2 & 1+5 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-3 & y-2 & z+5 \\ -6 & 6 & 0 \\ -6 & 0 & 6 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x+z+2 & y-2 & z+5 \\ -6 & 6 & 0 \\ 0 & 0 & 6 \end{vmatrix} = 0 \Rightarrow x + y + z = 0$$

Let $S = (-1, 4, -3)$. Show that $SA = SB = SC$

Centroid of $\Delta\, ABC = \left(\dfrac{3-3-3}{3}, \dfrac{2+8+2}{3}, \dfrac{-5-5+1}{3}\right) = (-1, 4, -3)$.

17. (*i*) The equations of the planes are $x + 2y + 3z = 5$, $3x + 3y + z = 9$

Let 'θ' be the angle between the given planes

$$\theta = \cos^{-1}\left(\frac{1(3) + 2(3) + 3(1)}{\sqrt{1+4+9}\,\sqrt{9+9+1}}\right) = \cos^{-1}\left(\frac{12}{\sqrt{14}\,\sqrt{19}}\right) \Rightarrow \theta = \cos^{-1}\left(\frac{12}{\sqrt{266}}\right)$$

The other angle between the planes is $180^\circ - \theta$ *i.e.*, $180^\circ - \cos^{-1}\left(\dfrac{12}{\sqrt{266}}\right)$

(*ii*) Given plane equations are $2x - 3y + 4z + 11 = 0$, $3x - 2y - 3z + 27 = 0$

Let 'θ' be the angle between the given planes

$$\theta = \cos^{-1}\left(\frac{2(3) - 3(-2) + 4(-3)}{\sqrt{4+1+16}\,\sqrt{9+4+9}}\right) \quad \theta = \cos^{-1}\left(\frac{-12+12}{\sqrt{21}\,\sqrt{22}}\right) = \cos^{-1}(0) = \theta = 90^0$$

(*iii*) Given plane are $2x - y + z = 0, x + y + 2z = 7$

Let 'θ' be the angle between the given planes

$$\theta = \cos^{-1}\left(\frac{2(1) - 1(1) + 1(2)}{\sqrt{4+1+1}\,\sqrt{1+1+4}}\right) \quad \theta = \cos^{-1}\left(\frac{3}{6}\right) = \cos^{-1}(1/2) \quad \theta = \frac{\pi}{3}$$

Other angle is $\pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$

(*iv*) Given planes are $x+y-5z+6=0$, $2x+3y+6z+5=0$

Let 'θ' be the angle between the given planes $\theta=\cos^{-1}\left(\dfrac{1(2)+1(3)-5(6)}{\sqrt{1+1+25}\sqrt{4+9+36}}\right)$

$$\theta=\cos^{-1}\left(\frac{-25}{\sqrt{27}\sqrt{49}}\right)\quad \theta=\cos^{-1}\left(\frac{-25}{7\sqrt{27}}\right)=\cos^{-1}\left(\frac{-25}{21\sqrt{3}}\right)$$

18. From theorem in Art 9.13 :

If the line through $A=(x_1, y_1, z_1)$ and $B=(x_2, y_2, z_2)$ intersect the plane $ax+by+cz+d=0$ in C, then

$(C\,;\,A, B)=-(ax_1+by_1+cz_1+d):(ax_2+by_2+cz_2+d)$

(*i*) Here $A=(1,2,0)$, $B=(1,2,-3)$ and the plane is $2x-3y+4z+5=0$

$\therefore$ $(C\,;\,A,B)=-(2,-6+0+5):(2-6-12+5)=1:11$

$\therefore$ A, B lie in different half spaces

(*ii*) Proceed as above.

19. Let $Oxyz$ and $OXYZ$ be two coordinate frames having the same origin O.

Let the equations of the plane w.r.t the frames of reference be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \ldots (1) \qquad \frac{X}{a_1}+\frac{Y}{b_1}+\frac{Z}{c_1}=1 \qquad \ldots (2)$$

Distance of the origin from the plane is fixed.

$$\therefore\ \frac{1}{\sqrt{\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}\right)}}=\frac{1}{\sqrt{\left(\dfrac{1}{a_1^2}+\dfrac{1}{b_1^2}+\dfrac{1}{c_1^2}\right)}} \Rightarrow a^{-2}+b^{-2}+c^{-2}=a_1^{-2}+b_1^{-2}+c_1^{-2}$$

Note. Observe translation of axes. Vide Art 11.2 of Chapter 11.

20. Let the given plane be $\dfrac{x}{p}+\dfrac{y}{q}+\dfrac{z}{r}=1$

Since the plane meets the axes in A, B, C we have $A=(p,0,0), B=(0,q,0), C=(0,0,r)$

$\therefore$ Centroid of $\Delta\ ABC=(a,b,c)$

$\Rightarrow$ $(p/3, q/3, r/3)=(a,b,c) \Rightarrow p=3a, q=3b, r=3c$

$\therefore$ Equation to the plane is $\dfrac{x}{3a}+\dfrac{y}{3b}+\dfrac{z}{3c}=1 \ \Rightarrow\ \dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=3$

21. Let a plane be $ax+by+cz=1$ ... (1)

Intercepts of (1) on the axes are $1/a, 1/b, 1/c$ respectively.

Given $a+b+c=K$, a constant $(\neq 0)$ $\Rightarrow$ $a\left(\dfrac{1}{k}\right)+b\left(\dfrac{1}{k}\right)+c\left(\dfrac{1}{k}\right)=0$

Comparing the equation with (1), it is clear that the variable plane $ax+by+cz=1$ passes through the fixed point $(1/k, 1/k, 1/k)$.

22. Let a plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ .... (1)

Plane (1) is at a constant distance $3p$ from the origin

$$\Rightarrow \frac{1}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}=3p \quad \Rightarrow \frac{1}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}=\frac{1}{9p^2}$$

But (1) cuts the axes in $A=(a,0,0)$, $B=(0,b,0)$ and $C=(0,0,c)$

$\therefore$ Centroid of $\Delta\, ABC=(x_1, y_1, z_1)=(a/3, b/3, c/3)$

$$\Rightarrow a=3x_1, b=3y_1, c=3z_1 \Rightarrow \frac{1}{9x_1^2}+\frac{1}{9y_1^2}+\frac{1}{9z_1^2}=\frac{1}{9p^2}$$

$\therefore$ Locus of the centroid of $\Delta\, ABC$ is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2} \Rightarrow x^{-2}+y^{-2}+z^{-2}=p^{-2}$.

23. Proceed as above using article 8.36.

24. Distance $=\frac{|3-5/2|}{\sqrt{(4+4+1)}}=\frac{1}{6}$.

25. Given point are $A=(1,3,2)$, $B=(-5,0,2)$, $C=(1,1,-4)$

Let $P=(2,3,4)$. $AB\times AC$ is a vector perpendicular to $\overleftrightarrow{ABC}$

$$\text{Now } AB\times AC=\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -6 & -3 & 0 \\ 0 & -2 & -6 \end{vmatrix}=\bar{i}\,(18-0)-\bar{j}\,(36-0)+\bar{k}\,(12-0)$$

$AB\times AC=18\,\bar{i}-36\bar{j}+12\bar{k}$. Let n be the unit vector along $AB\times AC$

$$\therefore\ n=\frac{(18,-36,12)}{\sqrt{36\,(9+36+4)}}=\frac{(18,-36,12)}{42}. \qquad \text{Also } PB=(-7,-3,-2)$$

$\therefore$ Perpendicular distance of p from $\overleftrightarrow{ABC}=PM$

$$=|n\,.\,PB|=\frac{|(18,-36,12)\,.\,(-7,-3,-2)|}{42}=\frac{|-126+108-24|}{42}=\frac{42}{42}=1$$

26. Let a required plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ... (1)

Intercepts of the plane (1) are a, b, c. Given that $a+b+c=0$ ... (2)

Plane (1) passes through $(6,-4,3),(0,4,-3)$

$\Rightarrow \dfrac{6}{a}-\dfrac{4}{b}+\dfrac{3}{c}=1$ ... (3) $0-\dfrac{4}{b}-\dfrac{3}{c}=1$ ... (4)

From (3), (4) : $a=3$. from (2) : $b+c=-3$ ... (5)

From (4), (5) : $b=-2$, $c=-1$; $b=6$, $c=-9$. Hence etc.

27. $P=(x_1, y_1, z_1)$

$$\left(\frac{x_1+y_1+z_1}{\sqrt{3}}\right)^2+\left(\frac{x_1+y_1-2z_1}{\sqrt{6}}\right)^2+\left(\frac{x_1-y_1}{\sqrt{2}}\right)^2=5$$

$\Rightarrow x_1^2+y_1^2+z_1^2=5$ $\Rightarrow$ The locus of P is $x^2+y^2+z^2=5$.

EXERCISE 3 (*b*)

1. (*i*) Given plane is $3x-y+z=10$.

Any plane parallel to this will be the form $3x-y+z=k$.

Since it passes through $(1,-2,-3)$, we have $3+2-3=k$. $\Rightarrow k=2$.

$\therefore$ The required plane is $3x-y+z-2=0$

2. Given planes are $2x+y+3z-2=0$ and $x-y+z+4=0$

Any plane passing through the point of intersection of the planes is given by $(2x+y+3z-2)+k(x-y+z+4)=0$

$\Rightarrow x(2+k)+y(1-k)+z(3+k)+(-2+4k)=0$ ...(1)

Distance of (1) from origin is $\left|\dfrac{0+0+0+(-2+4k)}{\sqrt{(2+k)^2+(1-k)^2+(3+k)^2}}\right|=2$ units by data

$(-2+4k)^2=4\left[(2+k)^2+(1-k)^2+(3+k)^2\right]$

$\Rightarrow 4+16k^2-16k=4\left[4+k^2+4k+1+k^2-2k+9+k^2+6k\right]$

$\Rightarrow 4k^2-48k-52=0 \Rightarrow k^2-12k-13=0 \Rightarrow (k-13)(k+1)=0 \Rightarrow k=-1$

Required planes are $15x-12y+16z+50=0$ and $x+2y+2z-6=0$

3. (*i*), (*ii*) Proceed as in W.Ex. 4.

(*iii*) Let the equation of the plane through given planes be

$(x+y+z-1)+\lambda(2x+3y-z+4)=0$, λ a fixed number.

$\Rightarrow (1+2\lambda)x+(1-3\lambda)y+(1-\lambda)z-1+4\lambda=0$ ... (1)

d.rs. of x-axis are 1, 0, 0. Since (1) is parallel to x-axis, the normal to (1) is perpendicular to x-axis.

$\therefore 1(1+2\lambda)+(1+3\lambda).0+(1-\lambda).0=0 \Rightarrow \lambda=-1/2$

$\therefore$ Equation to the required plane is $(x+y+z-1)-1/2(2x+3y-z+4)=0$, etc.

4. Given planes are $ax+by+cz+d=0$ ... (1) $a_1x+b_1y+c_1z+d_1=0$ ... (2)

Any plane passing through the line of intersection of the two planes is

$(ax+by+cz+d)+k\,(a_1x+b_1y+c_1z+d_1)=0$

$$x\,(a+a_1k)+y\,(b+b_1k)+z\,(c+c_1k)+(d+d_1k)=0 \qquad \text{... (3)}$$

If this plane is $\perp$ to xy plane then we have

$$0\,(a+a_1k)+0\,(b+b_1k)+1\,(c+c_1k)=0 \Rightarrow k=\frac{-c}{c_1}$$

$\therefore$ Required plane is $(ac_1+a_1c)\,x+(bc_1+b_1c)\,y+(dc_1-d_1c)=0$

5. Given planes are $x+y+z=1$ ... (1) $2x+3y+4z=5$ ... (2) $x-y+z=0$ ... (3)

Any plane passing through the line of intersection of the two planes is

$(x+y+z-1)+k\,(2x+3y+4z-5)=0$

$\Rightarrow x\,(1+2k)+y\,(1+3k)+z\,(1+4k)+(-1-5k)=0$ ... (4)

If this is $\perp_r$ to (3) then $(1+2k)\,(1)=(1+3k)\,(-1)+(1+4k)\,(1)=0$

$3k+1=0 \Rightarrow k=-1/3$

Substituting in (4), we get $x-z+4=0$

8. Let a plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ... (1)

Let the plane (1) be at a distance $p\,(\neq 0)$ from the origin

$$\therefore \frac{1}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}=p \Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2} \qquad \text{... (2)}$$

Since (1) meets the axes in A, B, C we have $A=(a,0,0)$, $B=(0,b,0)$, $C=(0,0,c)$

Equations to the planes through A, B, C and parallel to the coordinate planes are $x=a, y=b, z=c$.

Let (α, β, γ) be the point of intersection of the planes

$$\therefore \alpha=a, \beta=b, \gamma=c \qquad \therefore \text{ From (2), } \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{1}{p^2}$$

$\therefore$ Locus of (α, β, γ) is $x^{-2}+y^{-2}+z^{-2}=p^{-2}$.

9. (*i*) Given planes are $-x+2y-2z+19=0$ ... (1) and $4x-3y+12z+3=0$ ... (2)

[making the sign before the constant term as positive]

$\therefore$ Equations to the bisecting planes between (1) and (2) are

$$\frac{-x+2y-2z+19}{\sqrt{(1+4+4)}}=+\frac{4x-3y+12z+3}{\sqrt{(16+9+144)}}, \quad \frac{-x+2y-2z+19}{\sqrt{(1+4+4)}}=-\frac{4x-3y+12z+3}{\sqrt{(16+9+144)}}$$

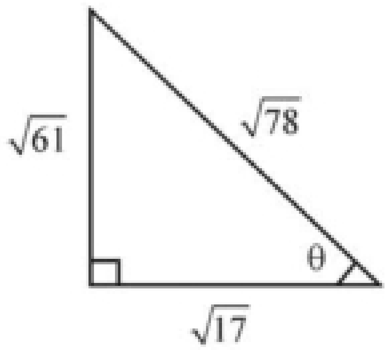
i.e., $-13x-26y-26z+247=12x-9y+36z+9$

$-13x-26y-26z+247=-12x+9y-36z-9$

i.e., $25x+17y+62z=238$ ... (3) $x+35y-10z=256$ ... (4)

Let θ be the acute angle between the planes (1) and (4).

$$\therefore\ \cos\theta=\left|\frac{(-1)\,1+(-2)\,35+(-2)\,(-10)}{\sqrt{9}\,.\,\sqrt{1+1225+100}}\right|=\frac{51}{3\,.\,\sqrt{1326}}=\frac{\sqrt{17}}{\sqrt{78}}$$



$$\therefore\ \tan\theta=\frac{\sqrt{61}}{\sqrt{17}}>1\ \Rightarrow\ \theta>\frac{\pi}{4}.$$

$\therefore$ Equation (3) represents the plane bisecting the acute angle between the planes (1) and (2) and equation (4) represents the plane bisecting the obtuse angle between the planes (1) and (2).

Also equation (3) represents the plane bisecting the angle contaiing the origin between the planes (1) and (2) and equation (4) represents the plane bisecting the angle not containing origin between the planes (1) and (2).

(*ii*) Given planes are $3x-6y+2z+5=0$ ... (1) and $-4x+12y-3z+3=0$ ... (2)

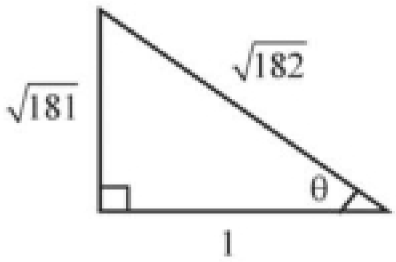
$\therefore$ Equations to the bisecting planes between the planes (1) and (2) are

$$\frac{3x-6y+2z+5}{\sqrt{9+36+4}}=+\frac{-4x+12y-3z+3}{\sqrt{16+144+9}},\ \frac{3x-6y+2z+5}{\sqrt{9+36+4}}=-\frac{-4x+12y-3z+3}{\sqrt{16+144+9}}$$

i.e., $67x-162y+47z+44=0$ ... (3) $11x+6y+5z+86=0$ ... (4)

Let θ be the acute angle between the planes (1) and (4).

$$\therefore\ \cos\theta=\frac{3\,(11)+(-6)\,6+2\,(5)}{\sqrt{9+36+4}\,.\,\sqrt{121+36+25}}=\frac{1}{\sqrt{182}}.$$



$$\therefore\ \tan\theta=\sqrt{181}>1\ \Rightarrow\ \theta=\frac{\pi}{4}$$

$\therefore$ Equation (3) represents the plane bisecting the acute angle between the planes (1) and (2) and equation (4) represents the plane bisecting the obtuse angle between the planes (1) and (2).

Also equation (3) represents the plane bisecting the angle containing the origin between the planes (1) and (2) and equation (4) represents the plane bisecting the angle not containing the origin between the planes (1) and (2).

10. Proceed as in Ex. 9.

11. A plane through the intersection of the planes

$x+y+z-1=0$, $2x+3y+z+4=0$ is $2x+3y-z+4+\lambda\,(x+y+z-1)=0$ ... (1)

λ being an unknown constant.

d.rs. of normal to (1) are $2+\lambda, 3+\lambda, -1+\lambda$.

But d.rs. of x-axis are 1, 0, 0

If the plane (1) is parallel to x-axis, then the normal to the plane (1) is perpendicular to x-axis. $\therefore\ 1(2+\lambda)+0+0=0\Rightarrow\lambda=-2$.

$\therefore$ Equation to the plane parallel to x-axis and passing through the intersection of the given planes is $y-3z+6=0$.

Similarly we can find the other two planes.

EXERCISE 3 (*c*)

1. (*i*) Given equation is $2x^2-3y^2+4z^2+xy+6zx-yz=0$

Let the equation be $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ ... (1)

comparing the given equation to (*i*), $a=2, b=-3, c=4$,

$2f=-1,\ 2g=6,\ 2h=1,\quad f=-1/2,\ g=3,\ h=1/2$

$\therefore\ abc+2fgh-af^2-bg^2-ch^2=0$

$\Rightarrow\ 2(-3)(4)+2(-1/2)(3)(1/2)-2(-1/2)^2+3(3)^2-4(1/2)^2=0$

$\Rightarrow\ 24+(-3/2)\dfrac{-2}{4}+27-1=0\quad \Rightarrow\ -48-3-1+54-2=0\quad \Rightarrow\ 0=0$

$f^2=\dfrac{1}{4}, bc=-12\ \Rightarrow\ f^2>bc;\qquad g^2=9, ac=8\Rightarrow g^2>ac$

$h^2=\dfrac{1}{4},\ ab=-6\ \Rightarrow\ h^2>ab$

$\therefore$ Given equation represents pair of planes, let 'θ' be the acute angle between the planes

$$\cos\theta=\left|\frac{a+b+c}{\sqrt{(a+b+c)^2+4(f^2+g^2+h^2-ab-bc-ca)}}\right|$$

$$\cos\theta=\left|\frac{2-3+4}{\sqrt{(2-3+4)^2+4\left(\frac{1}{4}+9+\frac{1}{4}+6-8+12\right)}}\right|=\left|\frac{3}{\sqrt{9+4\left(\frac{39}{2}\right)}}\right|=\left|\frac{3}{\sqrt{9+78}}\right|=\left(\frac{3}{\sqrt{87}}\right)$$

$$\cos\theta=\left(\frac{3}{\sqrt{87}}\right)\Rightarrow\theta=\cos^{-1}\left(\frac{3}{\sqrt{87}}\right)=\cos^{-1}\sqrt{\frac{3}{29}}$$

(*ii*) Given plane equation is $2x^2-2y^2+4z^2+2yz+6zx+3xy=0$

Let the given equation be $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ ... (1)

Comparing the given equation to (1) $a=2\ b=-2\ c=4$,

$2f=2,\ 2h=3,\ 2g=6,\quad f=1,\ h=3/2,\ g=3$

$\therefore\ abc+2fgh-af^2-bg^2-ch^2=0$

$\Rightarrow\ 2(2)(4)+2(1)(3)(3/2)-2(1)+2(9)+\left(\dfrac{9}{4}\right)=0$

$\Rightarrow -16+2\times\dfrac{9}{2}-2+18-9=0$

$f^2 = 1, bc = 8 \Rightarrow f^2 > bc$; $g^2 = 9, ac = 8 \Rightarrow g^2 > ac$

$h^2 = 9/4, ab = -4 \Rightarrow h^2 > ab$

$\therefore$ Given equation represents pair of planes.

Let 'θ' be the acute angle between the planes.

$$\cos\theta = \left| \frac{a+b+c}{\sqrt{(a+b+c)^2 + 4(f^2+g^2+h^2-ab-bc-ca)}} \right|$$

$$= \left| \frac{2-2+4}{\sqrt{(2-2+4)^2 + 4\left(1+9+\frac{9}{4}+4+8-8\right)}} \right|$$

$$\cos\theta = \left(\frac{4}{\sqrt{81}}\right) = \frac{4}{9} \quad \Rightarrow \theta = \mathrm{Cos}^{-1}(4/9)$$

2. Let $H \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

Given that $H = 0$ represents a pair of intersecting planes.

Let $H \equiv (l_1x + m_1y + n_1z + d_1)(l_2x + m_2y + n_2z + d_2)$

Where $(l_1, m_1, n_1) \neq (0,0,0)$ and $(l_2, m_2, n_2) \neq (0,0,0)$

$\therefore\ l_1l_2 = a, m_1m_2 = b, n_1n_2 = c, d_1d_2 = 0$,

$l_1m_2 + l_2m_1 = 2h, m_1n_2 + m_2n_1 = 2f, n_1l_2 + n_2l_1 = 2g_1$

$l_1d_2 + l_2d_1 = 0,\ m_1d_2 + m_2d_1 = 0,\ n_1d_2 + n_2d_1 = 0$

Now $d_1d_2 = 0 \Rightarrow d_1 = 0$ or $d_2 = 0$

$d_2 = 0 \Rightarrow l_2d_1 = 0,\ m_2d_1 = 0,\ n_2d_1 = 0 \Rightarrow d_1 = 0$

($\because$ atleast one of l_2, m_2, n_2 is not equal to zero)

Similarly $d_1 = 0 \Rightarrow d_2 = 0\ \therefore\ d_1 = d_2 = 0$

Consider $\begin{bmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{bmatrix} \begin{bmatrix} l_2 & m_2 & n_2 \\ l_1 & m_1 & n_1 \\ 0 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 2l_1l_2 & l_1m_2 + l_2m_1 & l_1n_2 + l_2n_1 \\ l_2m_1 + l_1m_2 & 2m_1m_2 & m_1n_2 + m_2n_1 \\ n_1l_2 + n_2l_1 & n_1m_2 + n_2m_1 & 2n_1n_2 \end{bmatrix} = \begin{bmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{bmatrix}$$

$$\therefore \begin{vmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} \begin{vmatrix} l_2 & m_2 & n_2 \\ l_1 & m_1 & n_1 \\ 0 & 0 & 0 \end{vmatrix} = 8 \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$\Rightarrow\ 0 \times 0 = 8(abc - af^2 - ch^2 + fgh + fgh - bg^2)$

$\Rightarrow\ abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

Given $a+b+c \neq 0$. Let $\theta\,(<\pi/2)$ be an angle between the planes

$$\therefore\ \tan\theta = \left|\frac{\sqrt{\left[\left(\sum m_1n_2 - m_2n_1\right)^2\right]}}{l_1l_2+m_1m_2+n_1n_2}\right| = \left|\frac{\sqrt{\left[\sum\left\{(l_1m_2+l_2m_1)^2 - 4l_1l_2m_1m_2\right\}\right]}}{a+b+c}\right|$$

$$= \left|\frac{\sqrt{\left[4h^2-4ab+4f^2-4bc+4g^2-4ac\right]}}{a+b+c}\right| = \left|\frac{2\sqrt{(f^2+g^2+h^2-bc-ca-ab)}}{a+b+c}\right|$$

$$\therefore\ \theta = \tan^{-1}\left|\frac{2\sqrt{(f^2+g^2+h^2-bc-ca-ab)}}{a+b+c}\right|$$

3. Let $l_1x+m_1y+n_1z=0$, $l_2x+m_2y+n_2z=0$ be the planes given by

$\theta\,(x,y,z) \equiv ax^2+by^2cz^2+2fyz+2gzx+2hxy$

$\therefore l_1l_2 = a,\ m_1m_2 = b,\ n_1n_2 = c,\ l_1m_2+l_2m_1 = 2h,\ \ m_1n_2+m_2n_1 = 2f,\ \ n_1l_2+n_2l_1 = 2g$

Product of the perpendiculars from (α,β,γ) to the planes

$$= \left|\frac{l_1\alpha+m_1\beta+n_1\gamma}{\sqrt{(l_1^2+m_1^2+n_1^2)}}\right|\left|\frac{l_2\alpha+m_2\beta+n_2\gamma}{\sqrt{(l_2^2+m_2^2+n_2^2)}}\right|$$

$$= \left|\frac{a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta}{(a^2+b^2+c^2+l_1^2m_2^2+l_2^2m_1^2+m_1^2n_2^2+n_1^2l_2^2+n_2^2l_1^2)}\right|$$

$$= \frac{|\,\phi\,(\alpha,\beta,\gamma)\,|}{\sqrt{(a^2+b^2+c^2+4h^2-4ab+4f^2-4bc+4g^2-4ac)}}$$

4. Given $a^2+b^2+c^2-2ab-2bc-2ca>0$ ... (1)

$$\frac{a}{y-z}+\frac{b}{z-x}+\frac{c}{x-y}=0 \Rightarrow ax^2+by^2+cz^2+(c-a-b)\,xy+(a-b-c)\,yz+(b-c-a)\,zx=0$$

$$\text{Since } (i) \begin{vmatrix} a & \frac{c-a-b}{2} & \frac{b-c-a}{2} \\ \frac{c-a-b}{2} & b & \frac{a-b-c}{2} \\ \frac{b-c-a}{2} & \frac{a-b-c}{2} & c \end{vmatrix}$$

$$= -8\begin{vmatrix} -2a & a+b-c & a+c-b \\ a+b-c & -2b & b+c-a \\ a+c-b & b+c-a & -2c \end{vmatrix}$$

$$= -8\begin{vmatrix} 0 & 0 & 0 \\ a+b-c & -2b & b+c-a \\ a+c-b & b+c-a & -2c \end{vmatrix} = 0, \ (D=0)$$

(ii) $\left(\frac{a-b-c}{2}\right)^2 - bc = \frac{a^2+b^2+c^2-2bc-2ca}{4} > 0$ using (1) $(f^2 - bc)$

(iii) $\left(\frac{b-c-a}{2}\right)^2 - ac > 0$ $(g^2 - ac)$ (iv) $\left(\frac{c-a-b}{2}\right)^2 - ab > 0$ $(h^2 - ab)$

given equation represents a pair of intersecting planes.

5. Given equation is $x^2 - 2y^2 - z^2 - xy + 3yz - 6x + 3y + 9 = 0$

$\Rightarrow x^2 + x(-6-y) + (-2y^2 - z^2 + 3yz + 3y + 9) = 0$

$\Rightarrow x = \frac{-(-6-y) \pm \sqrt{(-6-y)^2 - 4(-2y^2 - z^2 - 3yz + 3y + 9)}}{2}$

$\Rightarrow 2x = (6+y) \pm \sqrt{y^2 + 36 + 12y + 8y^2 + 4z^2 - 12yz - 36 - 12y}$

$\Rightarrow 2x = (6+y) \pm \sqrt{9y^2 + 4z^2 - 12yz} \Rightarrow 2x = (6+y) \pm \sqrt{(3y-2z)^2}$

$\Rightarrow 2x = (6+y) \pm (3y - 2z)$

$\Rightarrow 2x - 6 - y - 3y + 2z = 0, \ 2x - 6 - y + 3y - 2z = 0$

$\Rightarrow 2x - 4y + 2z - 6 = 0, 2x + 2y - 2z - 6 = 0$

$\Rightarrow x - 2y + z - 3 = 0, x + y - z - 3 = 0$

$\cos\theta = \left(\frac{1(1) - 2(1) + 1(-1)}{\sqrt{1+4+1}\sqrt{1+1+1}}\right) \Rightarrow \cos\theta = \left(\frac{-2}{\sqrt{6}\sqrt{3}}\right) = \left(\frac{-2}{3\sqrt{2}}\right)$

$\cos\theta = (-\sqrt{2}/3)$

6. Given equation is $x^2 + 4y^2 + 4z^2 + 4xy + 8yz + 4zx - 9x - 18y - 18z + 18 = 0$

$x^2 + 4y^2 + 4z^2 + 4xy + 8yz + 4zx = (x + 2y + 2z)^2$

$x^2 + 4y^2 + 4z^2 + 4xy + 8yz + 4zx - 9x - 18y - 18z + 18 = (x + 2y + 2z + k)(x + 2y + 2z + l)$

where $k + l = -9$ ... (1) $2k + 2l = -18$ $lk = 18$

$(k-l)^2 = (k+l)^2 - 4lk = 81 - 4(18) = 81 - 72$ $(k-l)^2 = 9$

$k - l = \pm 3$ ... (2)

from (1), (2) $k = -3$ $l = -6$

$\therefore$ Given equation represents the planes $x + 2y + 2z - 3 = 0, \ x + 2y + 2z - 6 = 0$

$\therefore$ Distance between the parallel planes $= \frac{|-6+3|}{\sqrt{1+4+4}} = \frac{3}{3} = 1$

UNIT - II

Right Line

Exercise 4 (*a*)

1. (*a*) Equations of the line through (α, β, γ) and parallel to $\overline{X'X}$ are

$$\frac{x-\alpha}{1}=\frac{y-\beta}{0}=\frac{z-\gamma}{0} \Rightarrow y-\beta=0, z-\gamma=0.$$

(*b*) Equations to the line are $m(x-a)=l(z-b), y=c \Rightarrow \frac{x-a}{l}=\frac{y-c}{0}=\frac{z-b}{m}$

since d.rs. of the y-axis are 0, 1, 0 and since $l.0+0.1+m.0=0$, given line is perpendicular to the y-axis.

2. If L is equally inclined to the axes, its d.cs. can be taken as $\cos\alpha, \cos\alpha, \cos\alpha$ *i.e.,* its d.rs. can be taken as 1, 1, 1.

$\therefore$ Required line is $\frac{x-3}{1}=\frac{y-1}{1}=\frac{z-2}{1} \Rightarrow x-3=y-1=z-2$

3. Given points are $P=(-2,1,3), Q=(1,1,4)$

Drs. of the line $\overline{PQ}$ are $1+2, 1-1, 4-3=3,0,1$ or $-3,0,-1$.

Using the formula $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$.

We get the equation of the form $\frac{x+2}{-3}=\frac{z-3}{-1}, y-1=0$.

4. Equations to the line through $(1,2,3), (4,0,4)$ are

$$\frac{x-1}{4-1}=\frac{y-2}{0-2}=\frac{z-3}{4-3} \quad i.e., \quad \frac{x-1}{3}=\frac{y-2}{-2}=\frac{z-3}{1} \qquad \text{... (1)}$$

Since $(-2,4,2)$ and $(7,-2,5)$ satisfy (1), the four given points are collinear.

5. Equations of the lines are given by $\frac{x-1}{1+1}=\frac{y-1}{1-5}=\frac{z-1}{1-5}$; $\frac{x+2}{-4}=\frac{y-4}{2}=\frac{z-1}{-4}$

$$\Rightarrow \frac{x-1}{2}=\frac{y-1}{-4}=\frac{z-1}{-4}; \quad \frac{x+2}{-4}=\frac{y-4}{2}=\frac{z-1}{-4}$$

$$\Rightarrow \frac{x-1}{1}=\frac{y-1}{-2}=\frac{z-1}{-2}; \quad \frac{x+2}{-2}=\frac{y-4}{1}=\frac{z-1}{-2}$$

6. Here $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC} \Rightarrow$ ABCD is a parallogram.

7. (*ii*) The equation of the line joining $(2,-3,1)$, $(3,-4,-5)$ is given by

$$\frac{x-2}{3-2}=\frac{y+3}{-4+3}=\frac{z-1}{-5-1} \Rightarrow \frac{x-2}{1}=\frac{y+3}{-1}=\frac{z-1}{-6}=t \text{ (say)}$$

Any point on it is given by $(t+2, -t-3, -6t+1)$

If this intersects the plane $2x+y+z=7$

$2(t+2)+(-t-3)+(-6t+1)=7$, $2t-t-6t+4-3+1-7=0 \Rightarrow -5t-5=0 \Rightarrow t=-1$.

The point of intersection is $(1,-2,7)$.

8. Given lines are $\dfrac{x-3}{3}=\dfrac{2-y}{4}=\dfrac{z+1}{1}$ $(=t \text{ say})$ ... (1)

and $2x+4y+3z+3=0=x+2y+3z$ ... (2)

A point on (1) is $(3t+3, 2-4t, t-1)$

If this point lies on (2), then $6t+6+8-16t+3t-3+3=0 \Rightarrow t=2$.

$\therefore$ Lines (1) and (2) intersect in the point $(9,-6,1)$.

Note that $(9,-6,1)$ satisfies $x+2y+3z=0$.

9. Given lines are $\dfrac{x-1}{-3}=\dfrac{y-2}{2}=\dfrac{z-3}{2}$ $(=s \text{ say})$ ... (1)

and $\dfrac{x-1}{3}=\dfrac{y-5}{1}=\dfrac{z}{-5}$ $(=t \text{ say})$ ... (2)

A point P on line (1) is $(-3s+1, 2s+2, 2s+3)$ and

a point Q on line (2) is $(3t+1, t+5, -5t)$

$P=Q \Rightarrow 3t+1=-3s+1, t+5=2s+2, -5t=2s+3 \Rightarrow s=1,\ t=-1$.

$\therefore$ Lines (1) and (2) intersect in $(-2,4,5)$.

10. Since d.rs. of lines are known, proceed as in W.Ex.3.

11. (*i*) Given lines are perpendicular $\Rightarrow (-3)(3k)+2k(1)+2(7)=0 \Rightarrow k=2$

12. Given lines are $\dfrac{x-1}{1}=\dfrac{y+4}{2}=\dfrac{x-5}{-2}$ ... (1) and $\dfrac{x-1}{4}=\dfrac{y+4}{3}=\dfrac{z-5}{12}$ ... (2)

A point on line (1) can be taken as $(t+1, 2t-4, -2t+5)$ and a point on line (2) can be taken as $(4s+1, 3s-4, 12s+5)$.

Let $t+1=4s+1, 2t-4=3s-4, -2t+5=12s+5$ *i.e.,* $t=4s,\ t=\dfrac{3s}{2},\ t=-6s \Rightarrow t=s=0$

$\Rightarrow$ lines (1) and (2) intersect in $(1,-4,5)$.

(It can be seen to be obvious from the equations of the lines)

d.rs. of (1) and (2) are respectively $1,2,-2\,;\,4,3,12$.

$\therefore$ d.cs. of (1) and (2) can be taken as $\dfrac{1}{3},\dfrac{2}{3},\dfrac{-2}{3};\dfrac{4}{13},\dfrac{3}{13},\dfrac{12}{13}$

$\therefore$ d.rs. of the bisectors of the angles between (1) and (2) are

$$\frac{1}{3}+\frac{4}{13},\frac{2}{3}+\frac{3}{13},\frac{-2}{3}+\frac{12}{13};\ \frac{1}{3}-\frac{4}{13},\frac{2}{3}-\frac{3}{13},\frac{-2}{3}-\frac{12}{13}$$

i.e., $25,35,10\,;\,1,17,-62$ *i.e.,* $5,7,2\,;\,-1,-17,62$.

$\therefore$ Equations to the bisectors of the angle between lines (1) and (2) are

$$\frac{x-1}{5}=\frac{y+4}{7}=\frac{z-2}{5};\quad \frac{x-1}{-1}=\frac{y+4}{-17}=\frac{z-5}{62}.$$

13. Equations to the line through $P=(3,-4,5)$ and having d.cs. proportional to $2,1,-2$ are

$$\frac{x-3}{2}=\frac{y+4}{1}=\frac{z-5}{-2}\quad (=t \text{ say}) \qquad \ldots (1)$$

Given plane is $2x+5y-6z=16$ ... (2)

A point on (1) is $Q=(2t+3,t-4,-2t+5)$.

If Q lies on (2), $4t+6+5t-20+12t-30=16 \Rightarrow t=20/7$.

$$\therefore\ Q=\left(\frac{40}{7}+3,\frac{20}{7}-4,-\frac{40}{7}+5\right).$$

$$\therefore\ PQ^2=\left(\frac{40}{7}\right)^2+\left(\frac{20}{7}\right)^2+\left(\frac{-40}{7}\right)^2=\frac{3600}{49}\Rightarrow PQ=\frac{60}{7}.$$

14. Let a plane through the line $2x+3y-5z-4=0=3x-4y+5z-6$

be $2x+3y-5z-4+\lambda(3x-4y+5z-6)=0$

i.e., $(2+3\lambda)x+(3-4\lambda)y+(-5+5\lambda)z-4-6\lambda=0$

If this plane is parallel to x-axis (d.cs. 1, 0, 0), $(2+3\lambda).1+0+0=0 \Rightarrow \lambda=-2/3$.

$\therefore$ Equation to the required plane is

$$0.x+\left(3+\frac{8}{3}\right)y+\left(-5-\frac{10}{3}\right)z-4+\frac{12}{3}=0 \Rightarrow 17y-25z=0.$$

Similarly other two planes can be found out.

15. Let d.rs. of the line $x-2y+z-2=0=4x+3y-z+1$ be l,m,n.

$$\therefore\ \left.\begin{array}{r} l-2m+n=0\\ 4l+3m-n=0\end{array}\right\}\ \frac{l}{-1}=\frac{m}{5}=\frac{n}{11}.$$

$\therefore$ Equation to the plane through (1, 1, 1) and perpendicular to the given line is $-1(x-1)+5(y-1)+11(z-1)=0 \Rightarrow x-5y-11z+15=0$.

16. Given line is $x+y+z+1=0=4x+y-2z+2$. Let l,m,n are the drs. of the line.

Then we have $l+m+n=0$ and $4l+m-2n=0$

Solving by the method of Cross multiplication $\frac{l}{-3}=\frac{m}{6}=\frac{n}{-3}$. drs are $(-1,2,-1)$

To get any point on the line put $z=0$

$x+y+1=0$ and $4x+y+2=0$

Solving we get $x=-\frac{1}{3}$ and $y=-\frac{2}{3}$. Then any point on the line is $\left(-\frac{1}{3},-\frac{2}{3},0\right)$

Equation of the line is $\frac{x+\frac{1}{3}}{-1}=\frac{y+\frac{2}{3}}{2}=\frac{z-0}{-1}$

17. Given lines are $\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}, \frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}$

The drs. of the lines are l_1, m_1, n_1 and l_2, m_2, n_2.

drs.of any line $\perp$ lar to these lines is given by $n_1m_2-n_2m_1, n_1l_2-n_2l_1, l_1m_2-l_2m_1$.

Given (x_1, y_1, z_1) is a point on the required line.

Required line is $\frac{x-x_1}{m_1n_2-m_2n_1}=\frac{y-y_1}{n_1l_2-l_1n_2}=\frac{z-z_1}{l_1m_2-l_2m_1}$

18. Let drs. of normal to the given plane be l, m, n

Since the required plane is parallel to the lines $\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}$ and $\frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}$.

We have $l_1l+m_1m+n_1n=0$ and $l_2l+m_2m+n_2n=0$

Now proceed further.

19. Equations to the line through (a,b,c) and (a',b',c') are $\frac{x-a}{a-a'}=\frac{y-b}{b-b'}=\frac{z-c}{c-c'}$ ... (1)

Since P is the distance of the origin from the point (a,b,c),

$p^2=a^2+b^2+c^2$ similarly $p'^2=a'^2+b'^2+c'^2$

$\therefore \quad (PP')^2=(a^2+b^2+c^2)(a'^2+b'^2+c'^2)$

But given that $PP'=aa'+bb'+cc'$

$\therefore \quad (a^2+b^2+c^2)(a'^2+b'^2+c'^2)-(aa'+bb'+cc')^2=0$

By using Lagrange's identity, we have $(ab'-a'b)^2+(bc'-b'c)^2+(ca'-c'a)^2=0$

$\Rightarrow ab'-a'b=0, bc'-b'c=0, ca'-c'a=0 \Rightarrow \frac{a'}{a}=\frac{b'}{b}=\frac{c'}{c}$ (= k say).

Now $\frac{0-a}{a-a'}=\frac{-a}{a-ak}=\frac{-a}{-a(k-1)}=\frac{1}{k-1}$, $\frac{0-b}{b-b'}=\frac{-b}{b-bk}=\frac{1}{k-1}$ and $\frac{0-c}{c-c'}=\frac{-c}{c-k}=\frac{1}{k-1}$

$\Rightarrow \frac{0-a}{a-a'}=\frac{0-b}{b-b'}=\frac{0-c}{c-c'}$

$\therefore$ The line (1) passes through the origin.

20. (*i*) Given point is $P=(1,3,4)$. Given plane π is $2x-y+z+3=0$

Let $Q=(x_1, y_1, z_1)$ be the image of P is π. $\therefore \overline{PQ}\perp\pi$. $\therefore$ d.rs. of $\overline{PQ}$ are $(2,-1,1)$

Equation of PQ is $\frac{x-1}{2}=\frac{y-3}{-1}=\frac{3-4}{1}=t$ (say)

Any point on $\overline{PQ}$ is $R=(2t+1,-t+3,t+4)$

If R is the mid point of $\overline{PQ}$ is then $R\in\pi$.

Then $2(2t+1)-(-t+3)+t+4+3=0$, $6t+6=0\Rightarrow t=-1$,

$\Rightarrow R = (-1, 4, 3)$. R is the mid point of $\overline{PQ} \Rightarrow \dfrac{1+x_1}{2} = -1$

$\Rightarrow x_1 = -3$; $\dfrac{y_1+3}{2} = 4 \Rightarrow y_1 = 5$

$\dfrac{4+z_1}{2} = 3 \Rightarrow z_1 = 2$. Image $= (-3, 5, 2)$

(*ii*) Let $P = (1, 6, 3)$ Let L be the line $\dfrac{x}{1} = \dfrac{y-1}{2} = \dfrac{z-2}{3}$ ($= t$ say)

Let $R = (t, 2t+1, 3t+2)$ be a point on L.

$PR \perp L \Rightarrow 1\,(t-1) + 2\,(2t+1-6) + 3\,(3t+2-3) = 0 \Rightarrow 14t = 14 \Rightarrow t = 1$

$\therefore$ $R = (1, 3, 5)$.

Let $Q = (x_1, y_1, z_1)$ be the image of P in L.

$\therefore$ R is the mid point $PQ \Rightarrow \dfrac{x_1+1}{2} = 1, \dfrac{y_1+6}{2} = 3, \dfrac{z_1+3}{2} = 5 \Rightarrow Q = (1, 0, 7)$.

(*iii*) Proceed as in W. Ex.5.

(*iv*) Let $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4} = t$ ... (1) be the given line

Given plane is $x + y + z = 1$ ... (2)

Any point on (1) is $(2t+1, 3t+2, 4t+3)$

If this lies on (1) then $(2t+1) + (3t+2) + (4t+3) = 1$; $9t + 5 = 0$; $t = 5/9$

The line intersects in the point $\left(\dfrac{-10}{9}+1, \dfrac{-15}{9}+2, \dfrac{-20}{9}+3\right) = \left(\dfrac{-1}{9}, \dfrac{3}{9}, \dfrac{7}{9}\right)$

$\therefore$ Image $P = \left(\dfrac{-1}{9}, \dfrac{-6}{9}, \dfrac{-11}{9}\right)$ in (2) is itself clearly (1, 2, 3) is a point in (1) equation of the line through (1, 2, 3) and $\perp r$ to the (2) is $\dfrac{x-1}{1} = \dfrac{y-2}{1} = \dfrac{z-3}{1} = r$ (say)

Any point on the line is $(r+1, r+2, r+3)$

If this lies on (2) $r+1+r+2+3 = 1 \Rightarrow 3r = 5 \Rightarrow r = 5/3$

$\therefore$ The foot of (1, 2, 3) in (2) is $\left(\dfrac{-5}{3}+1, \dfrac{-5}{3}+2, \dfrac{-5}{3}+3\right) = \left(\dfrac{-2}{3}, \dfrac{1}{3}, \dfrac{4}{3}\right)$

Let (x_1, y_1, z_1) be the image of (1, 2, 3) in (2) then

$\dfrac{1+x_1}{2} = \dfrac{-2}{3} \Rightarrow 1 + x_1 = \dfrac{-4}{3} \Rightarrow x_1 = -7/3$; $\dfrac{2+y_1}{2} = 1/3 \Rightarrow 2 + y_1 = \dfrac{2}{3} \Rightarrow y_1 = -4/3$

$\dfrac{z_1+3}{2} = \dfrac{4}{3} \Rightarrow 3 + z_1 = \dfrac{8}{3} \Rightarrow z_1 = -\dfrac{1}{3}$

The image (1, 2, 3) in (1) is $Q = \left(\dfrac{-7}{3}, \dfrac{-4}{3}, \dfrac{-1}{3}\right)$ equation of the image of (1) in (2) is the line joining P and Q.

21. Let l, m, n be d.rs. of the line $3x+2y-z-4=0=x-2y-2z-5$

$\therefore \left.\begin{matrix} 3l+2m-n=0 \\ l-2m-2n=0 \end{matrix}\right\} \dfrac{l}{-6}=\dfrac{m}{5}=\dfrac{n}{-8}$. Hence etc.

22. Proceed as W. Ex.3.

23. Given lines can be taken as $\dfrac{x-b}{a}=\dfrac{y-d}{c}=\dfrac{z}{1}$ and $\dfrac{x-b_1}{a_1}=\dfrac{y-d_1}{c_1}=\dfrac{z}{1}$. Hence etc.

24. The line $x=ay+b, y=cz+d$ passes throgh $(3,1,-3) \Rightarrow 3=a+b, 1=-3c+d$

Similarly we have $4=2a+b, 2=-4c+d$. Solving, $a=1, b=2, c=+1, d=-2$.

$\therefore$ Equations of the line through the points $(3,1,-3)$, $(4,2,-4)$ are $x=y+2, y=-z-2$.

Clearly this line passes through the point $(5,3,-5)$.

25. Let $P=(1,6,3)$. Let L be the line $\dfrac{x}{1}=\dfrac{y-1}{2}=\dfrac{z-2}{3}$ ($=t$ say)

Let $R=(t, 2t+1, 3t+2)$ be a point on L.

If R is the foot of the perpendicular from P to L, then

$$1(t-1)+2(2t+5)+3(3t-1)=0 \Rightarrow t=1 \quad \therefore \quad R=(1,3,5).$$

$\therefore$ Equations of the perpendicular line from P to L are

$\dfrac{x-1}{0}=\dfrac{y-6}{-3}=\dfrac{z-3}{2}$ *i.e.,* $x-1=0$, $\dfrac{y-6}{-3}=\dfrac{z-3}{2}$. Also $PR=\sqrt{0+9+4}=\sqrt{13}$.

26. Let $P=(x_1, y_1, z_1)$ and π be the plane $lx+my+nz=p$.

$$P \in \pi \Rightarrow lx_1+my_1+nz_1=p \qquad \text{... (1)}$$

Equation to $\overleftrightarrow{OP}\ \dfrac{x}{x_1}=\dfrac{y}{y_1}=\dfrac{z}{z_1}$. Let $Q=(x_2, y_2, z_2)$.

$$Q \in \overleftrightarrow{OP} \text{ such that } OP \,.\, OQ = P^2 \qquad \text{... (2)}$$

$\therefore \dfrac{x_2}{x_1}=\dfrac{y_2}{y_1}=\dfrac{z_2}{z_1}\left(=\dfrac{1}{t}\text{ say}\right)$. $\therefore$ From (1), $t\,(lx_2+my_2+nz_2)=p$ ... (3)

Also from (2), $\sqrt{\left[(x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2)\right]}=p^2$

$\Rightarrow \sqrt{\left[t^2(x_2^2+y_2^2+z_2^2)\right]}=p^2 \Rightarrow t^2(x_2^2+y_2^2+z_2^2)=p^2$

$\Rightarrow \dfrac{p}{lx_2+my_2+nz_2}(x_2^2+y_2^2+z_2^2)=p^2 \Rightarrow p\,(lx_2+my_2+nz_2)=(x_2^2+y_2^2+z_2^2)$

$\therefore$ Locus of Q is $p\,(lx+my+nz)=x^2+y^2+z^2$.

27. Proceed in W. Ex. 4 (with Note).

EXERCISE 4 (*b*)

1. Let L be the line $\dfrac{x+1}{-1}=\dfrac{y+2}{3}=\dfrac{z+5}{5}$ and π be the plane $x+2y-z=0$.

Clearly the point $(-1,-2,-5)\in L$. Since $-1+2(-2)-(-5)=0, (-1,-2,-5)\in\pi$.

Also $(-1)1+3(2)+5(-1)=0 \Rightarrow L\subset\pi$.

2. $1.3+(-2)4+5.1=0 \Rightarrow$ etc.

3. Since d.rs. of the given line are same as d.rs. of a normal to the given plane, the given line is perpendicular to the given plane.

5. Let L_1 be the line $\dfrac{x-3}{4}=\dfrac{y-2}{-5}=\dfrac{z-4}{-1}$ and L_2 be the line $\dfrac{x+2}{-4}=\dfrac{y}{5}=\dfrac{z-3}{1}$.

Given $L_1 \parallel L_2$. Let π be the plane containing L_1, L_2.

Let π be $a(x-3)+b(y-2)+c(z-4)=0$ ... (1) where $4a-5b-c=0$ ... (2)

From the equations to L_2,

we notice that $(-2,0,3)\in L_2$ and hence $(-2,0,3)\in\pi$.

Since $(3,2,4)\in\pi$, we can have $a(-2-3)+b(0-2)+c(3-4)=0$

i.e., $5a+2b+c=0$ ... (3)

Solving (2) and (3), $\dfrac{a}{-3}=\dfrac{b}{-9}=\dfrac{c}{33}$ *i.e.,* $\dfrac{a}{1}=\dfrac{b}{3}=\dfrac{c}{11}$.

$$\therefore \text{ Equation to } \pi \text{ is } \begin{vmatrix} x-3 & y-2 & z-4 \\ 4 & -5 & -1 \\ 5 & 2 & 1 \end{vmatrix}=0$$

[eliminating a,b,c from equation (1), (2) and (3)]

i.e., $x+3y-11z+35=0$.

6. Required plane is $\begin{vmatrix} x-1 & y+1 & z-3 \\ 2 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix}=0$ *i.e.,* $-9x+2y+5z-4=0$.

7. (*i*) Required plane is $\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 1 & 0 & 0 \end{vmatrix}=0$ *i.e.,* $4y-3z+1=0$.

8. Equation to z-axis can be taken as $\dfrac{x-0}{0}=\dfrac{y-0}{0}=\dfrac{z-0}{1}$

Required plane contains z-axis and is perpendicular to the line $\dfrac{x+5}{\sin\theta}=\dfrac{y+2}{\cos\theta}=\dfrac{z-7}{0}$.

$\therefore$ Required plane is $\begin{vmatrix} x-0 & y-0 & z-0 \\ 0 & 0 & 1 \\ \sin\theta & \cos\theta & 0 \end{vmatrix} = 0$ *i.e.,* $x\cos\theta + y\sin\theta = 0$.

9. (*i*) Let $P = (1, 0, -1)$ and $Q = (0, -8, 1)$. Let the given line be L.

$\therefore$ Equations to L can be taken as $\dfrac{x+1}{1} = \dfrac{y-1}{-2} = \dfrac{z+2}{3}$

Let d.rs. of normal to the required plane a, b, c.

$\therefore$ Equation to the required plane is $a(x-1) + b(y-0) + c(z+1) = 0$ ... (1)

where $a(1-0) + b(0+8) + c(-1-1) = 0$ ... (2)

i.e., $a + 8b - 2c = 0$ and $a(1) + b(-2) + c(3) = 0$

i.e., $a - 2b + 3c = 0$ ... (3)

$\therefore$ Required plane is $\begin{vmatrix} x-1 & y & z+1 \\ 1 & 8 & -2 \\ 1 & -2 & 3 \end{vmatrix} = 0$ *i.e.,* $4x - y - 2z - 6 = 0$.

10. Given line is $x = py + q = rz + s$ *i.e.,* $\dfrac{x-0}{1} = \dfrac{y-(-q/p)}{1/q} = \dfrac{z-(-s/r)}{1/r}$

Required plane passes through the given line and hence its equation is

$a(x-0) + b[y-(-q/p)] + c[z-(-s/r)] = 0$ ... (1)

where $a \,.\, 1 + b \,.\, \dfrac{1}{p} + c \,.\, \dfrac{1}{r} = 0$ ... (2)

since the required plane passes through (x_1, y_1, z_1), from (1),

$a(x_1 - 0) + b[y_1 - (-q/p)] + c[z_1 - (-s/r)] = 0$ ... (3)

Eliminating a, b, c from equations (1), (3) and (2), equation to the required plane is

$$\begin{vmatrix} x-0 & y+q/p & z+s/r \\ x_1-0 & y_1+q/p & z_1+s/r \\ 1 & 1/p & 1/r \end{vmatrix} = 0 \quad i.e., \quad \begin{vmatrix} x & py+q & rz+s \\ x_1 & py_1+q & rz_1+s \\ 1 & 1 & 1 \end{vmatrix} = 0$$

11. (*i*) Proceed as in W. Ex.3 before the Exercise.

(*ii*) As in above Ex. 10, required plane is $\begin{vmatrix} x+1 & y-3 & z+2 \\ 0+1 & 7-3 & -7+2 \\ -3 & 1 & 1 \end{vmatrix} = 0$

i.e., $14(x+1) + 14(y-3) + 14(z+2) = 0$ *i.e.,* $x + y + z = 0$ ... (1)

Given line is $6x = 14 - 2y = 3z + 21$ *i.e.,* $\dfrac{x-0}{1} = \dfrac{y-7}{-3} = \dfrac{z+7}{2}$ ... (2)

Since $0+7-7=0$ and $1(1)+1(-3)+1(2)=0$, the line (2) lies in plane (1).

12. Let the plane through the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ be

$l_1(x-x_1)+m_1(y-y_1)+n_1(z-z_1)=0$ ... (1) where $ll_1+mm_1+nn_1=0$ ... (2)

Given line is $ax+by+cz+d=0=a_1x+b_1y+c_1z+d_1$ ... (3)

Let p,q,r be d.rs. of line (3)

$$\therefore \left.\begin{aligned} ap+bq+cr&=0\\ \text{and } a_1p+b_1q+c_1r&=0\end{aligned}\right\} \quad \therefore \frac{p}{bc_1-b_1c}=\frac{q}{ca_1-c_1a}=\frac{r}{ab_1-a_1b}$$

If (1) is parallel to the line (3), then

$(bc_1-b_1c)\,l_1+(ca_1-c_1a)\,m_1+(ab_1-a_1b)\,n_1=0$ ... (4)

Eliminating l_1, m_1, n_1 from (1), (2) and (4), equation to the required plane is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l & m & n \\ bc_1-b_1c & ca_1-c_1a & ab_1-a_1b \end{vmatrix}=0$$

13. (*a*) Let the required plane be $2x-5y-2z-6+\lambda(2x+3y-z-5)=0$ ... (1)

where λ is an unknown constant

Plane (1) is parallel to the line $\frac{x}{1}=\frac{y}{-6}=\frac{z}{7}$

If $1(2+2\lambda)-6(-5+3\lambda)+7(2-\lambda)=0$ *i.e.,* $\lambda=2$, etc.

(*b*) Any point on the given line is $(2t+1, 4t+3, 3t+2)$.

Line through this point and $(3,8,2)$ is parallel to $3x+2y-2z+5=0$

$\Rightarrow 3(2t-2)+2(4t-5)-2(3t)=0 \Rightarrow t=2$

Point on the line is $(5,11,8)$. $\therefore$ Required distance $=\sqrt{4+9+36}=7$

14. Let l,m,n be d.rs. of the required line.

Since the required line is parallel to the planes

$2x+3y+4z=11$ and $3x+4y+5z=12$

we have $2l+3m+4n=0$ and $3l+4m+5n$. Hence etc.

15. Given lines are $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ ... (1)

and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ ... (2)

Let the equation to the plane containing (1) be

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ ... (3)

where $l_1a+m_1b+n_1c=0$ ... (4)

If the plane (3) is parallel to the line (2), then $l_2 a + m_2 b + n_2 c = 0$ ... (5)

Eliminating a, b, c from (3), (4) and (5), we get the equation to the plane containing the line (1) and parallel to the line (2).

Similarly we can get the other plane.

EXERCISE 4 (*c*)

1. (*i*) Proceed as in W. Ex. 1.

(*ii*) Given lines are $\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}$ ... (1) $\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$ ... (2)

$\therefore$ (1) passes through the point $(a-d, a, a+d)$ and

(2) passes through the point $(b-c, b, b+c)$.

D.rs. of (1) are $\alpha-\delta, \alpha, \alpha+\delta$ and d.rs. of (2) are $\beta-\gamma, \beta, \beta+\gamma$.

For (1) and (2) to be coplanar, $\begin{vmatrix} a-d-b-c & a-b & a+d-b-c \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix}$

$$= \begin{vmatrix} c_1-c_2 & & c_3-c_2 \\ c-d & a-b & d-c \\ -\delta & \alpha & \delta \\ -\gamma & \beta & \gamma \end{vmatrix} = -\begin{vmatrix} d-c & a-b & d-c \\ \delta & \alpha & \delta \\ \gamma & \beta & \gamma \end{vmatrix} = 0$$

$\therefore$ Lines (1) and (2) are coplanar and (1) and (2) are not parallel.

$\therefore$ (1) and (2) intersect *i.e.,* (1) and (2) are coplanar.

The plane containing the lines (1) and (2) is $\begin{vmatrix} x-a+d & y-a & z-a-d \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix} = 0$.

2. Given lines are $x = ay + b = cz + d,\ x = py + q = rz + s$

i.e., $\frac{x-0}{1} = \frac{y-(-b/a)}{1/a} = \frac{z-(-d/c)}{1/c}$, $\frac{x-0}{1} = \frac{y-(-q/p)}{1/p} = \frac{z-(-s/r)}{1/r}$.

Lines are coplanar

$$\Rightarrow \begin{vmatrix} 0-0 & \frac{q}{p}-\frac{b}{a} & \frac{s}{r}-\frac{d}{c} \\ 1 & \frac{1}{a} & \frac{1}{c} \\ 1 & \frac{1}{p} & \frac{1}{r} \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 0 & \frac{aqbp}{ap} & \frac{cs-dr}{cr} \\ 0 & \frac{1}{a}-\frac{1}{p} & \frac{1}{c}-\frac{1}{r} \\ 1 & \frac{1}{p} & \frac{1}{r} \end{vmatrix} = 0$$

$$\Rightarrow \frac{(aq-bp)(r-c)}{apcr}=\frac{(p-a)(cs-dr)}{apcr} \quad \Rightarrow (r-c)(aq-bp)=(p-a)(cs-dr)$$

$\therefore$ If $(r-c)(aq-bp)=(p-a)(cs-dr)$, given lines are coplanar.

3. Let a, b, c be d.rs. of a normal to the plane π containing the lines

$$\frac{x}{m}=\frac{y}{n}=\frac{z}{l} \text{ and } \frac{x}{n}=\frac{y}{l}=\frac{z}{m}.$$

$$\left.\begin{array}{l}\therefore \quad am+bn+cl=0 \\ \text{and } an+bl+cm=0\end{array}\right\} \therefore \frac{A}{mn-l^2}=\frac{b}{nl-m^2}=\frac{c}{lm-n^2}.$$

$\therefore$ Equation to the plane through the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ and perpendicular to the plane π is

$$\begin{vmatrix} x & y & z \\ l & m & n \\ mn-l^2 & nl-m^2 & lm-n^2 \end{vmatrix}=0 \quad i.e., \sum x\,(lm^2-mn^2-ln^2+nm^2)=0$$

$$i.e., \quad \sum x\,(m-n)(lm+mn+nl)=0$$

$$i.e., \quad \sum x\,(m-n)=0) \quad \text{(Cancelling } lm+mn+nl \neq 0\text{)}.$$

4. Let L be the required line with d.rs. a, b, c.

Let $\frac{x-5}{1}=\frac{y}{1}=\frac{z-5}{1}$ ($=p$ say) and $\frac{x+5}{1}=\frac{y}{1}=\frac{z+5}{1}$ ($=q$ say).

Let $P=(p+5, p, p+5)$ and $Q=(q-5, q, 2q-5)$.

$\therefore$ d.rs. of $\overrightarrow{PQ}$ are $(p-q+10, p-q, p-2q+10)$

If $PQ \in L$ and since L is parallel to the line $\frac{x-5}{2}=\frac{y-5}{1}=\frac{z-10}{3}$,

we have $\frac{p-q+10}{2}=\frac{p-q}{1}=\frac{p-2q+10}{3}$ $i.e.,$ $\left.\begin{array}{r} p-q-10=0 \\ 2p-q-10=0\end{array}\right\} \therefore p=0, q=-10$

$\therefore p=(5,0,5)$. $\quad \therefore$ Equations to L are $\frac{x-5}{2}=\frac{y-0}{1}=\frac{z-5}{3}$ $\quad i.e.,$ $\frac{x-5}{2}=\frac{y}{1}=\frac{z-5}{3}$.

5. Proceed as in W. Ex. 1 (2nd method).

6. The vectors along the lines through O with d.rs. $1, -1, 1; 2, -3, 0; 1, 0, 3$ are $(1, -1, 1), (2, -3, 0), (1, 0, 3)$.

$$\therefore [(1,-1,1),(2,-3,0),(1,0,3)]=\begin{vmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \\ 1 & 0 & 3 \end{vmatrix}=0$$

$\Rightarrow$ the three lines are coplanar.

7. Given lines are $x+2y-5z+9=0$, $3x-y+2z-5=0$ and $4x-5y+z+3=0$, $2x+3y-z-3=0$.

Since $\begin{vmatrix} 1 & 2 & -5 & 9 \\ 3 & -1 & 2 & -5 \\ 4 & -5 & 1 & 3 \\ 2 & 3 & -1 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -5 & 9 \\ 0 & -7 & 17 & -32 \\ 0 & -13 & 21 & -33 \\ 0 & -1 & 9 & -21 \end{vmatrix} \begin{matrix} R_2-3R_1 \\ R_3-4R_1 \\ R_4-2R_1 \end{matrix}$

$= \begin{vmatrix} 7 & 17 & 32 \\ 13 & 21 & 33 \\ 1 & 9 & 21 \end{vmatrix} = \begin{vmatrix} 0 & -46 & -115 \\ 0 & -96 & -240 \\ 1 & 9 & 21 \end{vmatrix} \begin{matrix} R_1-7R_3 \\ R_2-13R_3 \end{matrix}$

$= 46\times 240 - 96\times 115 = 0$, given lines are coplanar.

Consider $x+2y-5z+9=0$ ... (1)

$3x-y+2z-5=0$ ... (2) $4x-5y+z+3=0$ ... (3)

$-3\times(1)+(2): -7y+17z-32=0$ ... (4)

$-4\times(1)+(3): -13y+21z-33=0$ ... (5)

$\therefore \dfrac{y}{-561+672} = \dfrac{z}{416-231} = \dfrac{1}{-147+221}$ *i.e.,* $y=\dfrac{3}{2},\ z=\dfrac{5}{2}$

$\therefore$ From (1), $x+3-\dfrac{25}{2}+9=0 \Rightarrow x=\dfrac{1}{2}$

8. Ex. 13(*b*) of Exercise 10 (*b*)

9. Let L be the required line

Let $P=(2,-3,1)$. Given line is $\dfrac{x-2}{2}=\dfrac{y}{-3}=\dfrac{z-2}{-1}$ ($=t$ say)

Let $Q=(2t+2,-3t,-t+2)$. Let L meet the given line in Q.

Since $P\in L$, d.rs. of $\overrightarrow{PQ}$ are d.rs. of L.

$\therefore$ d.rs. of L are $2t, -3t+3, -t+1$.

But L is parallel to the plane $2x+y-z=6$.

$\therefore$ $2\,2t)+1(-3t+3)-1(-t+1)=0 \Rightarrow t=-1$. $\therefore Q=(0,3,3)$.

$\therefore$ Equations to L are $\dfrac{x-2}{0-2}=\dfrac{y+3}{3+3}=\dfrac{z-1}{3-1}$, etc.

10. Proceed as in W. Ex. 3.

11. Given lines are $2x-3y+4z+1=0=3x+2y+4z-5$ ... (1)

$2x-4y+z+6=0=3x+4y+z-3$ ... (2)

Let the equation to the plane containing (1)

and passing through (0, 0, 0) be $(2x-3y+4z+1)+\mu(3x+2y+4z-5)=0$

$(0-0+0+1)+\mu(0+0+0-5)=0$

$-5\lambda+1=0$; $\mu=1/5$

$(2x-3y+4z+1)+1/5\,(3x+2y+4z-5)=0$

$10x-15y+20z+5)+3x+2y=4z-5=0$

$13x-13y+24z=0$ is the required plane.

Let equation to the plane containing (2) and passing through (0, 0, 0) be

$(2x-4y+z+6)+\mu\,(3x-4y+z-3)=0$

$(0-0+0+6)+\mu(0-0+0-3)=0$; $\mu=2$

$(2x-4y+z+6)+2\,(3x-4y+z-3)=0$

$8x-12y+3z=0$ is the required plane.

$\therefore$ Equations of required line are $13x-13y+24z=0$, $8x-12y+3z=0$

12, 13, 14. Proceed as in W. Ex. 8.

15. Proceed as in W. Ex. 9.

16. Proceed as in Ex. 13 taking d.rs. of the required line as 4, 1, 1.

17. Proceed as in Ex. 1(i).

18. Let $P=(2,3,4)$. Let L be the required line. Given line is $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$ $(=t$ say)

Let $Q=(t,t,t)$, a point on the given line. Since $P\in L$, d.rs. of $\overleftrightarrow{PQ}$ are d.rs. of L.

$\therefore$ d.rs. of L are $t-2, t-3, t-4$.

Since L is perpendicular to x-axis, we have $1\,(t-2)+0\,(t-3)+0\,(t-4)=0\Rightarrow t=2$

$\therefore Q=(2,2,2)$. $\therefore$ Equations to L are $\dfrac{x-2}{0}=\dfrac{y-3}{1}=\dfrac{z-4}{2}$, etc.

19. Let L be the required line and let $P=(2,2,2)$.

Given lines are $\dfrac{x-2}{2}=\dfrac{y+2}{3}=\dfrac{z+4}{4}$ ... (1) and $x=2y=3z$ ... (2)

From (1), $3x-2y-10=0$, $4y-3z+20=0$ and

From (2), $x-2y=0$, $2y-3z=0$.

Since L intersects (1) and (2), Equations to L can be taken as

$3x-2y-10+\lambda_1\,(4y-3z+20)=0$, $x-2y+\lambda_1\,(2y-3z)=0$

for some fixed values of λ_1 and λ_2.

Since $P\in L$, $6-4-10+\lambda_1\,(8-6+20)=0$ and $2-4+\lambda_2\,(4-6)=0$

$\Rightarrow \lambda_1=4/11$, $\lambda_2=-1$.

$\therefore$ Equations to L are $3x-2y-10+\dfrac{4}{11}(4y+3z+20)=0$,

$x-2y-1\,(2y-3z)=0$ *i.e.,* etc.

EXERCISE 4 (*d*)

1. Proceed as in W. Ex. 1.

2. (*i*), (*ii*). Proceed as in W.Ex. 1.

3. Given $A = -\bar{i} + 2\bar{j} - 3\bar{k} = (-1, 2, -3)$, $B = -16\bar{i} + 6\bar{j} + 4\bar{k} = (-16, 6, 4)$,

$C = \bar{i} - \bar{j} + 3\bar{k} = (1, -1, 3)$ and $D = 4\bar{i} + 9\bar{j} + 7\bar{k} = (4, 9, 7)$

Now $\overline{AB} = (-16+1, 6-2, 4+3) = (-15, 4, 7)$,

$\overline{CD} = (4-1, 9+1, 7-3) = (3, 10, 4)$ and $\overline{AB} \times \overline{CD} = (-15, 4, 7) \times (3, 10, 4)$

$= (16-70, 21+60, -150-12) = (-54, 81, -162)$

S.D. between the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD} = \left| \overline{AC} \cdot \dfrac{\overline{AB} \times \overline{CD}}{|\overline{AB} \times \overline{CD}|} \right|$

$$= \frac{|(1+1, -1-2, 3+3) \cdot (-54, 81, -162)|}{\sqrt{(54^2 + 81^2 + 162^2)}} = \frac{1-108-243-972|}{27\sqrt{(4+9+36)}} = \frac{1323}{27 \times 7} = 7.$$

4. Proceed as in W. Ex. 2.

5. Let L_1 be the line $\dfrac{x-2}{3} = \dfrac{y-3}{4} = \dfrac{z-4}{5}$ and L_2 be the line $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$.

Let $A = (2, 3, 4)$ and $B = (1, 2, 3)$. Clearly $A \in L_1$ and $B \in L_2$.

A vector along L_1 is (3, 4, 5) and a vector along L_2 is (2, 3, 4).

$\therefore$ $(3, 4, 5) \times (2, 3, 4) = (16-15, 10-12, 9-8) = (1, -2, 1)$

is a vector perpendicualr to L_1 and L_2

$\therefore$ S.D. between L_1 and L_2 $= \left| \overline{AB} \cdot \dfrac{(1, -2, 1)}{|(1, -2, 1)|} \right|$

$$= \frac{|(1-2, 2-3, 3-4) \cdot (1, -2, 1)|}{\sqrt{(1+4+1)}} = \frac{-1 \cdot 1 + (-1)(-2) + (-1)1}{\sqrt{6}} = 0$$

$\therefore$ L_1 and L_2 intersect and hence L_1, L_2 are coplanar.

OR : S.D. between

$$L_1 \text{ and } L_2 = \begin{vmatrix} 1-2 & 2-3 & 3-4 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} + \sqrt{[(16-15)^2 + (10-12)^2 + (9-8)^2]}$$

$$= \begin{vmatrix} -1 & -1 & -1 \\ 3 & 4 & 5 \\ -1 & -1 & -1 \end{vmatrix} + \sqrt{6} = 0 \quad R_3 - R_2$$

$\therefore$ L_1, L_2 are intersecting *i.e.,* L_1, L_2 *are coplanar.*

6. Equations to the line through $(-6, 1, -10), (-3, 7, 13)$ are

$$\frac{x+6}{3}=\frac{y-1}{6}=\frac{z+10}{-3} \quad i.e., \quad \frac{x+6}{1}=\frac{y-1}{2}=\frac{z+10}{-1} = t \text{ (say)} \quad ... \ (1)$$

Given line is $3x+2y-15z-144=0=3x-y-3z-42$ ... (2)

A point on line (2) is obtained by putting some value for z, say, -10 in the equations

$$\therefore \left.\begin{array}{r} 3x+2y+6=0 \\ 3x-y-12=0 \end{array}\right\} x=2, y=-6$$

$\therefore$ A point on the line (2) is $(2, -6, -10)$.

If l, m, n are d.rs. of line (2), then

$$\left.\begin{array}{r} 3l+2m+15n=0 \\ 3l-m-3n=0 \end{array}\right\} \quad \therefore \frac{l}{-6-15}=\frac{m}{-45+9}=\frac{n}{-3-6} \quad i.e., \ \frac{l}{7}=\frac{m}{12}=\frac{n}{3}.$$

$\therefore$ Equations to line (2) in the symmetrical form are $\dfrac{x-2}{7}=\dfrac{y+6}{12}=\dfrac{z+10}{3}$ $(=r \text{ say})$

A point P on (1) is $(t-6, 2t+1, -t-10)$ and a point Q on (2) is $(7r+2, 12r-6, 3r-10)$.

$\therefore$ d.rs. of $\overrightarrow{PQ}$ are $7r-t+8, 12r-2t-7, 3r+t$.

If $\overrightarrow{PQ}$ is perpendicular to (1) and (2), then

$1\,(7r-t+8)+2\,(12r-2t-7)-1\,(3r+t)=0$

$7\,(7r-t+8)+12\,(12r-2t-7)+3\,(3r+t)=0$

$$\Rightarrow \left.\begin{array}{r} 14r-3t-3=0 \\ 101r-14t-14=0 \end{array}\right\} \therefore \frac{r}{42-42}=\frac{t}{-303+196}=\frac{1}{-196+303} \quad \Rightarrow r=0, t=-1.$$

$\therefore P=(-7,-1,-9), \quad Q=(2,-6,-10)$.

Since PQ is the S.D. between lines (1) and (2), P is the nearest point on the line (1) which is nearest to the line (2).

Note : S.D. $=PQ=\sqrt{[81+25+1]}=\sqrt{107}$.

7. Proceed as in W. Ex. 3.

8. Proceed as in W. Ex. 2.

9. Given lines are $\dfrac{x+k}{12}=\dfrac{y}{6}=\dfrac{z}{-1}$ ... (1) and $x-y-2k=0=x-6z+6k$ ... (2)

Any plane through (2) and parallel to (1) is

$x-y-2k+\lambda\,(x-6z+6k)=0$, λ is an unknown constant

i.e., $(1+\lambda)\,x+(-1)\,y+(-6\lambda)\,z-2k+6k\lambda=0$ ... (3)

$\therefore \quad 12\,(1+\lambda)+6(-1)-1\,(-6\lambda)=0 \Rightarrow \lambda=-1/3$.

$\therefore$ Equation to the plane (3) is $2\dfrac{x}{3}-y+2z-2k-2k=0$ *i.e.,* $2x-3y+6z-12k=0$.

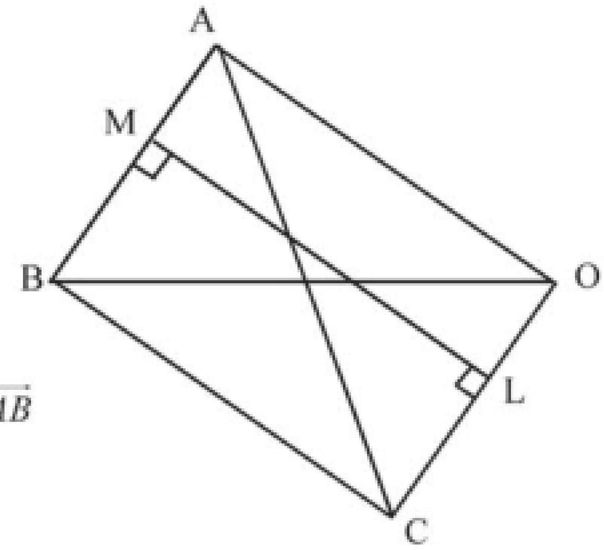
A point on (1) is $(-k, 0, 0)$. $\therefore$ S.D. between (1) and (2)

$$= \text{Distance of } (-k,0,0) \text{ from (3)} = \left|\frac{-2k-0+0-12k}{\sqrt{(4+9+36)}}\right| = 2k.$$

10. Proceed as in W. Ex.5 by putting $b = c = a$.

11. Let $OABC$ be the tetrahedron where O is the origin.

Let the planes $\overleftrightarrow{OBC}, \overleftrightarrow{OCA}, \overleftrightarrow{OAB}, \overleftrightarrow{ABC}$ be represented by the equations $y+z=0, z+x=0, x+y=0, x+y+z=a$.



Now equations to $\overleftrightarrow{OC}$ are $y+z=0=z+x$

i.e., $\dfrac{x}{1}=\dfrac{y}{1}=\dfrac{z}{-1}$ ... (1)

and equations to $\overleftrightarrow{AB}$ are $x+y=0=x+y+z=a$

i.e., $\dfrac{x}{1}=\dfrac{y}{-1}=\dfrac{z-a}{0}$ ... (2)

Let $\overleftrightarrow{LM}$ be the line of S.D. between $\overleftrightarrow{OC}$ and $\overleftrightarrow{AB}$ so that $L \in \overleftrightarrow{OC}$ and $M \in \overleftrightarrow{AB}$.

Let l, m, n be d.rs. of $\overleftrightarrow{LM}$.

$$\therefore \quad \left.\begin{array}{l} l+m-n=0 \\ l-m+0n=0 \end{array}\right\} \Rightarrow \frac{l}{1}=\frac{m}{1}=\frac{n}{2}$$

A point on $\overleftrightarrow{OC}$ is $O(0, 0, 0)$. Let a point on $\overleftrightarrow{AB}$ be $A(0,0,a)$.

$$\therefore \quad LM = \left|\overline{\mathrm{OA}} \cdot \frac{\overline{\mathrm{LM}}}{|\mathrm{LM}|}\right| = \left|(0,0,a) \cdot \frac{(1,1,2)}{\sqrt{1+1+4}}\right| = \frac{2a}{\sqrt{6}}$$

Similarly we can find S.D.s. between other pairs of opposite edges.

12, 13. Proceed as in W. Ex. 6.

14. Express the given line in the form $\dfrac{x-2}{1}=\dfrac{y+3}{-2}=\dfrac{z-0}{1}$ and proceed as above.

15. Proceed as in W. Ex. 6.

EXERCISE 4 (*e*)

1. Proceed as in Art. 10. 17.

2. Let $A=(1,2,3)$ and $B=(2,-1,4)$. Now OAB is the given triangle.

Let A_1, B_1 be the respective projections of A, B in XY plane

$\therefore$ ΔOA_1B_1 is the projection of ΔOAB in XY plane.

Since $A_1=(1,2,0), B_1(2,-1,0)$,

Area of $\Delta\, OA_1B_1 = \frac{1}{2}\left|\overline{OA_1}\times\overline{OB_1}\right| = \frac{1}{2}\left|(1,2,0)\times(2,-1,0)\right|$

$$= \frac{1}{2}\left|(0,0,-5)\right| = \frac{1}{2}\sqrt{(0+0+25)} = \frac{5}{2} \text{ sq. units}$$

Similarly area of the projection of $\Delta\, OAB$ in XY plane

$= \frac{1}{2}\left|\ 0,2,3\ \right|\times(0,-1,4) = \frac{11}{2}$ sq. units. and area of the projection of $\Delta\, OAB$ in ZX plane.

$= 1/2\,|\,(1,0,3)\times(2,0,4)\,| = 1$ sq. unit.

$\therefore\quad (\text{Area of } \Delta\, OAB)^2 = \left(\frac{5}{2}\right)^2 + \left(\frac{11}{2}\right)^2 + (1)^2 = \frac{150}{4}$

$\Rightarrow\quad$ Area of $\Delta\, OAB = \frac{\sqrt{150}}{2}$ sq. units.

3. (*ii*) Let $A=(1,2,1), B=(3,2,5), C=(2,-1,0)$ and $D=(-1,0,1)$ be the vertices of the tetrahedron $ABCD$.

$\therefore$ Volume of the tetrahedron $ABCD = \frac{1}{6}\left[\!\left[\overline{AB}\ \ \overline{AC}\ \ \overline{AD}\right]\!\right]$

$$= \frac{1}{6}\left|[(2,0,4),(1,-3,-1),(-2,-2,0)]\right| = \frac{1}{6}\left(+\text{ value of }\begin{vmatrix}2 & 0 & 4\\ 1 & -3 & -1\\ -2 & -2 & 0\end{vmatrix}\right) = 6 \text{ cu. units.}$$

4. Since $\frac{1}{6}\begin{vmatrix}1-6 & 2+4 & -5-4\\ -1-6 & -2+4 & -3-4\\ 0-6 & 0+4 & -4-4\end{vmatrix} = \frac{1}{6}\begin{vmatrix}-5 & 6 & -9\\ -7 & 2 & -7\\ -6 & 4 & -8\end{vmatrix}$

$= \frac{1}{6}[-60-84+144] = 0$, the four points are coplanar.

5. Since $A\in\overrightarrow{OX}, B\in\overrightarrow{OY}, C\in\overrightarrow{OZ}$ such that $OA=a$.

$OB=b, OC=c$ we have $A=(a,0,0), B=(0,b,0), C=(0,0,c)$

$\therefore\ \overline{AB}=(-a,b,0)$ and $\overline{AC}=(-a,0,c)$

$\therefore$ Area of $\Delta\, ABC = \frac{1}{2}\left|\overline{AB}\times\overline{AC}\right| = \frac{1}{2}\left|(-a,b,0)\times(-a,0,c)\right|$

$= \frac{1}{2}\left|\ (bc,\ ca,\ ab)\ \right| = \frac{1}{2}\sqrt{[b^2c^2+c^2a^2+a^2b^2]}$ sq. units.

6. Given $P=(2,2,1)$. Let π be the plane containing P and perpendicular to $\overrightarrow{OP}$. Since d.rs. of $\overrightarrow{OP}$ are $2-0, 2-0, 1-0$, equation to π is $2(x-2)+2(y-2)+1(z-1)=0$ $\Rightarrow 2x+2y+z=9$.

Since π intersects the coordinate axes in $A, B, C,$ we have

$$A=\left(\frac{9}{2},0,0\right),\quad B=\left(0,\frac{9}{2},0\right),\quad C=(0,0,9)$$

Further since $\overline{\mathrm{AB}}=\left(-\frac{9}{2},\frac{9}{2},0\right)$ and $\overline{\mathrm{AC}}=\left(-\frac{9}{2},0,9\right)$,

area of $\Delta\, ABC=\frac{1}{2}|\overline{\mathrm{AB}}\times\overline{\mathrm{AC}}|=\frac{1}{2}\left|\left(-\frac{9}{2},\frac{9}{2},0\right)\times\left(-\frac{9}{2},0,9\right)\right|$

$$=\frac{1}{2}\left|\left(\frac{81}{2},\frac{81}{2},\frac{81}{4}\right)\right|=\frac{1}{2}\sqrt{\left[\frac{81^2}{4},\frac{81^2}{4},16\right]}\ =\frac{243}{8}\ \text{sq. units.}$$

7. Given $O=(0,0,0)$, $A=(2,6,3)$ and $(5,12,0)$

$\therefore$ Area of $\Delta\, OAB=\frac{1}{2}|\overline{\mathrm{OA}}\times\overline{\mathrm{AB}}|=\frac{1}{2}|(2,6,3)\times(5,12,0)|$

$$=\frac{1}{2}|(-36,15,-6)|=\frac{3}{2}\sqrt{[144+25+4]}\ =\frac{3}{2}\sqrt{173}$$

If OC is the altitude from O to AB, then $\frac{1}{2}OC\times AB$ = Area of $\Delta\, OAB$

$\therefore\ \frac{1}{2}OC\times\sqrt{[(5-2)^2+(12-6)^2+(0-3)^2}=\frac{3}{2}\sqrt{173}\quad\Rightarrow\ OC=\frac{3\sqrt{173}}{\sqrt{54}}=\sqrt{\frac{173}{6}}$.

9. Let $P=(x_1,y_1,z_1)$. Given $A=(a,0,0)$, $B=(0,b,0)$, $C=(0,0,c)$

since volume of the tetrahedron $PABC=\frac{kabc}{6}$.

$$\frac{1}{6}=\frac{kabc}{6}\Rightarrow\ \left|\,[(a-x_1,-y_1,-z_1),(-a,b,0),(-a,0,c)\,\right|=kabc$$

$$\Rightarrow\begin{vmatrix}a-x_1 & -y_1 & -z_1\\ -a & b & 0\\ -a & 0 & c\end{vmatrix}=k^2a^2b^2c^2$$

$$\Rightarrow\ (abc-bcx_1-cay_1-abz_1)^2=k^2a^2b^2c^2$$

$$\Rightarrow\ (bcx_1+cay_1+abz_1-abc-kabc)\,(bcx_1+cay_1+abz_1-abc-kabc)=0$$

$$\Rightarrow\ \left(\frac{x_1}{a}+\frac{y_1}{b}+\frac{z_1}{c}-1-k\right)\left(\frac{x_1}{a}+\frac{y_1}{b}+\frac{z_1}{c}-1+k\right)=0$$

$\therefore$ Locus of P is the pair of parallel planes $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1-k=0,\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1+k=0$

But equation to the plane $\overleftrightarrow{ABC}$ is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1=0$.

$\therefore$ Locus of P is the pair of parallel planes each parallel to the plane $\overleftrightarrow{ABC}$.

10. Let the variable plane be π with equation $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ... (1)

Clearly π forms a tetrahedron with the coordinate planes

$x=0$ ... (2), $y=0$ ... (3), $z=0$ ... (4)

Let $OABC$ be the tetrahedron where O is the point of intersection of the planes (2), (3), (4), A is the point of intersection of the planes (1), (3), (4), B is the point of intersection of the planes (1), (2), (4) and C is the point of intersection of the planes (1), (2), (3).

$\therefore$ $O=(0,0,0)$, $A=(a,0,0)$, $B=(0,b,0)$, $C=(0,0,c)$

$\therefore$ Centroid of the tetrahedron $=G=\left(\frac{a}{4},\frac{b}{4},\frac{c}{4}\right)$

since volume of the tetrahedron $OABC=64k^3$, $\frac{1}{6}\left|[\,\overline{\mathrm{OA}}\ \ \overline{\mathrm{OB}}\ \ \overline{\mathrm{OC}}\,]\right|=64k^3$

$$\Rightarrow \frac{1}{6}\times \text{mod of}\begin{vmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{vmatrix}=64k^3 \ \Rightarrow\ a^2b^2c^2=36\times 64\times 64k^6 \quad \text{... (5)}$$

Let $G=(x_1, y_1, z_1)$.

$\therefore$ $x_1=\frac{a}{4}, y_1=\frac{b}{4}, z_1=\frac{c}{4} \ \Rightarrow a=4x_1, b=4y_1, =4z_1$.

$\therefore$ From (5), $(64x_1\ y_1\ z_1)^2=36\times 64\times 64k^6 \ \Rightarrow\ x_1^2y_1^2z_1^2=36k^6$

$\therefore$ Locus of G is $x^2y^2z^2=36k^6$.

11. Given lines are $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}\ (=r_1 \text{ say})$ ... (1)

and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}\ (=r_2 \text{ say})$ ... (2)

For the given tetrahedron an edge of length r_1 lines on (1) and an edge of length r_2 lies on (2).

$\therefore$ Four vertices of the tetrahedron can be taken as

$(x_1, y_1, z_1), (l_1r_1+x_1, m_1r_1+y_1, n_1r_1+z_1)$, $(x_2, y_2, z_2), (l_2r_2+x_2, m_2r_2+y_2, n_2r_2+z_2)$.

$\therefore$ Volume of the tetrahedron

$$=+\text{ve value of } \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1\\ l_1r_1+x_1 & m_1r_1+y_1 & n_1r_1+z_1 & 1\\ x_2 & y_2 & z_2 & 1\\ l_2r_2+x_2 & m_2r_2+y_2 & n_2r_2+z_2 & 1\end{vmatrix}$$

$$= + \text{ ve value of } \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ l_1r_1 & m_1r_1 & n_1r_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ l_2r_2 & m_2r_2 & n_2r_2 & 1 \end{vmatrix} \begin{matrix} R_2 - R_1 \\ R_4 - R_3 \end{matrix}$$

$$= + \text{ ve value of } \frac{1}{6}r_1r_2\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ l_1 & m_1 & n_1 & 0 \\ x_2 & y_2 & z_2 & 1 \\ l_2 & m_2 & n_2 & 0 \end{vmatrix}$$

$$= + \text{ ve value of } \frac{1}{6}r_1r_2\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 & 0 \\ l_1 & m_1 & n_1 & 0 \\ x_2 & y_2 & z_2 & 1 \\ l_2 & m_2 & n_2 & 0 \end{vmatrix}$$

$$= + \text{ ve value of } \frac{1}{6}r_1r_2\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \text{ cu. units.}$$

EXERCISE 4 (*f*)

1. Proceed as in W. Ex. 1 and W. Ex. 2.

2. Given planes are $x - y + z + 1 = 0$, $\lambda x + 3y - 2z - 3 = 0$, $3x + \lambda y - z - 2 = 0$.
Clearly no two of the three planes are parallel

Coefficient matrix of the given equations is $\begin{vmatrix} 1 & -1 & 1 & 1 \\ \lambda & 3 & 2 & -3 \\ 3 & \lambda & -1 & -2 \end{vmatrix}$

(*i*) $\therefore \quad \Delta = \begin{vmatrix} 1 & -1 & 1 \\ \lambda & 3 & 2 \\ 3 & \lambda & -1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ \lambda+3 & 3 & 5 \\ \lambda+3 & \lambda & \lambda-1 \end{vmatrix} \begin{matrix} C_1 + C_2 \\ C_3 + C_2 \end{matrix}$

$= (\lambda + 3)(\lambda - 1) - 5(\lambda + 3) = (\lambda + 3)(\lambda - 6)$.

For the three given planes to intersect in a unique point, $\Delta \neq 0$.

$\therefore \ (\lambda + 3)(\lambda - 6) \neq 0 \Rightarrow \lambda \neq -3$ or 6.

$\therefore$ for all values of λ except -3 or 6, the three given planes intersct in a point *i.e.,* for a fixed value of $\lambda\,(\neq -3, 6)$ we get a unique point of intersection.

(*ii*) For the three given planes to form a triangular prism, Δ must be zero and one of $\Delta_1, \Delta_2, \Delta_3$ is not equal to zero.

$\therefore \ \Delta = 0 \Rightarrow (\lambda + 3)(\lambda - 6) = 0 \Rightarrow \lambda = -3$ or 6

Now $\Delta_1 = \begin{vmatrix} -1 & 1 & 1 \\ 3 & 2 & -3 \\ \lambda & -1 & -2 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ 3 & 5 & 0 \\ \lambda & \lambda-1 & \lambda-2 \end{vmatrix} \begin{matrix} C_1+C_2 \\ C_1+C_3 \end{matrix}$

$= S(\lambda-2) \neq 0$ for $\lambda = -3$ or 6.

3. Given planes are $x+ay+(b+c)z+d=0$, $x+by+(c+a)z+d=0$ and $x+cy+(a+b)z+d=0$. Clearly no two of the three planes are parallel.

Coefficient matrix of the given equations is $\begin{vmatrix} 1 & a & b+c & d \\ 1 & b & c+a & d \\ 1 & c & a+b & d \end{vmatrix}$

$\therefore\ \Delta = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} C_1+C_2$

$= (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} = 0$

Also $\Delta_1 = \begin{vmatrix} a & b+c & d \\ b & c+a & d \\ c & a+b & d \end{vmatrix} = \begin{vmatrix} a & a+b+c & d \\ b & a+b+c & d \\ c & a+b+c & d \end{vmatrix} C_1+C_2$

$= d\,(a+b+c)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix} = 0$

$\therefore$ The three given planes intersect in a unique line.

4. Proceed as in W. Ex. 3.

5. Given planes are $bx-ay-n=0$. $cy-bz-l=0$ and $az-cx-m=0$.
Clearly no two of the three planes are parallel.

Coefficient matrix of the given equations is $\begin{vmatrix} b & -a & 0 & -n \\ 0 & c & -b & -l \\ -c & 0 & a & -m \end{vmatrix}$

$\therefore\ \Delta \begin{vmatrix} b & -a & 0 \\ 0 & c & -b \\ -c & 0 & a \end{vmatrix} = abc - abc = 0$

Also $\Delta_1 = \begin{vmatrix} -a & 0 & -n \\ c & -b & -l \\ 0 & a & -m \end{vmatrix} \begin{matrix} = -a\,(bm+al) - can \\ = -a\,(al+bm+cn). \end{matrix}$

If $al+bm+cn=0$, then $\Delta_1=0$ implying that the three given planes intersect in a line.

If l_1, m_1, n_1 are d.rs. of the common line of intersection of the planes, then

$$\frac{l_1}{ab}=\frac{m_1}{b^2}=\frac{n_1}{bc} \Rightarrow \frac{l_1}{a}=\frac{m_1}{b}=\frac{m_1}{b}=\frac{n_1}{c}$$

$\Rightarrow$ d.rs. of the common line are a, b, c.

6. Given planes are $x-cy-bz=0, cx-y+az=0$ and $bx+ay-z=0$.

Clearly no two of the three planes are parallel.

Coefficient matrix of the given equation is $\begin{vmatrix} 1 & -c & -b & 0 \\ c & -1 & a & 0 \\ b & a & -1 & 0 \end{vmatrix}$

$$\therefore \ \Delta=\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix}=1-a^2-b^2-c^2-2abc$$

If $a^2+b^2+c^2+2abc=1$, then $\Delta=0$, Also $\Delta_1=0$.

$\therefore$ The given planes pass through a unique line if $a^2+b^2+c^2+2abc=1$.

Let l, m, n be d.cs. of the common line

$\therefore \ l-cm-bn=0, \ cl-m+an=0, \ bl+am-n=0$

Solving two by two equations at a time, we get

$$\frac{l}{ac+b}=\frac{m}{bc+a}=\frac{n}{1-c^2} \quad \ldots (1) \qquad \frac{l}{ab+c}=\frac{m}{1-b^2}=\frac{n}{a+bc} \quad \ldots (2)$$

$$\frac{l}{1-a^2}=\frac{m}{ab+c}=\frac{n}{ac+b} \quad \ldots (3)$$

From (2) and (3) :

$$\therefore \ \frac{l^2}{(ab+c)(1-a^2)}=\frac{m^2}{(1-b^2)(ab+c)} \qquad \frac{l^2}{1-a^2}=\frac{m^2}{1-b^2}$$

From (1) and (3) : $\dfrac{l^2}{1-a^2}=\dfrac{n^2}{1-c^2}$

$$\therefore \ \frac{l^2}{1-a^2}=\frac{m^2}{1-b^2}=\frac{n^2}{1-c^2} \Rightarrow \frac{l}{\sqrt{1-a^2}}=\frac{m}{\sqrt{1-b^2}}=\frac{n}{\sqrt{1-c^2}}$$

Since (0, 0, 0) lies on the three planes, the common line also passes through (0, 0, 0).

$\therefore$ Equations to the common line are $\dfrac{x}{\sqrt{1-a^2}}=\dfrac{y}{\sqrt{1-b^2}}=\dfrac{z}{\sqrt{1-c^2}}$

7. Bounding planes of the tetrahedron OABC, are $lx+my=0 \quad \ldots (1)$

$my+nz=0 \quad \ldots (2) \quad nz+lx=0 \quad \ldots (3)$ and $lx+my+nz=0 \quad \ldots (4)$

Solving, (1), (2), (3) : $O=(0, 0, 0)$

(2), (3), (4) : $C = (p/l, p/m, -p/n)$

(1), (3), (4) : $A = (-p/l, p/m, p/n)$

(1), (2), (4) : $B = (p/l, -p/m, p/n)$.

$\therefore$ Volume of the tetrahedron $OABC = \frac{1}{6}\left|[\overline{\text{OA}}\ \overline{\text{OB}}\ \overline{\text{OC}}]\right|$

$$= \frac{1}{6}\left|[(-p/l, p/m, p/n), (p/l, -p/m, p/n), (p/l, p/m, -p/n)]\right|$$

$$= \frac{1}{6}\left(+ \text{ve value of } \begin{vmatrix} \frac{-p}{l} & \frac{p}{m} & \frac{p}{n} \\ \frac{p}{l} & \frac{-p}{m} & \frac{p}{n} \\ \frac{p}{l} & \frac{p}{m} & \frac{-p}{n} \end{vmatrix} \right) = \frac{2p^3}{3\,lmn} \text{ cu. units.}$$

8. Let $P = (x_1, y_1, z_1)$. Any plane through the line $y = 1, z = -1$ is

$$y - 1 + \lambda\,(z + 1) = 0 \qquad \dots (1)$$

If P lies on (1), then $y_1 - 1 + \lambda\,(z_1 + 1) = 0 \Rightarrow \lambda = \frac{y_1 - 1}{z_1 + 1}$.

$\therefore$ Equation to the plane through the line $y = 1, z = -1$ and containing.

P is $y - 1 - \frac{y_1 - 1}{z_1 + 1}(z + 1) = 0$

i.e., $\quad 0\,.\,x + (z_1 + 1)\,(y - 1) - (y_1 - 1)\,z - (y_1 + z_1) = 0 \qquad \dots (2)$

Similarly the other two planes are

$$-(z_1 + 1)\,x + 0\,.\,y + (x_1 + 1)\,z - (x_1 + z_1) = 0 \qquad \dots (3)$$

$$(y_1 + 1)\,x - (x_1 - 1)\,y + 0\,.\,z - (x_1 + y_1) = 0 \qquad \dots (4)$$

These planes (2), (3), (4) pass through a line

$$\Rightarrow \begin{vmatrix} 0 & z_1 + 1 & -(y_1 + 1) \\ -(z_1 - 1) & 0 & x_1 + 1 \\ y_1 + 1 & -(x_1 - 1) & 0 \end{vmatrix} = 0 \quad \Rightarrow 1 + x_1 y_1 + y_1 z_1 + z_1 x_1 = 0$$

(on simplification)

$\therefore$ Locus of P is $xy + yz + zx + 1 = 0$ *i.e.,*

P lies on the surface $xy + yz + zx + 1 = 0$.

Change of Axes

Exercise 5

1. Transferring the origin to $(-1, 2, -2)$ by the translation of axes, transformation equations are $x = X - 1, y = Y + 2, z = Z - 2$

The equation to the plane $x + 5y + 6z - 8 = 0$ can be transformed into

$$X - 1 + 5y + 10 + 6Z - 12 - 8 = 0 \qquad i.e., \ X + 5Y + 6Z - 11 = 0$$

For convenience, the transformed equation may be written as $x + 5y + 6z - 11 = 0$.

2. Proceed as in W. Ex. 2.

3. O is the origin. Let $A = (1, -2, 2)$, $B = (2, 2, 1)$, $C = (-2, 1, 2)$.

$\therefore$ d.cs. of $\overrightarrow{OA}$ are $\dfrac{1}{\sqrt{(1+4+4)}}, \dfrac{-2}{\sqrt{(1+4+4)}}, \dfrac{2}{\sqrt{(1+4+4)}}$

i.e., $\dfrac{1}{3}, \dfrac{-2}{3}, \dfrac{2}{3}$,

d.cs. of $\overrightarrow{OB}$ are $\dfrac{2}{3}, \dfrac{2}{3}, \dfrac{1}{3}$, and d.cs. of $\overrightarrow{OC}$ are $\dfrac{-2}{3}, \dfrac{1}{3}, \dfrac{2}{3}$

Let $l_1 = \dfrac{1}{3}, m_1 = \dfrac{-2}{3}, n_1 = \dfrac{2}{3}$; $l_2 = \dfrac{2}{3}, m_2 = \dfrac{2}{3}, n_2 = \dfrac{1}{3}$; $l_3 = \dfrac{-2}{3}, m_3 = \dfrac{1}{3}, n_3 = \dfrac{2}{3}$.

We observe that $l_1 l_2 + m_1 m_2 + n_1 n_2 = \dfrac{2}{9} - \dfrac{4}{9} + \dfrac{2}{9} = 0$,

$$l_2 l_3 + m_2 m_3 + n_2 n_3 = 0, \quad l_1 l_3 + m_1 m_3 + n_1 n_3 = 0$$

$\therefore$ $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ form a system of three mutually perpendicular lines.

Let $A \in \overrightarrow{OX}, B \in \overrightarrow{OY}$ and $C \in \overrightarrow{OZ}$

With rotation of axes through the same origin, the new frame of reference is *OXYZ*. The transformation equations are

$$x = l_1 X + l_2 Y + l_3 Z = \frac{1}{3}X + \frac{2}{3}Y - \frac{2}{3}Z$$

$$y = m_1 X + m_2 Y + m_3 Z = -\frac{2}{3}X + \frac{2}{3}Y + \frac{1}{3}Z$$

$$z = n_1 X + n_2 Y + n_3 Z = \frac{2}{3}X + \frac{1}{3}Y + \frac{2}{3}Z$$

i.e., $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$ Taking transpose of the matrices.

$$[xyz]=[XYZ]\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

We can write $4x^2+2y^2+3z^2+0xy+4yz-4zx$ in the form

$$[x\ y\ z]\begin{bmatrix} 4 & 0 & -2 \\ 0 & 2 & 2 \\ -2 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ which transforms to}$$

$$[X\ Y\ Z]\begin{bmatrix} \frac{1}{3} & \frac{-2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{-2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 4 & 0 & -2 \\ 0 & 2 & 2 \\ -2 & 2 & 3 \end{bmatrix}\times\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \\ \frac{-2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$=[X\ Y\ Z]\begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 1 \\ -4 & 2 & 4 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \\ \frac{-2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$=[X\ Y\ Z]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}=3Y^2+6Z^2.$$

Required transformed equation can be written as $3y^2+6z^2$.

4. Proceed as in Ex. 3.

5. Proceed as in Ex. 3 taking the d.rs. of the new axes as $1, 2, 2$; $2, -2, 1$; $2, 1, -2$ (with the same origin and rotation of axes).

6. Given $O_1=(1,-2,4)$ w.r.t. $OXYZ$ as the frame of reference, O_1XYZ is another frame of reference. Let the d.cs. of $\overrightarrow{O_1X}, \overrightarrow{O_1Y}, \overrightarrow{O_1Z}$ be $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$; w.r.t. $Oxyz$ as the frame of refenrece.

$$l_1=\frac{1}{3}, m_1=\frac{2}{3}, n_1=\frac{2}{3}; l_2=\frac{-14}{15}, m_2=\frac{2}{15}; l_3=\frac{2}{15}, m_3=\frac{-11}{15}, n_3=\frac{2}{3}$$

$P=(x,y,z)$ w.r.t. $Oxyz$ and $P=(X,Y,Z)$ w.r.t. O_1XYZ.

The transformation equations are

$$x = 1 + l_1X + l_2Y + l_3Z = 1 + \frac{1}{3}X - \frac{14}{15}Y + \frac{2}{15}Z,$$

$$y = -2 + m_1X + m_2Y + m_3Z = -2 + \frac{2}{3}X + \frac{2}{15}Y - \frac{11}{15}Z,$$

$$z = 4 + n_1X + n_2Y + n_3Z = 4 + \frac{2}{3}X + \frac{1}{3}Y + \frac{2}{3}Z,$$

w.r.t. $O\,xyz$ frame, equation to the plane is $2x - 3y - 2z + 5 = 0$.

Transformed equation w.r.t. O_1XYZ frame is

$$2\left(1 + \frac{1}{3}X - \frac{14}{15}Y + \frac{2}{15}Z\right) - 3\left(-2 + \frac{2}{3}X + \frac{2}{15}Y - \frac{11}{15}Z\right) \; -2\left(4 + \frac{2}{3}X + \frac{1}{3}Y + \frac{2}{3}Z\right) = 5$$

i.e., $\left(\frac{2}{3} - \frac{6}{3} - \frac{4}{3}\right)X + \left(-\frac{28}{15} - \frac{6}{15} - \frac{2}{3}\right)Y + \left(\frac{4}{15} + \frac{33}{15} - \frac{4}{3}\right)Z = 5$

i.e., $-\frac{8}{3}X - \frac{44}{15}Y + \frac{17}{15}Z = 5$ *i.e.*, $40X + 44Y - 17Z + 75 = 0$.

7. *O xyz, O XYZ* are two frames of reference connected by the equations

$$x = \frac{X}{\sqrt{3}} + \frac{Y}{\sqrt{2}} + \frac{Z}{\sqrt{6}}, \quad y = \frac{X}{\sqrt{3}} - \frac{2Z}{\sqrt{6}}, \quad z = \frac{X}{\sqrt{3}} - \frac{Y}{\sqrt{2}} + \frac{Z}{\sqrt{6}}.$$

Equation to the plane w.r.t. $O\,xyz$ frame is $x + y + z = 0$.

Equation to the plane w.r.t. $O\,XYZ$ frame is

$$\frac{X}{\sqrt{3}} + \frac{Y}{\sqrt{2}} + \frac{Z}{\sqrt{6}} + \frac{X}{\sqrt{3}} - \frac{2Z}{\sqrt{6}} + \frac{X}{\sqrt{3}} - \frac{Y}{\sqrt{2}} + \frac{Z}{\sqrt{6}} = 0 \; \textit{i.e.,} \; \sqrt{3}X = 0 \qquad \textit{i.e.,} \; X = 0 \; \ldots \; (1)$$

Equation to the surface w.r.t. $O\,xyz$ frame is $yz + zx + xy + 1 = 0$.

$\therefore$ Equation to the plane w.r.t. $O\,XYZ$ frame is

$$\left(\frac{X}{\sqrt{3}} - \frac{2Z}{\sqrt{6}}\right)\left(\frac{X}{\sqrt{3}} - \frac{Y}{\sqrt{2}} + \frac{Z}{\sqrt{6}}\right) + \left(\frac{X}{\sqrt{3}} - \frac{Y}{\sqrt{2}} + \frac{Z}{\sqrt{6}}\right)$$

$$\times\left(\frac{X}{\sqrt{3}} - \frac{Y}{\sqrt{2}} + \frac{Z}{\sqrt{6}}\right) + \left(\frac{X}{\sqrt{3}} + \frac{Y}{\sqrt{2}} + \frac{Z}{\sqrt{6}}\right)\left(\frac{X}{\sqrt{3}} - \frac{2Z}{\sqrt{6}}\right) + 1 = 0$$

i.e., $2X^2 - Y^2 - Z^2 + 2 = 0$ ... (2) (on simplification).

The section of the plane $x + y + z = 0$ with the surface

$yz + zx + xy + 1 = 0$ is the section of the plane $X = 0$ with the surface

$2X^2 - Y^2 - Z^2 + 2 = 0$ *i.e.*, $Y^2 + Z^2 = (\sqrt{2})^2$

Clearly it is a circle in YZ plane with radius $\sqrt{2}$.

8. Since l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 are the d.cs. of three mutually perpendicular rays $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$,

$l_1^2 + m_1^2 + n_1^2 = l_2^2 + m_2^2 + n_2^2 = l_3^2 + m_3^2 + n_3^2 = 1$,

$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$, $l_2 l_3 + m_2 m_3 + n_2 n_3 = 0$, $l_1 l_3 + m_1 m_3 + n_1 n_3 = 0$

d.rs. of $\overrightarrow{OP}$ are $l_1 + l_2 + l_3, m_1 + m_2 + m_3, n_1 + n_2 + n_3$.

If $(\overrightarrow{OP}, \overrightarrow{OA}) = \alpha$, $(\overrightarrow{OP}, \overrightarrow{OB}) = \beta$, $(\overrightarrow{OP}, \overrightarrow{OC}) = \gamma$, then

$$\cos\alpha = \frac{l_1(l_1 + l_2 + l_3) + m_1(m_1 + m_2 + m_3) + n_1(n_1 + n_2 + n_3)}{\sqrt{(l_1^2 + m_1^2 + n_1^2)}\,.\,\sqrt{(l_1 + l_2 + l_3)^2 + (m_1 + m_2 + m_3)^2 + (n_1 + n_2 + n_3)^2}}$$

$$= \frac{1 + 0 + 0}{1\sqrt{3 + 0 + 0}} = \frac{1}{\sqrt{3}}, \qquad \cos\alpha = \frac{1}{\sqrt{3}} \text{ and } \cos\gamma = \frac{1}{\sqrt{3}}. \quad \therefore \ \alpha = \beta = \gamma \cos^{-1}\frac{1}{\sqrt{3}}.$$

9. Given l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3, are three mutually perpendicular lines. So we can have $l_1 + l_2 + l_3$; $m_1 + m_2 + m_3$; $n_1 + n_2 + n_3$ as d.cs. of three mutually perpendicular lines.

$\therefore \ l_1^2 + m_1^2 + n_1^2 = l_2^2 + m_2^2 + n_2^2 = l_3^2 + m_3^2 + n_3^2 = 1$;

$l_1^2 + l_2^2 + l_3^2 = m_1^2 + m_2^2 + m_3^2 = n_1^2 + n_2^2 + n_3^2 = 1$;

$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \quad \ldots (1) \qquad l_2 l_3 + m_2 m_3 + n_2 n_3 = 0 \quad \ldots (2)$

$l_1 l_3 + m_1 m_3 + n_1 n_3 = 0 \quad \ldots (3) \qquad l_1 m_1 + l_2 m_2 + l_3 m_3 = 0 \quad \ldots (4)$

$m_1 n_1 + m_2 n_2 + m_3 n_3 = 0 \quad \ldots (5) \qquad l_1 n_1 + l_2 n_2 + l_3 n_3 = 0 \quad \ldots (6)$

From (1) and (3) : $\dfrac{l_1}{m_2 n_3 - m_3 n_2} = \dfrac{m_1}{n_2 l_3 - n_3 l_2} = \dfrac{n_1}{l_2 m_3 - l_3 m_2}$

$$= \frac{\sqrt{(l_1^2 + m_1^2 + n_1^2)}}{\sqrt{\left[\sum (m_2 n_3 - m_3 n_2)^2\right]}} = \frac{\pm 1}{\sin 90^\circ} = \pm \left(\begin{array}{l} \because \text{ lines with d.cs. } l_2, m_2, n_2; \\ l_3, m_3, n_3 \text{ are perpendicular} \end{array} \right)$$

Let $l_1 = m_2 n_3 - m_3 n_2$ ... (7) etc. by taking each ratio = + 1.

From (4) and (6) : $\dfrac{l_1}{m_2 n_3 - m_3 n_2} = \dfrac{l_2}{m_2 n_1 - m_1 n_3} = \dfrac{l_3}{m_1 n_2 - m_2 n_1} = 1$

(as above) ... (8)

(*i*) L.H.S. $= l_1 l_2 l_3 = (m_2 n_3 - m_3 n_2)(-m_2 m_3 - n_2 n_3)$ (using (7), (2))

$$= -m_2^2 + m_3 n_3 + m_2 m_3^2 n_2 - m_2 n_2 n_3^2 + n_2^2 m_2 n_3$$

$$= m_3 n_2 (m_3^2 - n_3^2) - m_3 n_3 (m_2^2 - n_2^2) = \text{R.H.S.}$$

(*ii*) L.H.S. $= \sum l_1 m_1 n_1 (m_2 n_3 + m_3 n_2)$

$$= m_1 n_1 (m_2^2 n_3^2 - m_3^2 n_2^2) + m_2 n_2 (m_3^2 n_1^2 - m_1^2 n_3^2) + m_3 n_3 (m_1^2 n_2^2 - m_2^2 n_1^2)$$

$$= m_1 m_2 n_3^2 (m_2 n_1 - m_1 n_2) + m_3^2 n_2 n_1 (m_2 n_1 - m_1 n_2)$$

$-m_3 n_3 (m_2 n_1 - m_1 n_2)(m_1 n_2 + m_2 n_1)$ (rearranging the first four terms)

$$= (m_1 n_2 - m_2 n_1)\left[m_1 m_3 n_2 n_3 + m_2 m_3 n_1 n_3 - m_1 m_2 n_3^2 - m_3^2 n_2 n_1 \right]$$

$l_3 = [m_3 n_1 - m_1 n_3][m_2 n_3 - m_3 n_2] \quad = l_3 l_2 l_1$ using (8) = R.H.S.

(*iii*) Given $\dfrac{a}{l_1} + \dfrac{b}{m_1} + \dfrac{c}{n_1} = 0$, $\dfrac{a}{l_2} + \dfrac{b}{m_2} + \dfrac{c}{n_2} = 0$.

$\therefore \quad a m_1 n_1 + b n_1 l_1 + c l_1 m_1 = 0 \qquad \text{... (9)}$

$a m_2 n_2 + b n_2 l_2 + c l_2 m_2 = 0 \qquad \text{... (10)}$

$\Rightarrow a(m_1 n_1 + m_2 n_2) + b(n_1 l_1 + n_2 l_2) + c(l_1 m_1 + l_2 m_2) = 0$

$\Rightarrow -a m_3 n_3 - b l_3 n_3 - c l_3 m_3$ using (5), (6), (4) $\Rightarrow \dfrac{a}{l_3} + \dfrac{b}{m_3} + \dfrac{c}{n_3} = 0$ (dividing $-l_3 m_3 n_3$)

From (9) and (10) :

$$\frac{a}{l_1 l_2 (n_1 m_2 - n_2 m_1)} = \frac{b}{m_1 m_2 (l_1 n_2 - l_2 n_1)} = \frac{c}{n_1 n_2 (m_1 l_2 - m_2 l_1)}$$

(By solving (2), (3) for l_3, m_3, n_3)

$$\Rightarrow \frac{a}{l_1 l_2 l_3} = \frac{b}{m_1 m_2 m_3} = \frac{c}{n_1 n_2 n_3} \Rightarrow a : b : c = l_1 l_2 l_3 : m_1 m_2 m_3 : n_1 n_2 n_3$$

UNIT - III

The Sphere

Exercise 6 (*a*)

1. (*ii*) Given sphere is $2x^2+2y^2+2z^2-2x+4y+2z+1=0$

i.e., $x^2+y^2+z^2-x+2y+z+\frac{1}{2}=0$.

$\therefore$ For the sphere centre $=\left(\frac{1}{2},-1,-\frac{1}{2}\right)$ and radius $=\sqrt{\left(\frac{1}{4}+1+\frac{1}{4}-\frac{1}{2}\right)}=1$.

2. (*ii*) Equation of the sphere through the non-coplanar points $(4,-1,2)$, $(0,-2,3)$, $(1,5,-1)$, $(2,0,1)$ is

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 16+1+4 & 4 & -1 & 2 & 1 \\ 0+4+9 & 0 & -2 & 3 & 1 \\ 1+25+1 & 1 & 5 & -1 & 1 \\ 4+0+1 & 2 & 0 & 1 & 1 \end{vmatrix}=0$$

i.e., $$\begin{vmatrix} x^2+y^2+z^2-5 & x-2 & y & z-1 & 0 \\ 16 & 2 & -1 & 1 & 0 \\ 8 & -2 & -2 & 2 & 0 \\ 22 & -1 & 5 & -2 & 0 \\ 5 & 2 & 0 & 1 & 1 \end{vmatrix}=0, \quad \begin{matrix} R_1-R_5 \\ R_2-R_5 \\ R_3-R_5 \\ R_4-R_5 \end{matrix}$$

i.e., $$\begin{vmatrix} x^2+y^2+z^2-5 & x-2 & y & z-1 \\ 16 & 2 & -1 & 1 \\ 8 & -2 & -2 & 2 \\ 22 & -1 & 5 & -2 \end{vmatrix}=0$$

i.e., $$\begin{vmatrix} x^2+y^2+z^2-5-4z+4 & x+z-3 & y+z-1 & z-1 \\ 12 & 3 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 30 & -3 & 3 & -2 \end{vmatrix}=0, \quad \begin{matrix} C_1-4C_4 \\ C_2+C_4 \\ C_3+C_4 \end{matrix}$$

i.e., $$-2\begin{vmatrix} x^2+y^2+z^2-4z-1 & x+z-3 & y+z-1 \\ 12 & 3 & 0 \\ 30 & -3 & 3 \end{vmatrix}=0$$

$$i.e., \begin{vmatrix} x^2+y^2+z^2-4z-1-4x-4z+12 & x+z-3 & y+z-1 \\ 0 & 3 & 0 \\ 42 & -3 & 3 \end{vmatrix} = 0, \ C_1 - 4C_2$$

i.e., $3\,[3\,(x^2+y^2+z^2-4x-8z+11)-42\,(y+z-1)]=0$

i.e., $x^2+y^2+z^2-4x-14y-22z+25=0$.

OR **:** Let the equation to a sphere be $x^2+y^2+z^2-2uz+2vy+2wz+d=0$ ... (1)

If (1) passes through $(4,-1,2), (0,-2,3), (1,56,-1), (2,0,1)$, then

$16+1+4+8u-2v+4w+d=0$

$\Rightarrow \quad 8u-2v+4w+d=-21$ ... (2) $\quad -4v+6w+d=-13$ ... (3)

$2u+10v-2w+d=-27$ ... (4) $\quad 4u+2w+d=-5$ ... (5)

Solving (2), (3), (4), (5) for u, v, w, d we can get the equation of the sphere (1) through the four given points.

3. (*i*) Since $\begin{vmatrix} 4 & 0 & 1 & 1 \\ 10 & -4 & 9 & 1 \\ -5 & 6 & -11 & 1 \\ 1 & 2 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -2 & -2 & 0 \\ 9 & -6 & 6 & 0 \\ -6 & 4 & -14 & 0 \\ 1 & 2 & 3 & 1 \end{vmatrix} \begin{matrix} R_1-R_4 \\ R_2-R_4 \\ R_3-R_4 \end{matrix}$

$$= \begin{vmatrix} 3 & -2 & -2 \\ 9 & -6 & 6 \\ -6 & 4 & -14 \end{vmatrix} = \begin{vmatrix} 3 & -2 & -2 \\ 0 & 0 & 12 \\ -6 & 4 & -14 \end{vmatrix} \begin{matrix} R_2-3R_1 \\ =12-12=0 \end{matrix}.$$

given points are coplanar and hence no sphere can be found through the four given points.

(*ii*) The equation to the sphere concentric with the sphere

$x^2+y^2+z^2-2x-2y-2z=1$ is of the form

$x^2+y^2+z^2-2x-2y-2z=k$, the radius of this sphere = 3

$=\sqrt{1+1+1+k}=3 \Rightarrow k+3=9 \Rightarrow k=6$

the required sphere is $x^2+y^2+z^2-2x-2y-2z=6$

(*iii*) The given spheres are $x^2+y^2+z^2+3x-25y+7z-11=0$ ... (1)

and $x^2+y^2+z^2+2x-3y+5z-7=0$ ... (2)

the equation to the sphere concentric with (1) is of the form

$x^2+y^2+z^2+3x-5y+7z+k=0$ ... (3)

(3) passes through $(1, 3/2, -5/2)$ which is the centre of (2)

$\Rightarrow 1+\frac{9}{4}+\frac{25}{4}+3-\frac{15}{2}-\frac{35}{2}+k=0 \Rightarrow k=25/2$

$\therefore$ The required sphere is $x^2+y^2+z^2+3x-5y+7z+\frac{25}{2}=0$.

4. (*i*) Given faces of tetrahedron are

$x=0$ ... (1), $y=0$ ... (2), $z=0$ ... (3), $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ....(4)

Solving (1), (2), (3) : $O=(0,0,0)$. Solving (2), (3), (4) : $A=(a,0,0)$.

Solving (1), (3) (4) : $B=(0,b,0)$. Solving (1), (2), (4) : $C=(0,0,c)$.

Now *OABC* is the tetrahedron having faces (1), (2), (3), (4). Let the sphere *z*, circumscribing the tetraheron *OABC*, be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$.

$$O\in z \Rightarrow d=0\,;\quad A\in z\Rightarrow a^2+2ua=0\Rightarrow u=-\frac{a}{2},$$
$$B\in z\Rightarrow b^2+2vb=0\Rightarrow v=-\frac{b}{2},\quad C\in z\Rightarrow c^2+2wc=0\Rightarrow w=-\frac{c}{2}.$$

$\therefore$ Equation to the sphere *z* is $x^2+y^2+z^2+2\left(\frac{-a}{2}\right)x+2\left(\frac{-b}{2}\right)y+2\left(\frac{-c}{2}\right)z+0=0$

i.e., $x^2+y^2+z^2-ax-by-cz=0$.

(*ii*) We have to find the equation to the sphere passing through the points given, as in examples in Ex. 2.

5. Let the required sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ ... (1)

Its centre is $(-u,-v,-w)$.

Since the centre lies on the line $5y+2z=0=2x-3y$, we have

$5v+2w=0$ ... (2) $2u-3v=0$ ... (3)

Since (1) passes through $(2,-1,-1),(0,-2,-4)$, we have

$4u-2v-2w+d=-6$ ... (4) $-4v-8w+d=-20$ ... (5)

By solving the equations (2), (3), (4), (5) we get *u, v, w, d* and hence the equation to the sphere (1) as $x^2+y^2+z^2-6x-4y+10z+12=0$

6. Let $P(x_1,y_1,z_1)$ be the point such that the sum of the squares of its distances from the planes $x+y+z=0$, $x-2y+z=0$, $x-z=0$ is k^2.

$\therefore\ \frac{(x_1+y_1+z_1)^2}{3}+\frac{(x_1-2y_1+z_1)^2}{6}+\frac{(x_1-z_1)^2}{2}=k^2$ *i.e.*, $x_1^2+y_1^2+z_1^2=k^2$.

$\therefore$ *P* lies on the sphere $x_1^2+y_1^2+z_1^2=k^2$.

7. Let $A=(x_2,y_2,z_2), B=(x_3,y_3,z_3)$ be the two fixed points.

Let $P=(x_1,y_1,z_1)$ such that $PA:PB=k^2:1$

$\therefore\ PA^2=k^2PB^2$

$\Rightarrow\ (x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2 = k^4[x_1-x_3)^2+(y_1-y_3)^2+(z_1-z_3)^2]$

$\Rightarrow\ k^4-1(x_1^2+y_1^2+z_1^2)-2(k^4x_3-x_2)x_1-2(k^4y_3-y_2)y_1$

$-2(k^4z_3-z_2)z_1-(x_2^2+y_2^2+z_2^2+k^4x_3^2+k^4y_3^2+k^4z_3^2)=0$

$\Rightarrow$ p lies on the sphere

$$x^2+y^2+z^2-\frac{2\,(k^4x_3-x_2)}{k^4-1}x-\frac{2\,(k^4y_3-y_2)}{k^4-1}y-\frac{2\,(k^4z_3-z_2)}{k^4-1}z$$

$$-\frac{(x_2^2+y_2^2+z_2^2+k^4x_3^2+k^4y_3^2+k^4z_3^2)}{k^4-1}=0$$

8. Let the equation to the sphere the

$$\sum x^2+y^2+z^2+2ux+2vy+2wz+d\ =0 \qquad \text{... (1)}$$

The given points are $A=(1,-4,3)$, $B=(1,-5,2)$, $C=(1,-3,0)$

$A \in \sum \quad \Rightarrow 2u-8v+6wz+d=-26$ ... (1)

$B \in \sum \quad \Rightarrow 2u-10v+4wz+d=-30$ ... (2)

$C \in \sum \quad \Rightarrow 2u-6v+d=-10$ ... (3)

The centre $(-u,-v,-w)$ lies on the plane $x+y+z=0$

$\Rightarrow u+v+w=0$ ... (4)

Solving (1), (2), (3), (4) we get u, v, w, d.

$$u=-2, v=7/2, w=-3/2, d=15.$$

9. Let the required sphere be $x^2+y^2+z^2+2uz+2vy+2wz+d=0$.

Since it passes through (1, 0, 0), (0, 1, 0), (0, 0, 1) we have

$$2u+d=-1, 2v+d=-1, 2w+d=-1 \ \ i.e., \quad u=-\left(\frac{1+d}{2}\right), v=-\left(\frac{1+d}{2}\right), w=-\left(\frac{1+d}{2}\right).$$

If r is the radius of the sphere, then

$$r^2=u^2+v^2+w^2-d=\frac{3}{4}(1+d)^2-d=\frac{3+3d^2+2d}{4}$$

$$=\frac{1}{4}\left[3\left(d^2+\frac{2}{3}d+1\right)\right]=\frac{1}{4}\left[3\left(d+\frac{1}{3}\right)^2+\frac{8}{3}\right].$$

For r to be least, r^2 must be least and d must be $-\dfrac{1}{3}$.

$\therefore\ r^2=\dfrac{1}{4}\left[0+\dfrac{8}{3}\right]=\dfrac{2}{3}$ when $d=-\dfrac{1}{3}$. $\qquad \therefore u=\dfrac{-1}{3}, v=\dfrac{-1}{3}, w=\dfrac{-1}{3}$.

$\therefore$ Equation to the sphere is $3\,(x^2+y^2+z^2)-2x-2y-2z-1=0$.

10. Let the sphere through O, A, B, C be $x^2+y^2+z^2+2ux+2vy+2wz=0$ ... (1)

Since $A \in x-$axis. from (1), $x=-2u \quad \therefore A=(-2u,0,0)$

Similarly $B=(0,-2v,0)$ and $C=(0,0,-2w)$

$\therefore$ Equation to the plane $\overline{ABC}$ is $\dfrac{x}{-2u}+\dfrac{y}{-2v}+\dfrac{z}{-2w}=1$

Since the plane passes through the point (a,b,c) we have

$$\frac{x}{-2u}+\frac{y}{-2v}+\frac{z}{-2w}=1 \Rightarrow \frac{a}{-u}+\frac{b}{-v}+\frac{c}{-w}=2$$

$\therefore$ The centre $(-u,-v,-w)$ of the sphere $OABC$ lies on $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$

11. Let $P(x_1, y_1, z_1)$ be the foot of the perpendicular from the origin to the plane through (a, b, c).

$\therefore$ d.rs. of $\overrightarrow{OP}$ are x_1-0, y_1-0, z_1-0 *i.e.,* x_1, y_1, z_1.

Equation to the plane through P and perpendicular to OP is

$x_1(x-x_1)+y_1(y-y_1)+z_1(z-z_1)-0$.

But it passes through (a, b, c).

$\therefore x_1(a-x_1)+y_1(b-y_1)+z_1(c-z_1)-0 \Rightarrow x_1^2+y_1^2+z_1^2-ax_1-by_1-cz_1=0$

$\therefore$ P lies on the sphere $x^2+y^2+z^2-ax-by-cz=0$.

12. Proceed as in Solved Example 7. Take $x_1=\frac{a}{4}, y_1=\frac{b}{4}, z_1=\frac{c}{4}$.

13. Let a sphere through O, A, B, C be $x^2+y^2+z^2-ax-by-cz=0$.

Let (x_1, y_1, z_1) be the centre of the above sphere

$\therefore x_1=\frac{a}{2}, y_1=\frac{b}{2}, z_1=\frac{c}{2}$ *i.e.,* $a=2x_1, b=2y_1, c=2z_1$.

But the volume of the the tetrahedron $OABC=\frac{1}{6}\left|[\overline{OA}\ \overline{OB}\ \overline{OC}]\right|$

$$\therefore \frac{1}{36}\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}^2=\frac{16\times 7}{9} \Rightarrow a^2b^2c^2=64\times 7$$

$\Rightarrow 64x_1^2y_1^2z_1^2=64\times 7 \Rightarrow x_1^2y_1^2z_1^2=7$

$\therefore$ Centre of the sphere through O, A, B, C lies on $x^2y^2z^2=7$.

EXERCISE 6 (*b*)

1. Let the equation of the required sphere through the given circle be

$x^2+y^2+z^2-9+\lambda(2x+3y+4z-5)=0$, λ being a fixed number.

Since this sphere passes through the point (1, 2, 3),

We have $1^2+2^2+3^2-9+\lambda(2.1+3.2+4.3-5)=0$ *i.e.,* $15\lambda+5=0$ *i.e.,* $\lambda=-\frac{1}{3}$.

$\therefore$ Equation to the required sphere is $x^2+y^2+z^2-9-\frac{1}{3}(2x+3y+4z-5)=0$

i.e., $3(x^2+y^2+z^2)-2x-3y-4z-22=0$.

2. (*i*) Given sphere is $x^2+y^2+z^2=a^2$ ... (1)

$\therefore$ Centre of the sphere $=O(0,0,0)$.

A plane cuts the sphere (1) in a circle with centre $P(\alpha,\beta,\gamma)$.

$\therefore$ $\overrightarrow{OP}$ is perpendicular to the plane

d.rs. of $\overrightarrow{OP}$ are $\alpha-0,\beta-0,\gamma-0$ *i.e.,* α,β,γ.

Also the plane passes through $P(\alpha,\beta,\gamma)$.

$\therefore$ Equation to the required plane is $\alpha(x-\alpha)+\beta(y-\beta)+\gamma(z-\gamma)=0$.

(*ii*) A circle is formed by a plane section to the sphere

$x^2+y^2+z^2-2x+4y-6z-3=0$ ... (1)

Centre of (1) $=C=(1,-2,3)$.

Let the centre of the required circle be M. $\therefore$ $M=(2,-3,4)$

$\therefore$ d.rs. of $\overrightarrow{CM}$ are $2-1,-3+2,4-3$ *i.e.,* $1,-1,1$

$\therefore$ Equation to the plane having the circle with centre M is

$1(x-2)-1(y+3)+1(z-4)=0$ *i.e.,* $x-y+z-9=0$ ... (2)

$\therefore$ Equation to the required circle is given by (1) and (2).

3. (*iii*) Given sphere is $x^2+y^2+z^2+2x-2y-4z-19=0$ ... (1)

and given plane is $x+2y+2z+7=0$ ... (2)

$\therefore$ Centre of (1) $=C=(-1,+1,2)$

and radius of (1) $=\sqrt{(1+1+4+19)}=5$.

Let M be the centre of the circle formed by the intersection of the plane (2) with the sphere (1). $\therefore$ $\overrightarrow{CM}$ is perpendicular to (2).

$\therefore$ Equation to $\overrightarrow{CM}$ is $\dfrac{x+1}{1}=\dfrac{y-1}{2}=\dfrac{z-2}{2}$ $(=t\text{ say})$ ... (3)

If $M=(t-1,2t+1,2t+2)$, then $M\in(2)$.

$\therefore$ $(t-1)+(2t+1)2+(2t+2)2+7=0 \Rightarrow t=-4/3$.

$\therefore$ $M=(-7/3,-5/3,-2/3)$. Also $CM=\dfrac{16}{9}+\dfrac{64}{9}+\dfrac{64}{9}=4$.

$\therefore$ radius of the circle $=\sqrt{5^2-CM^2}=\sqrt{25-16}=3$.

4. Given sphere is $x^2+y^2+z^2+2uz+2vy+2wz+d=0$ ... (1)

and given plane of intersection of (1) is $lx+my+nz=0$ ... (2)

Let C be the centre, r be the radius of the sphere and M be the centre of the circle formed by the intersection of (1) by (2).

$\therefore$ $C=(-u,-v,-w)$, $r_1=\sqrt{(u^2+v^2+w^2-d)}$ and

$$CM=\frac{|l(-u)+m(-v)+n(-w)|}{\sqrt{(l^2+m^2+n^2)}}.$$

But r is the radius of the circle. $\therefore\ CM^2 + r^2 = r_1^2$

$$\Rightarrow \frac{(lu+mv+nw)^2}{l^2+m^2+n^2} + r^2 = u^2+v^2+w^2-d$$

$$\Rightarrow (r^2+d)(l^2+m^2+n^2) = (l^2+m^2+n^2)(u^2+v^2+w^2) - (lu+mv+nw)^2$$

$$\Rightarrow (r^2+d)(l^2+m^2+n^2) = (mw-nv)^2 + (nu-lw)^2 + (lv+mu)^2$$

(using Lagrange's Identity)

5. (*i*) Consider the circle $x^2+y^2=a^2$, $z=0$ as the plane section of the sphere $x^2+y^2+z^2=a^2$ with the plane $Z=0$. Hence etc.

(*ii*) Centre of the sphere $x^2+y^2+z^2-2x+4y-6z+5=0$ is $(1,-2,+3)$. Hence etc. [Proceed as in (*i*)].

(*iii*) [Proceed as in (*i*)].

6. A circle is always given by a sphere through it and a plane containing it.

$\therefore$ The sphere through $(a,0,0), (0,a,0), (0,0,a)$ and (0, 0, 0)

(fourth point is taken arbitrarily so that the four points are non-coplanar) is

$x^2+y^2+z^2-ax-ay-az=0$ ... (1)

Its centre is $\left(\frac{a}{2},\frac{a}{2},\frac{a}{2}\right)$

The plane through $(a,0,0), (0,a,0), (0,0,a)$ is

$$\begin{vmatrix} x & y & z & 1 \\ a & 0 & 0 & 1 \\ 0 & a & 0 & 1 \\ 0 & 0 & a & 1 \end{vmatrix} = 0 \qquad i.e., \ x+y+z=a \qquad \text{... (2)}$$

(*OR* : Equation to the plane through $(a,0,0), (0,a,0), (0,0,a)$ is

$\frac{x}{a}+\frac{y}{a}+\frac{z}{a}=1$ *i.e.*, $x+y+z=a$)

$\therefore$ Required circle is given by the equations (1) and (2).

If C is the centre of (1) and M is the centre of the circle formed by the intersection of (2) with (1), the equations to $\overleftrightarrow{CM}$ are $\frac{x-a/2}{1} = \frac{y-a/2}{1} = \frac{z-a/2}{1}$ ($= t$ say)

Let M be $\left(t+\frac{a}{2}, t+\frac{a}{2}, t+\frac{a}{2}\right)$.

Since M lies on (2), $t+\frac{a}{2}, t+\frac{a}{2}, t+\frac{a}{2}=a \Rightarrow t=-\frac{a}{6}$

$\therefore \quad M=\left(-\frac{a}{6}+\frac{a}{2},-\frac{a}{6}+\frac{a}{2},-\frac{a}{6}+\frac{a}{2}\right)=\left(\frac{a}{3},\frac{a}{3},\frac{a}{3}\right)$

7. Proceed as in W. Ex. 2. Page 368

8. Given ends of diameter are (1, 2, 3) (2, 3, 4)

Equation of sphere with ends of its diameter $(x_1, y_1, z_1), (x_2, y_2, z_2)$ is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

$$\Rightarrow (x-1)(x-2)+(y-2)(y-3)+(z-3)(z-4)=0$$

$$\Rightarrow x^2-3x+2+y^2-5xy+6+z^2-7z+12=0$$

$$\Rightarrow x^2+y^2+z^2-3x-5y-7z+20=0$$

9. Equation to the sphere with $(2,-1,4)$ and $(-2,2,-2)$ as the ends of a diameter is $(x-2)(x+2)+(y+1)(y-2)+(z-4)(z+2)=0$ *i.e.,* $x^2+y^2+z^2-y-2z-14=0$ ... (1)

Sphere (1) with the plane $2x+y-z-3=0$ (2) is a circle. Its radius can be found out as in Ex. 2 and hence the area of the circle can be found out.

10. Given sphere is $x^2+y^2+z^2=a^2$ ... (1) Its centre is $O=(0,0,0)$.

Let (x_1, y_1, z_1) be the centre of the plane section of (1).

$\therefore$ Equation of this plane is $x_1(x-x_1)+y_1(y-y_1)+z_1(z-z_1)=0$.

If this plane passes through (α, β, γ), then $x_1(\alpha-x_1)+y_1(\beta-y_1)+z_1(\gamma-z_1)=0$.

$\therefore$ The centres of the plane sections of the sphere (1) lie on the sphere $x(\alpha-x)+y(\beta-y)+z(\gamma-z)=0$ *i.e.,* $x(x-\alpha)+y(y-\beta)+z(z-\gamma)=0$.

11. Given equations of the circle are $x^2+y^2+z^2+x+y+z-4=0$; $x+y+z=0$

Proceed as in Ex. 3

12. Let the equation to the plane $\overleftrightarrow{PQR}$ be $lx+my+nz=p$ ... (1)

Given sphere is $x^2+y^2+z^2=a^2$ ... (2)

For the sphere centre $O=(0,0,0)$ and radius $=a$.

$\overrightarrow{OP}, \overrightarrow{OQ}, \overrightarrow{OR}$ are three mutually perpendicular rays with d.cs.

$l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$.

Since $OP=OQ=OR=a$, we have

$$P=(l_1a, m_1a, n_1a),\ Q=(l_2a, m_2a, n_2a),\ R=(l_3a, m_3a, n_3a).$$

Since P, Q, R lie on the plane (1),

$$ll_1a+mm_1a+nn_1a=P \Rightarrow ll_1+mm_1+nn_1=\frac{p}{a} \quad \text{... (2)}$$

$$ll_2a + mm_2a + nn_2a = P \Rightarrow ll_2 + mm_2 + nn_2 = \frac{p}{a} \qquad \text{... (3)}$$

$$ll_3a + mm_3a + nn_3a = P \Rightarrow ll_3 + mm_3 + nn_3 = \frac{p}{a} \qquad \text{... (5)}$$

$l_1 \times (3) + l_2 \times (4) + l_3 \times (5)$:

$$l\,(l_1^2 + l_2^2 + l_3^3) + m\,(l_1m_1 + l_2m_2 + l_3m_3) + n\,(l_1n_1 + l_2n_2 + l_3n_3) = \frac{p}{a}(l_1 + l_2 + l_3).$$

$$\Rightarrow \quad l\,.\,1 + m\,.\,0 + n\,.\,0 = \frac{p}{a}(l_1 + l_2 + l_3) \Rightarrow l = \frac{p}{a}(l_1 + l_2 + l_3)$$

Similarly $m = \dfrac{p\,(m_1 + m_2 + m_3)}{a}$, $n = \dfrac{p\,(n_1 + n_2 + n_3)}{a}$

$\therefore$ From (1), equation to the plane $\overleftrightarrow{PQR}$ is

$$\frac{p}{a}(l_1 + l_2 + l_3)\,x + \frac{p}{a}(m_1 + m_2 + m_3)\,y + \frac{p}{a}(n_1 + n_2 + n_3)\,z = p$$

$$\Rightarrow (l_1 + l_2 + l_3)\,x + (m_1 + m_2 + m_3)\,y + (n_1 + n_2 + n_3)\,z = a \qquad \text{... (6)}$$

Let OM be perpendicular to the plane (2) and r be the radius of the circle through P, Q, R given by (2) and (6).

But $OM = \dfrac{|\,0 + 0 + 0 + a\,|}{[(l_1 + l_2 + l_3)^2 + (m_1 + m_2 + m_3)^2 + (n_1 + n_2 + n_3)^2]} = \dfrac{a}{\sqrt{3}}$.

$\therefore \dfrac{a^2}{3} + r^2 = (\text{ radius of the sphere })^2 = a^2 \quad \Rightarrow r = \sqrt{\dfrac{2}{3}}\,a$.

13. Given circle is $x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 = 0 = x^2 + y^2 + z^2 + 4x + 5y - 6z + 2$

i.e., $x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 = 0$ ... (1)

$3x + 4y - 5z - 3 = 0$ ... (2)

A sphere through the circle is

$x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 + \lambda\,(3x + 4y - 5z - 3) = 0$, λ being a fixed number.

Centre of the sphere is $\left(\dfrac{2 - 3\lambda}{2}, \dfrac{3 - 4\lambda}{2}, \dfrac{5\lambda - 4}{2}\right)$. If it lies on the plane $4x - 5y - z = 3$,

then $\dfrac{4\,(2 - 3\lambda)}{2} - \dfrac{5\,(3 - 3\lambda)}{2} - \dfrac{(5\lambda - 4)}{2} = 3 \Rightarrow \lambda = 3$

$\therefore$ Equation to the sphere with centre on the plane $4x - 5y - z = 3$ is

$x^2 + y^2 + z^2 + 7x + 9y - 11z - 1 = 0$.

14. (*i*) Given circle is $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$ ... (1) $x + y + z = 3$ ... (2)

A sphere through the circle is $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 + \lambda\,(x + y + z - 3) = 0$ ... (3)

λ being a fixed number.

$\therefore$ Centre of the sphere (3) $\left(\frac{4-\lambda}{2}, \frac{-\lambda-6}{2}, \frac{8-\lambda}{2}\right)$.

If the centre lies on (2), $\frac{4-\lambda}{2}+\frac{-\lambda-6}{2}+\frac{8-\lambda}{2}=3 \Rightarrow \lambda=0$.

$\therefore$ Centre of (3) = $(2,-3,4)$ and sphere (3) is same as sphere (1).
Also centre of (1) lies on (2).
$\therefore$ Circle given by (1), (2) is a great circle.
Note : If sphere (1) and (3) are different then the circle will be a small circle.

15. Given equations are $x^2+y^2+z^2=9$; $x-2y+2z-5=0$.

Equation of circle is $(x^2+y^2+z^2-9)+\lambda(x-2y+2z-5)=0$

$x^2+y^2+z^2+\lambda x-2\lambda y+2\lambda z-5\lambda-9=0$ ... (1)

centre $=\left(\frac{-\lambda}{2}, \lambda, -\lambda\right)$ By data (1) is a great circle

$\therefore$ Centre lies on the plane $x-2y+2z-5=0$

$(-\lambda/2)-2(\lambda)+2(-\lambda)-5=0$; $-\lambda-4\lambda-4\lambda-10=0$; $-9\lambda-10=0$; $\lambda=-10/9$.

sub in (1) $9(x^2+y^2+z^2)-10x+20y-20z-31=0$. Centre $=\left(\frac{5}{9}, \frac{-10}{9}, \frac{10}{9}\right)$

16. Given spheres are $x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$ ... (1)

$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$ ... (2)

$\therefore$ Plane of intersection of (1) and (2) is

$2(u_1-u_2)x+2(v_1-v_2)y+2(w_1-w_2)z+d_1-d_2=0$ ... (3)

Since (1) intersects (2) in a great circle, the centre of (2)

i.e., $(-u_2, v_2, -w_2)$ lies on (3).

$\therefore$ $2(u_1-u_2)(-u_2)+2(v_1-v_2)(-v_2)+2(w_1-w_2)(-w_2)+d_1-d_2=0$

i.e., $2(u_1u_2+v_1v_2+w_1w_2)-d_1=2(u_2^2+v_2^2+w_2^2)-d_2$

17. Given circles are $x^2+y^2+z^2-2x+3y+4z-5=0$, $5y+6z+1=0$ ... (1)

$x^2+y^2+z^2-3x-4y+5z-6=0$, $x+2y-7z=0$ ... (2)

A sphere through (1) is $x^2+y^2+z^2-2x+3y+4z-5+\lambda(5y+6z+1)=0$ ... (3)

and a sphere through (2) is $x^2+y^2+z^2-3x-4y+5z-6+\mu(x+2z-7z)=0$ ... (4)

for a fixed value λ and for a fixed value μ

(3) and (4) represent the same sphere if

$\frac{-2}{-3+\mu}=\frac{3+5\lambda}{-4+2\mu}=\frac{4+6\lambda}{5-7\mu}=\frac{-5+\lambda}{-6}=1$ *i.e.,* $\mu=1$, $\lambda=-1$.

Clearly all the ratios are equal for $\lambda = -1,\ \mu = 1$.

$\therefore$ Equation to the sphere on which circles (1) and lie is

$x^2+y^2+z^2-2x+3y+4z-5-5y-6z-1=0$ *i.e.,* $x^2+y^2+z^2-2x-2y-2z-6=0$.

[The sphere can also be got by putting $\mu = 1$ in (4)].

18. Given circles are $x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0,\ l_1x+m_1y+n_1z-p_1=0$ ... (1)

$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0,\ l_2x+m_2y+n_2z-p_2=0$ ... (2)

A sphere through the circle (1) is :

$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1+\lambda\,(l_1x+m_1y+n_1z-p_1)=0$ ... (3)

and a sphere through the circle (2) is

$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2+\mu\,(l_2x+m_2y+n_2z-p_2)=0$ ... (4)

for a fixed value λ and a fixed value μ (3) and (4) represent the same sphere if

$$\frac{2u_1+\lambda l_1}{2u_2+\mu l_2}=\frac{2v_1+\lambda m_1}{2v_2+\mu m_2}=\frac{2w_1+\lambda n_1}{2w_2+\mu n_2}=\frac{d_1+\lambda p_1}{d_2+\mu p_2}=1$$

$\Rightarrow l_1\lambda-l_2\mu+2\,(u_1-u_2)=0$ ... (5) $\quad m_1\lambda-m_2\mu+2\,(v_1-v_2)=0$ ... (6)

$n_1\lambda-n_2\mu+2\,(w_1-w_2)=0$ ... (7) $\quad p_1\lambda-p_2\mu+(d_2-d_1)=0$ ... (8)

Eliminating λ, μ from (5), (6), (7); (5), (6), (8); (5), (7), (8); (6), (7), (8), we get

$$\begin{vmatrix} l_1 & -l_2 & 2\,(u_1-u_2) \\ m_1 & -m_2 & 2\,(v_1-v_2) \\ n_1 & -n_2 & 2\,(w_1-w_2) \end{vmatrix}=0, \text{ etc. } \textit{i.e., } \begin{vmatrix} 2\,(u_1-u_2) & 2\,(v_1-v_2) & 2\,(w_1-w_2) \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0, \text{ etc.}$$

EXERCISE 6 (*c*)

1. (*i*) Given line is $\dfrac{x+3}{4}=\dfrac{y+4}{3}=\dfrac{z-8}{-5}$ ($= t$ say)

Any point on the line is $(4t-3, 3t-4, 8-5t)$.

If this point lies on the sphere $x^2+y^2+z^2+2x-10y=23$, then

$(4t-3)^2+(3t-4)^2+(8-5t)^2+2\,(4t-3)-10\,(3t-4)=23$

$\Rightarrow t^2-3t+2=0 \ \Rightarrow t=1, 2$.

$\therefore$ The point of intersection of the given line with the given sphere are $(1,-1,3), (5,2,-2)$.

2. Let a sphere be $S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0$

Let O be the fixed point and the axes be three mutually perpendicular lines.

If $S=0$ intersects axis $(y=0, z=0)$ then $x^2+2ux+d=0$ ... (1)

If $S=0$ meets x-axis in $(x_1, 0, 0)$ and $(x_2, 0, 0)$ then

$x_1+x_2=-2u, x_1x_2=d$. $\qquad \therefore\ (x_1-x_2)^2=4u^2-4d$.

Similarly we can have $(y_1-y_2)^2=4v^2-4d$ and $(z_1-z_2)^2=4w^2-4d$

$\therefore$ Sum of the squares of the intercepts on the axes

$= 4u^2 - 4d + 4v^2 - 4d + 4w^2 - 4d$,

a constant since for a given sphere u, v, w, d are constant.

3. Given sphere is $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$.

The tangent plane at $(-1, 4, -2)$ to the sphere is

$S_1 = x(-1) + y(4) + z(-2) - x + 1 - 2y - 8 + z - 2 - 3 = 0$ *i.e.,* $2x - 2y + z + 12 = 0$.

4. Given sphere is $3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0$ ... (1)

Its centre $= \left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right)$ and radius $= \sqrt{\left(\frac{1}{9} + \frac{1}{4} + \frac{4}{9} + 22\right)} = \frac{\sqrt{293}}{6}$ ... (2)

Given plane is $4x + 9y + 14z - 64 = 0$. Distance of $\left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right)$ from (2)

$$\frac{\left|4\left(\frac{1}{3}\right) + 9\left(\frac{1}{2}\right) + 14\left(\frac{2}{3}\right) - 64\right|}{\sqrt{(16 + 81 + 196)}} = \frac{\sqrt{293}}{6} \quad \text{... (3)}$$

$\therefore$ The given plane (2) touches the given sphere (1)

$\therefore$ Equation to the line through the centre and perpendicular to (2) is

$$\frac{x - 1/3}{4} = \frac{y - 1/2}{9} - \frac{z - 2/3}{14} \ (= t \text{ say})$$

A point on this line is $\left(4t + \frac{1}{3}, 9t + \frac{1}{2}, 14t + \frac{2}{3}\right)$.

If this is the point of contact of (2) with (1),

then $4\left(4t + \frac{1}{3}\right) + 9\left(9t + \frac{1}{2}\right) + 14\left(14t + \frac{2}{3}\right) - 64 = 0 \Rightarrow t = 1/6$.

$\therefore$ Point of contact = (1, 2, 3).

5. Given sphere is $x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$ ... (1)

Its centre is (1, 1, 1) and radius $= \sqrt{(1+1+1+6)} = 3$.

Given plane is $x + y + z - k\sqrt{3} = 0$. If this touches the sphere (1), then

$$\left|\frac{1(1) + 1(1) + 1(1) - \sqrt{3}k}{\sqrt{(1+1+1)}}\right| = 3$$

$\Rightarrow \left|3 - k\sqrt{3}\right| = 3\sqrt{3} \Rightarrow 3 - k\sqrt{3} = \pm 3\sqrt{3} \Rightarrow k = 3 \pm \sqrt{3}$.

6. Equation to the tangent plane at $(1, 1, -1)$ to the sphere

$S \equiv x^2 + y^2 + z^2 - x + 3y + 2z - 3 = 0$ is $S_1 = 0$

i.e., $x.1 + y.1 + z.(-1) - \frac{1}{2}x - \frac{1}{2}.1 + \frac{3}{2}y + \frac{3}{2}.1 + z - 1 - 3 = 0$

i.e., $x+5y-6=0$.

Let the sphere touching the given sphere at $(1,1,-1)$ and passing through the origin be

$x^2+y^2+z^2-x+3y+2z-3+\lambda(x+5y-6)=0$, for a fixed value, λ.

$\therefore\ 0+0+0-0+0+0-3+\lambda(0+0-6)=0 \Rightarrow \lambda=-1/2$

$\therefore$ Equation to the required sphere is $x^2+y^2+z^2-x+3y+2z-3-\dfrac{1}{2}(x+5y-6)=0$.

7. (*i*) Let the equation of the sphere passing through the circle $x^2+y^2+z^2=5$, $x+2y+3z-3=0$ and touchiong the plane $4x+3y-15=0$ be $x^2+y^2+z^2-5+\lambda(x+2y+3z-3)=0$

for a fixed value of λ. Its centre $=\left(-\dfrac{\lambda}{2},-\lambda,\dfrac{-3}{2}\lambda\right)$.

The perpendicular distance from $\left(-\dfrac{\lambda}{2},-\lambda,\dfrac{-3}{2}\lambda\right)$ to the plane $4x+3y-15=0$ is equal to the radius

$$\Rightarrow \left|\frac{4\left(-\frac{\lambda}{2}\right)+3(-\lambda)-15}{\sqrt{(16+9)}}\right|=\sqrt{\left(\frac{\lambda^2}{4}+\lambda^2+\frac{9}{4}\lambda^2+5+3\lambda\right)}$$

$\Rightarrow 5\lambda^2-6\lambda-8=0 \quad \Rightarrow \lambda=-\dfrac{4}{5}, 2$.

$\therefore$ Equations to the required spheres are $5(x^2+y^2+z^2)-4x-8y-12z-13=0$,

$x^2+y^2+z^2+2x+4y+6z-11=0$.

(*ii*), (*iii*) Proceed as in Ex. 7 (*i*).

8. The tangent line to a circle is the line of intersection of the tangent plane to the sphere at the given point and the plane of the circle.

Given sphere is $3x^2+3y^2+3z^2-2x-3y-4z-22=0$

i.e., $x^2+y^2+z^2-\dfrac{2}{3}x+y+\dfrac{4}{3}z-\dfrac{22}{3}=0$ ... (1)

and plane of the circle is $3x+4y+5z-26=0$ ... (2)

Equation to the tangent plane to the sphere at (1, 2, 3) is

$x.1+y.2+z.3-\dfrac{1}{3}x+\dfrac{1}{3}.1-\dfrac{1}{2}.y-\dfrac{1}{2}.2-\dfrac{2}{3}z-\dfrac{2}{3}.3-\dfrac{22}{3}=0$

$\Rightarrow 4x+9y+14z-64=0$ ... (3)

$\therefore$ Equations to the tangent line to the circle at (1, 2, 3) are

$4x+9y+14z-64=0=3x+4y+5z-26$.

Expressing the equation in symmetrical form, equations to the tangent line to the circle at (1, 2, 3) are $\dfrac{x-1}{1}=\dfrac{y-2}{-2}=\dfrac{z-3}{1}$.

9. (*i*) Given line is $\dfrac{x-5}{2}=\dfrac{y-1}{-2}=\dfrac{z-1}{1}$

i.e., $2x-2y-12=0$, $x-2z-3=0$; *i.e.,* $x+y-6=0$, $x-2z-3=0$.

Now proceed as in Ex. 9 (*ii*), worked below.

(*ii*) Let a plane through the line $x+y=6$, $x-2z=3$ and touching the sphere $x^2+y^2+z^2=9$ be $x+y-6+\lambda(x-2z-3)=0$

i.e., $x(\lambda+1)+y-2\lambda z-6-3\lambda=0$, for a fixed value of λ.

$$\therefore \quad \left|\frac{(\lambda+1)\,0+0-2\lambda\,(0)-6-3\lambda}{\sqrt{\{(\lambda+1)^2+1+(-2\lambda)^2\}}}\right|=3$$

$$\Rightarrow \quad 9\lambda^2+36\lambda+36=45\lambda^2+18\lambda+18 \Rightarrow \quad 2\lambda^2-\lambda-1=0 \quad \Rightarrow \quad \lambda=-\frac{1}{2}, 1.$$

$\therefore$ Equations to the required tangent planes are $2x+y-2z-9=0$, $x+2y+2z-9=0$.

10. Let the equation of the sphere having origin as its centre and touching the line

$\dfrac{x+1}{1}=\dfrac{y-2}{-2}=\dfrac{z+3}{2}$ ($=t$ say) be $x^2+y^2+z^2=a^2$ ... (1)

A point P on the line is $(t-1, 2-2t, 2t-3)$.

If P is the point of contact of the line with (1), then

$(t-1)^2+(2-2t)^2+(2t-3)^2=a^2 \quad \Rightarrow 9t^2-22t+14=a^2$ ... (2)

Also $(t-1)\,1+(2-2t)\,(-2)+(2t-3)\,2=0 \quad \Rightarrow t=\dfrac{11}{9}$

$\therefore$ From (2), $9\left(\dfrac{11}{9}\right)^2-22\left(\dfrac{11}{9}\right)+14=a^2 \quad \Rightarrow a^2=\dfrac{5}{9}$.

$\therefore$ Equation to the required sphere is $9(x^2+y^2+z^2)=5$.

11. Given sphere is $x^2+y^2+z^2=r^2$ ... (1)

Let the equation to the tangent plane to the sphere (1) and making intercepts *a, b, c* on the axes be $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$ ... (2)

$\therefore$ The perpendicular distance from the centre (0, 0, 0) of the pshere from (2) is r

$$\Rightarrow \left|\frac{0+0+0-1}{\sqrt{\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}\right)}}\right|=r \qquad \Rightarrow a^{-2}+b^{-2}+c^{-2}=r^{-2}.$$

12. Let the equation to a tangent plane to the sphere

$x^2+y^2+z^2-4x+2y-6z+5=0$ and parallel to the plane

$2x+2y-z=0$ be $2x+2y-z+\lambda=0$, for a fixed λ.

$$\therefore \quad \left|\frac{2\,(2)+2\,(-1)-3+\lambda}{\sqrt{(4+4+1)}}\right| = \sqrt{(4+1+9-5)} \Rightarrow \lambda^2-2\lambda-80=0 \Rightarrow \lambda=-8, 10.$$

$\therefore$ Equations to the required planes are $2x+2y-z-8=0$, $2x+2y-z+10=0$.

13. Let the sphere through the points $(4, 1, 0), (2, -3, 4), (1, 0, 0)$ be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad \ldots (1)$$

$\therefore$ $8u+2v+d=-17$ ... (2) $4u-6v+8w+d=-29$ ... (3)

$2u+d=-1$ ... (4) $\therefore$ From (4), $2u=-d-1$ ... (5)

From (2), $-4d-4+2v+d=-17$ *i.e.,* $2v=3d-13$ ... (6)

From (3), (5), (6) : $-2d-2-9d+39+8w+d=-29$ *i.e.,* $w=\dfrac{5d}{4}-\dfrac{33}{4}$ ... (7)

If (1) touches the plane $2x+2y-z-11=0$, then

$$\left|\frac{-2u-2v+w-11}{\sqrt{(4+4+1)}}\right| = \sqrt{(u^2+v^2+w^2-d)} \Rightarrow (2u+2v-w+11)^2 = 9\,(u^2+v^2+w^2-d)$$

$$\Rightarrow \frac{1}{9}\left(d+1-3d+13+\frac{5d}{4}-\frac{33}{4}-11\right)^2$$

$$=\left[\frac{d^2+2d+1}{4}+\frac{9d^2-78d+169}{4}+\frac{25d^2-330d-1089}{16}-d\right]$$

$$\Rightarrow 64d^2-664d+1720=0 \quad \Rightarrow 8d^2-83d+215=0$$

$$\Rightarrow (d-5)\,(8d-43)=0 \quad \Rightarrow d=5 \text{ or } \frac{43}{8}$$

$$\therefore \Rightarrow \left.\begin{array}{c} d=5 \\ 2u=-6 \\ 2v=2 \\ w=-2 \end{array}\right\} \Rightarrow \begin{array}{c} d=\dfrac{43}{8} \\ 2u=\dfrac{51}{8} \\ 2v=\dfrac{25}{8} \\ 2w=\dfrac{49}{16} \end{array}$$

Hence the required spheres are $x^2+y^2+z^2-6x+2y-4z+5=0$,

$16\,(x^2+y^2+z^2)-102x+50y-49z+86=0$.

14. (*i*) Let the equation to a sphere touching the axes be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad (d>0) \qquad \dots (1)$$

(1) touches x-axis $(y=0, z=0)$ at points whose x-coordinates are given by $x^2+2ux+d=0$.

Since the two values of x are equal, $4u^2-4d=0 \Rightarrow u^2=d$.

Similarly $v^2=d$ and $w^2=d$. $\quad \therefore u=v=w=\pm\sqrt{d}$.

But radius of (1) $=a$. $\quad \therefore u^2+v^2+w^2-d=a^2$.

$$\Rightarrow 2d=a^2 \Rightarrow d=\frac{a^2}{2} \qquad \therefore u=v=w=\pm\frac{a}{\sqrt{2}}$$

$\therefore$ Equations to the required sphere are

$$2(x^2+y^2+z^2)\pm 2\sqrt{2}\,ax\pm 2\sqrt{2}\,ay\pm 2\sqrt{2}\,az+a^2=0.$$

(*ii*) Let the required sphere be $x^2+y^2+z^2+2ux+2vz+2wz+d=0$ ... (1)

Since the required sphere is inscribed in the tetrahedron founded by the planes $x=0, y=0, z=0$ and $x+2y+2z-1=0$.

We have $\left|\dfrac{-u}{1}\right|=\sqrt{(u^2+v^2+w^2-d)} \Rightarrow v^2+w^2=d$.

Similarly $w^2+u^2=d, u^2+v^2=d$.

$$\therefore (u^2+v^2+w^2)=3d \Rightarrow u^2=\frac{d}{2} \Rightarrow u=\pm\frac{\sqrt{d}}{2}=\frac{d}{2}=v=w.$$

Also $\left|\dfrac{-u-v-2w-1}{3}\right|=\sqrt{(u^2+v^2+w^2-d)} \Rightarrow (u+2v+2w+1)=9u^2$

$$\Rightarrow (5u+1)^2=9u^2 \Rightarrow 16u^2+10u+1=0 \Rightarrow u=-\frac{1}{8},-\frac{1}{2}$$

$$\Rightarrow -\sqrt{\frac{d}{2}}=-\frac{1}{8},-\frac{1}{2}\left(\text{taking } u=-\sqrt{\frac{d}{2}} \text{ only}\right) \Rightarrow d=\frac{1}{32},\frac{1}{2}$$

$\therefore$ Equation to the required sphere is

$$x^2+y^2+z^2+2\left(-\frac{1}{8}\right)x+2\left(-\frac{1}{8}\right)y+2\left(-\frac{1}{8}\right)z+\frac{1}{32}=0$$

Since $d=\dfrac{1}{2}$ is inadmissible.

15. For the required sphere given tangent plane is $3x+2y-z+2=0$ at $A=(1,-2,1)$.

Let AB be a diameter of the required sphere

$\therefore$ Equation to $\overleftrightarrow{AB}$ is $\dfrac{x-1}{3}=\dfrac{y+2}{2}=\dfrac{z-1}{-1}$ $(=r$ say)

($\because \overleftrightarrow{AB}$ is perpendicular to the tangent plane at A)

Let $B = (3r+1, 2r-2, -r+1)$.

$\therefore$ Equation to the required sphere is

$$(x-1)(x-3r-1)+(y+2)(y-2r+2)+(z-1)(z+r-1)=0$$

But it passes through the origin.

$\therefore$ $(0-1)(0-3r-1)+(0+2)(0-2r+2)+(0-1)(0+r-1)=0 \Rightarrow -2r=-6 \qquad \Rightarrow r=3$.

$\therefore$ Equation to the required sphere is

$$(x-1)(x-10)+(y+2)(y-4)+(z-1)(z+2)=0$$

$$\Rightarrow x^2+y^2+z^2-11x-2y+z=0.$$

16. Given spheres are $x^2+y^2+z^2+2x-4y-6z+10=0$ ... (1)

$x^2+y^2+z^2-6x-4y-12z+40=0$ ... (2)

Let A, B be the centres and r_1, r_2 be the radii of the two spheres

(1), (2) respectively $A=(-1,2,3)$ $B=(3,2,6)$

$r_1=\sqrt{1+4+9-10}=2$; $r_2=\sqrt{9+4+36-40}=3$

$AB=\sqrt{(3+1)^2+(2-2)^2+(6-3)^2}=5$ $r_1+r_2=5$ $AB=r_1+r_2$

$\therefore$ Two two circles touch externally.

17. Given spheres are $x^2+y^2+z^2-25=0$ ... (1)

$x^2+y^2+z^2-24x-40y-18z+225=0$ ... (2)

Let A, B be the centres and r_1, r_2 be the radii of the spheres (1) and (2) respectively.

$\therefore$ $A=(0,0,0)$; $B=(12,20,9)$; $r_1=5$; $r_2=\sqrt{(144+400+81-225}=20$.

Now $AB=\sqrt{(144+400+81)}=25=r_1+r_2$.

$\therefore$ The two spheres touch externally, say, at P.

$\therefore$ $(P; A, B)=r_1:r_2=5:20=1:4$ $\therefore$ $P=\left(\frac{12}{5}, 4, \frac{9}{5}\right)$.

18. Given spheres are $x^2+y^2+z^2-64=0$ ... (1)

and $x^2+y^2+z^2-12x+4y-6z+48=0$ ... (2)

Let A,B be the centres and r_1, r_2 be the radii of (1) and (2) respectively.

$\therefore$ $A=(0,0,0)$; $r_1=8$, $B=(6,-2,3)$; $r_2=\sqrt{(36+4+9-48)}=1$

Now $AB=\sqrt{(36+4+9)}=7=r_1-r_2$.

$\therefore$ The two spheres touch internally, say, at P.

$\therefore$ $(P; A, B)=r_1:r_2=8:-1$ $\therefore$ $P=\left(\frac{48}{7}, \frac{-16}{7}, \frac{24}{7}\right)$.

19. Let a sphere be $x^2+y^2+z^2+2uz+2vy+2wz+d=0$ ... (1)

If (1) passes through $(0,0,c)$, then $c^2+2cw+d=0$ ... (2)

If (1) touches the plane $Z=0$, then $\left|\frac{-w}{1}\right|=\sqrt{(u^2+v^2+w^2-d)}$

$\Rightarrow\ u^2+v^2=d$ ... (3)

From (2) and (3), $c^2+2cw+u^2+v^2=0$ *i.e.*, $(-u^2)+(-v^2)-2c(-w)+c^2=0$

$\therefore$ Locus of the centre $(-u,-v,-w)$ of (1) is $x^2+y^2-2cz+c^2=0$.

20. Let a sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ ... (1)

If (1) passes through $(0,0,a)$, then $a^2+2aw+d=0$ ... (2)

If (1) intersects the line *x*-axis $(y=0,\ z=0)$,

then the points of intersection are given by $x^2+2ux+d=0$.
But (1) touches *x*-axis.

$\therefore\ 4u^2-4d=0\Rightarrow d=u^2$ ... (3)

$\therefore$ From (1) and (3), $a^2+2awu^2+u^2=0$ *i.e.*, $(-u)^2-2a(-w)+a^2=0$

$\therefore$ Locus of the centre $(-u,-v,-w)$ of (1) is $x^2-2az+a^2=0$.

EXERCISE 6 (*d*)

1. (*i*) Given equation of sphere is $x^2+y^2+z^2-2x+4y+6z-11=0$ ... (1)

Given point is $(3,-1,5)$ is equation of plane of contact is $s_1=0$

$xx_1+yy_1+zz_1-1(x+x_1)+2(y+y_1)+3(z+z_1)-11=0$

$3x-y+5z-(x+3)+2(y-1)+3(z+5)-11=0$

$3x-y+5z-x-3+2y-2+3z+15-11=0 \quad\Rightarrow\quad 2x+y+8z-1=0$

2. Given sphere is $x^2+y^2+z^2-2x+4y+6z-11=0$

Given point is $(0,-1,1)$ equation of polar plane is $S_1=0$.

$xx_1+yy_1+zz_1-(x+x_1)+2(y+y_1)+3(z+z_1)-11=0$

$0+y(-1)+z(1)-(x+0)+2(y-1)+3(z+1)-11=0$

$-y+z-x+2y-2+3z+3-11=0 \qquad -x+y+4z-10=0$

3. Given sphere is $x^2+y^2+z^2=9$. Given plane is $x-y+5z-3=0$.

Then pole of the plane is $\left(\frac{a^2l}{p},\frac{a^2m}{p},\frac{a^2n}{p}\right)\Rightarrow\left(\frac{9\,(1)}{3},\frac{9\,(-1)}{3},\frac{9\,(5)}{3}\right)=(3,-3,15)$

4. Given plane is $x - y - z + 9 = 0$ ... (1)

Let $P(x_1, y_1, z_1)$ be the pole of (1) w.r.t the sphere

$x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0$ ... (2)

$\therefore$ Polar plane of $P(x_1, y_1, z_1)$ w.r.t. (2) is

$xx_1 + yy_1 + zz_1 - (x + x_1) + 2(y + y_1) - 3(z + z_1) + 5 = 0$

$\Rightarrow (x_1 - 1)x + (y_1 + 2)y + (z_1 - 3)z - (x_1 - 2y_1 + 3z_1 - 5) = 0$ ... (3)

since (1) and (3) represent the same polar plane.

$$\frac{x_1 - 1}{1} = \frac{y_1 + 2}{-1} = \frac{z_1 - 3}{-1} = \frac{-(x_1 - 2y_1 + 3z_1 - 5)}{+9} = t$$

$\therefore (x_1, y_1, z_1) = (t + 1, -t - 2, -t + 3)\ -(x_1 - 2y_1 + 3z_1 - 5) = 9t$

$\Rightarrow -[(t + 1) - 2(-t - 2) + 3(-t + 3) - 5] = 9t$

$\Rightarrow -[t + 1 + 2t + 4 - 3t + 9 - 5] = 9t \quad \Rightarrow -9 = 9t \quad \Rightarrow t = -1$

sub $t = -1$ in $(t + 1, -t - 2, \ -t + 3)$ $\quad \therefore$ Pole $= (0, -1, 4)$

5. Given sphere $x^2 + y^2 + z^2 - 6x + 2y - 3z + 1 = 0$. Given points $P(1, -1, 2)$ $Q(-2, 0, 4)$

$$S_{12} = x_1x_2 + y_1y_2 + z_1z_2 - 3(x_1 + x_2) + (y_1 + y_2) - \frac{3}{2}(z_1 = z_2) = 1 = 0$$

$$\Rightarrow 1(-2) + (-1)(0) + (2)(4) - 3(1 - 2) + (-1 + 0) - \frac{3}{2}(2 + 4) + 1 = 0$$

$$\Rightarrow -2 + 0 + 8 + 3 - 1 - \frac{3}{2}(6) + 1 = 0 \quad \Rightarrow 12 - 3 - 9 = 0 \quad \Rightarrow 12 - 12 = 0 \quad \Rightarrow 0 = 0$$

$\therefore$ P and Q are conugate points.

6. Polar plane of the point $\mathrm{P}(-2,3,0)$ w.r.t. the sphere $x^2 + y^2 + z^2 + 4x - 5z - 3 = 0$ is

$$x(-2) + y(3) + z(0) + 2(x - 2) - \frac{5}{2}(z - 0) - 3 = 0 \ \ i.e.\ \ 6y - 5z - 14 = 0 \qquad .... (1)$$

Clearly it passes through $\mathrm{Q}(3,4,2)$.

Hence P, Q are conjugate points w.r.t. the plane (1).

7. Polar plane of the origin w.r.t the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is

$x.0 + y.0 + z.0 + u(x + 0) + v(y + 0) + w(z + 0) + d = 0$ *i.e.,* $uv + vy + wz + d = 0$ ... (1)

Centre and radius of the sphere $x^2 + y^2 + z^2 = a^2$ ... (2) are (0, 0, 0) and a.

Since (1) touches the sphere (2), $\dfrac{u.0 + v.0 + w.0 + d}{\sqrt{u^2 + v^2 + w^2}} = a$

$\Rightarrow \lambda^2 = a^2(u^2 + v^2 + w^2)$.

8. Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$. Let a sphere be $x^2 + y^2 + z^2 = a^2$ ... (1)

centre of (1) is $O = (0, 0, 0)$

$\therefore\ OP = \sqrt{(x_1^2 + y_1^2 + z_1^2)}$ and $OQ = \sqrt{(x_2^2 + y_2^2 + z_2^2)}$.

Polar plane of P w.r.t. (1) is $xx_1 + yy_1 + zz_1 - a^2 = 0$ ... (2)

$\therefore$ Distance of Q from the polar plane of P *i.e.*,

distance of Q from (2) is $\left| \dfrac{x_1x_2 + y_1y_2 + z_1z_2 - a^2}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}} \right|$

Similarly distance of P from the polar plane of Q is $\left| \dfrac{x_2x_1 + y_2y_1 + z_2z_1 + a^2}{\sqrt{(x_2^2 + y_2^2 + z_2^2)}} \right|$

$$\therefore\ \frac{\text{Distance of } Q \text{ from the polar plane of } P}{\text{Distance of } p \text{ from the polar plane of } Q} = \frac{\sqrt{(x_2^2 + y_2^2 + z_2^2)}}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}} = \frac{OQ}{OP}.$$

9. Given spheres are $x^2 + y^2 + z^2 - 2x - 6y - 7 = 0$ ... (1)

$x^2 + y^2 + z^2 - 4x - 4z = 0$ ... (2) Let $P = (x_1, y_1, z_1)$ be a point.

Polar plane of P w.r.t. (1) is $xx_1 + yy_1 + zz_1 - x - x_1 - 3y - 3y_1 - 7 = 0$

i.e., $(x_1 - 1)\,x + (y_1 - 3)\,y + z_1z - x_1 - 3y_1 - 7 = 0$ ... (3)

Polar plane of P w.r.t. (2) is $xx_1 + yy_1 + zz_1 - 2x - 2x_1 - 2z - 2z_1 = 0$

i.e., $(x_1 - 2)\,x + y_1y + (z_1 - 2)\,z - 2x_1 - 2z_1 = 0$ ... (4)

Polar planes of P w.r.t. (1) and (2) are perpendicular

$\Leftrightarrow\ (x_1 - 1)(x_1 - 2) + (y_1 - 3)\,y_1 + z_1\,(z_1 - 2) = 0 \ \Leftrightarrow\ x_1^2 + y_1^2 + z_1^2 - 2x_1 - 3y_1 - 2z_1 + 2 = 0$

$\therefore$ Locus of P is $x^2 + y^2 + z^2 - 3x - 3y - 2z + 2 = 0$.

10. (*ii*) Given line is $\dfrac{x+3}{1} = \dfrac{y+1}{2} = \dfrac{z-2}{3}$ ($= t$ say) ... (1)

Any point P on (1) is $(t - 3, 2t - 1, 3t + 2)$.

Given sphere is $x^2 + y^2 + z^2 = 1$ ... (2)

Polar plane of P w.r.t. (2) is $x\,(t - 3) + y\,(2t - 1) + z\,(3t + 2) = 1$

$\Rightarrow\ (-3x - y + 2z - 1) + 1\,(x + 2y + 3z) = 0$

$\therefore$ For all t, the polar plane of P passes through the line

$-3x - y + 2z - 1 = 0 = x + 2y + 3z$ ... (3)

Put $x = 0$. $\therefore\ -y + 2z = 1,\ \ 2y + 3z = 0 \quad \therefore\ z = \dfrac{2}{7},\ y = \dfrac{-3}{7}$.

Also d.rs. l, m, n of (3) are given by : $\left.\begin{array}{r}-3l-m+2n=0\\ l+2m+3n=0\end{array}\right\}\ \therefore\ \frac{l}{-7}=\frac{m}{11}=\frac{n}{-5}$.

$\therefore$ Equtions to the polar line (1) are $\frac{x}{7}=\frac{y+3/7}{-11}=\frac{z-2/7}{5}$ ... (4)

Since $(1)(7)+2(-11)+(3)(5)=0$, lines (1) and (4) are perpendicular.

Note: $\frac{x}{1}=\frac{7y+3}{-11}=\frac{7z-2}{5}$ is the conjugate line of $\frac{x+3}{1}=\frac{y+1}{2}=\frac{z-2}{3}$.

EXERCISE 6 (*e*)

1. Given spheres are $x^2+y^2+z^2+6y+2z+8=0$ ... (1)

and $x^2+y^2+z^2+6x+8y+4z+20=0$ ... (2)

(1) and (2) cut orthogonally if $\frac{0.6}{2}+\frac{6.8}{2}+\frac{2.4}{2}=8+20$ is true.

Since it is true (1) and (2) cut orthogonally.

2. Let the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ cut each of the spheres

$x^2+y^2+z^2-(a^2+b^2+c^2)=0$, $x^2+y^2+z^2+2ax-a^2=0$,

$x^2+y^2+z^2+2by-b^2=0$, $x^2+y^2+z^2+2cz-c^2=0$ orthogonally.

$\therefore$ $2u.\frac{0}{2}+2v.\frac{0}{2}+2w.\frac{0}{2}=d-a^2-b^2-c^2$ ... (1)

$2u.\frac{2a}{2}+2v.\frac{0}{2}+2w.\frac{0}{2}=d-a^2$ ... (2)

$2u.\frac{2a}{2}+2v.\frac{2b}{2}+2w.\frac{0}{2}=d-b^2$ ... (3)

$2u.\frac{0}{2}+2v.\frac{0}{2}+2w.\frac{2c}{2}=d-c^2$ ... (4)

$\Rightarrow$ $d=a^2-b^2-c^2$, $2u=\frac{b^2+c^2}{a}$, $2v=\frac{c^2+a^2}{b}$, $2w=\frac{a^2+b^2}{c}$. Hence etc.

3. A sphere through the circle is $x^2+y^2+z^2-2ax+r^2=0$, $z=0$ is

$x^2+y^2+z^2-2ax+r^2+\lambda z=0$ ... (1) for a fixed λ.

A sphere through the circle is $x^2+y^2+z^2-r^2=0$, $y=0$ is

$x^2+y^2+z^2-r^2+\mu y=0$ ... (2) for a fixed μ.

If (1) and (2) were to cut orthogonally,

then $(-2a)\ \frac{0}{2}+0.\frac{\mu}{2}+\lambda.\frac{0}{2}=r^2-r^2$ must be true. But it is true for any λ and μ.

$\therefore$ Any sphere passing through the first circle and any sphere passing through the second circle cut orthogonally.

4. Let $S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0$,

$S' \equiv x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0$,

$S'' \equiv x^2+y^2+z^2+2u''x+2v''y+2w''z+d''=0$.

Let $S=0$ cut orthogonally $S'=0$ and $S''=0$.

$\therefore\ 2uu'+2vv'+2ww'=d+d' \cdots$ (1) $2uu''+2vv''+2ww''=d+d'' \cdots$ (2)

Given sphere is $\lambda S'+\mu S''=0$, $(\lambda+\mu\neq 0)$.

i.e., $$x^2+y^2+z^2+\frac{2(\lambda u'+\mu u'')}{\lambda+\mu}x+\frac{2(\lambda v'+\mu v'')}{\lambda+\mu}y+\frac{2(\lambda w'+\mu w'')}{\lambda+\mu}z+\frac{\lambda d'+\mu d''}{\lambda+\mu}=0$$

If this sphere were to be cut orthogonally by $S=0$, then

$$2u\,.\frac{2(\lambda u'+\mu u'')}{2(\lambda+\mu)}+2v\,.\frac{2(\lambda v'+\mu v'')}{2(\lambda+\mu)}+2w\,.\frac{2(\lambda w'+\mu w'')}{2(\lambda+\mu)}=d+\frac{\lambda d'+\mu d''}{\lambda+\mu}$$ must be true.

$$\text{L.H.S.} = \frac{\lambda(2uu'+2vv'+2ww')+\mu\ (2uu''+2vv''+2ww'')}{\lambda+\mu}$$

$$=\frac{\lambda(d+d')+\mu(d+d'')}{\lambda+\mu}$$ using (1) and (2)

$$=\frac{(\lambda+\mu)\,d+\lambda d'+\mu d''}{\lambda+\mu}=d+\frac{\lambda d'+\mu d''}{\lambda+\mu} = \text{R.H.S.}$$

Hence the sphere $S=0$ cutting orthogonally $S'=0$ and $S''=0$, cuts orthogonally $\lambda S'+\mu S''=0$.

5. Let $S \equiv x^2+y^2+z^2-r^2=0$ be a sphere.

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ be conjugate points w.r.t. $S=0$.

$\therefore$ The polar plane of B *i.e.,* $xx_2+yy_2+zz_2-r^2=0$ passes through A.

$\therefore\ x_1x_2+y_1y_2+z_1z_2-r^2=0$ ... (1)

The sphere having AB as diameter is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

i.e., $x^2+y^2+z^2-(x_1+x_2)\,x-(y_1+y_2)\,y-(z_1+z_2)\,z+x_1x_2+y_1y_2+z_1z_2=0$ ... (2)

If the sphere (2) was to cut $S=0$ orthogonally, then

$$-(x_1+x_2)\,.\frac{0}{2}-(y_1+y_2)\,.\frac{0}{2}-(z_1+z_2)\,.\frac{0}{2}=x_1x_2+y_1y_2+z_1z_2-r^2$$ must be true.

It is clearly true from (1).

Hence the sphere on AB as diameter cuts the sphere $S=0$ orthogonally.

6. A plane through the points

$A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$ is $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}-1=0$ ... (I)

The sphere through $O, A, B, C,$ is $x^2+y^2+z^2-ax-by-cz=0$ ... (II)

$\therefore$ the circle through the points $A, B, C,$ is given by the equations I, II.

Any sphere through the circle given by I, II is

$$x^2+y^2+z^2-ax-by-cz-\lambda\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1\right)=0 \qquad \text{... (1)}$$

For different values of λ, (1) represents a system of spheres.

If (1) intersects the sphere $x^2+y^2+z^2-2ax-2by-2cz=0$ orthogonally, then

$$\left(-\frac{\lambda}{a}-a\right)\left(\frac{-2a}{2}\right)+\left(-\frac{\lambda}{b}-b\right).\left(\frac{-2b}{2}\right)+\left(-\frac{\lambda}{c}-c\right)\left(\frac{-2c}{2}\right)=\lambda+0$$

$$\Rightarrow 2\lambda=-(a^2+b^2+c^2) \quad \Rightarrow \lambda=-\frac{(a^2+b^2+c^2)}{2}.$$

7. Let $S\equiv x^2+y^2+z^2+2uz+2vy+2wz+d=0$ cut the spheres

$x^2+y^2+z^2+x-3z-2=0$, $x^2+y^2+z^2+\frac{1}{2}x+\frac{3}{2}y+2=0$ orthogonally.

$$\therefore \quad 2u.\frac{1}{2}+2v.\frac{0}{2}+2w\left(\frac{-3}{2}\right)=d-2 \quad i.e., \quad u.-3w=d-2 \qquad \text{... (1)}$$

and $2u.\frac{1}{4}+2v.\frac{3}{4}+2w.0=d+2$ $\quad$ *i.e,* $\quad u+3v=2d+4$ $\qquad$ ... (2)

Since $S=0$ passes through (0, 3, 0), $(-2,-1,-4)$

we have $6v+d=-9$ $\qquad$ ... (3) $\quad -4u-2v-8w+d=-21$ $\quad$ ... (4)

From (3), $6v=-d-9 \Rightarrow v=\frac{-d-9}{6}$.

$\therefore$ from (2), $u+3\left(\frac{-d-9}{6}\right)=2d+4 \Rightarrow u=\frac{5d}{2}+\frac{17}{2}$.

From (1), $3w=\frac{5d}{2}+\frac{17}{2}-d+2=\frac{3d}{2}+\frac{21}{2} \Rightarrow w=\frac{d}{2}+\frac{7}{2}$

$\therefore$ From(4), $-10d-34+\frac{d}{3}+3-4d-28+d=-21 \Rightarrow d=-3$

The required sphere is $x^2+y^2+z^2+2x-2y+4z-3=0$.

8. Equation to the sphere with $(1,2,-3)$, $(5,0,0)$ as the ends of a diameter is

$$(x-1)(x-5)+(y-2)(y-0)+(z+3)(z-1)=0$$

$$\Rightarrow x^2+y^2+z^2-6x-2y+2z+2=0 \qquad \text{... (1)}$$

Its centre $=(3,1,-1)$ and radius $r_1=\sqrt{(9+1+1-2)}=3$.

Given sphere is $x^2+y^2+z^2-2x-4y-6z+10=0$ $\qquad$ ... (2)

Its centre = (1, 2, 3) and radius $r_2=\sqrt{(1+4+9-10)}=2$.

$\therefore$ d = distance between the centres of the sphere (1) and (2) $=\sqrt{(4+1+16)}=\sqrt{21}$.

If θ is an angle of intersection of the spheres, then

$$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2\, r_1 r_2} = \frac{9+4-21}{2.3.2} = -\frac{2}{3} \quad \Rightarrow \cos^{-1}\left(-\frac{2}{3}\right).$$

9. Given spheres are $x^2 + y^2 + z^2 - 2x - 4y - 6z - 11 = 0$ ... (1)

and $x^2 + y^2 + z^2 + 2x - y + 12z + 5 = 0$ ... (2)

$\therefore$ radical plane of (1) and (2) is $4x + 3y + 12z + 16 = 0$ ... (3)

$\therefore$ Any sphere through the circle (1), (3) is

$x^2 + y^2 + z^2 - 2x - 4y - 11 + \lambda\,(4x + 3y + 12z + 16) = 0$.

If this sphere passes through $(1, -1, -1)$, then

$1 + 1 + 1 - 2 + 4 - 1 + \lambda\,(4 - 3 - 12 + 16) = 0$ *i.e.,* $\lambda = 6/5$.

$\therefore$ Equation to the required sphere is $5\,(x^2 + y^2 + z^2) + 14x - 2y + 72z + 41 = 0$.

10. Given spheres are $S = x^2 + y^2 + z^2 + 4y = 0$ ... (1)

$S' = x^2 + y^2 + z^2 + 3x - 2y + 8z + 6 = 0$ ... (2)

$S'' = x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0$ ... (3)

Radical plane of (1) and (2) is $S - S' = 0$,

$\Rightarrow -3x + 6y - 8z - 6 = 0 \qquad \Rightarrow 3x - 6y + 8z + 6 = 0$

Radical plane of (3) and (1) is $x - y + z + 1 = 0$

The Radical line is the line of intersection of the radical planes and its equation is given by $3x - 6y + 8z + 6 = 0 = x - y + z + 1$ or $x - y + z + 1 = 0 = 3x - 6y + 8z + 6$

11. Given spheres are $x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0$ ... (1)

$x^2 + y^2 + z^2 + 4x + 4z + 4 = 0$ ... (2)

$x^2 + y^2 + z^2 + x + 6y - 4z - 2 = 0$ ... (3)

Let $P\,(x_1, y_1, z_1)$ be a point such that the lengths of the tangents to the spheres (1), (2), (3) are equal.

$$\therefore \quad x_1^2 + y_1^2 + z_1^2 + 2x_1 + 2y_1 + 2z_1 + 2 = x_1^2 + y_1^2 + z_1^2 + 4x_1 + 4z_1 + 4$$
$$= x_1^2 + y_1^2 + z_1^2 + x_1 + 6y_1 - 4z_1 - 2$$

$\Rightarrow 2x_1 - 2y_1 + 2z_1 + 2 = 0,\ 3x_1 - 6y_1 + 8z_1 + 6 = 0,\ x_1 - 4y_1 + 6z_1 + 4 = 0$

$\therefore$ Locus of P is the intersection of the planes

$x - y + z + 1 = 0,\ 3x - 6y + 8z + 6 = 0,\ \ x - 4y + 6z + 4 = 0$

$\therefore$ Equations to the line of intersection of the first two planes are

$\dfrac{x-0}{2} = \dfrac{y-1}{5} = \dfrac{z-0}{3}$ (Expressing in symmetrical form)

Note that this line of intersection lies in the third plane.

$(\because 0 - 4\,(1) + 6\,(0) + 4 = 0,\ 2\,.\,1 + (-4)\,5 + 6\,(3) = 0)$

$\therefore$ Locus of P is the line intersection of the three planes and its equations are

$\frac{x}{2} = \frac{y-1}{5} = \frac{z}{3}$.

12. Find the radical plane of spheres (1), (2) and also the radical plane of the spheres (1), (3). Then express the line of intersection of the radical planes in the symmetrical form.

EXERXISE 6 (*f*)

1. Given coaxial system of spheres is

$$x^2 + y^2 + z^2 - 20x + 30y - 40z + 29 + \lambda\,(2x - 3y + 4z) = 0$$

Its centre $= \left(\frac{20-2\lambda}{2}, \frac{3\lambda-30}{2}, \frac{40-4\lambda}{2}\right)$ and

radius $= \sqrt{\left[\left(\frac{20-2\lambda}{2}\right)^2 + \left(\frac{3\lambda-30}{2}\right)^2 + \left(\frac{40-4\lambda}{2}\right)^2 - 29\right]}$

For limiting points of the system, radius = 0.

$\therefore \left(\frac{20-2\lambda}{2}\right)^2 + \left(\frac{3\lambda-30}{2}\right)^2 + \left(\frac{40-4\lambda}{2}\right)^2 - 29 = 0$

$\Rightarrow \lambda^2 - 20\lambda + 96 = 0 \quad \Rightarrow \lambda = 12, 8$

$\therefore$ Limiting points are $(-2,3,-4), (2,-3,4)$

2. Any sphere of the given system is $x^2 + y^2 + z^2 + 2\lambda x + \lambda y + 3\lambda z - (5 + 3\lambda) = 0$ ... (1)

Its centre is $\left(-\lambda, \frac{-\lambda}{2}, \frac{-3\lambda}{2}\right)$ and radius

$= \sqrt{\left[\left(\frac{-2\lambda}{2}\right)^2 + \left(\frac{-\lambda}{2}\right)^2 + \left(\frac{-3\lambda}{2}\right)^2 + 5 + 3\lambda\right]} = \frac{\sqrt{14\lambda^2 + 12\lambda + 20}}{2}$

Given plane is $3x + 4y - 15 = 0$.

By data, sphere (1) touches the plane (2) $\Rightarrow \frac{|-3\lambda - 2\lambda - 15|}{\sqrt{3^2 + 4^2}} = \frac{\sqrt{14\lambda^2 + 12\lambda + 20}}{2}$

$\Rightarrow 4\,(\lambda + 3)^2 = 14\lambda^2 + 12\lambda + 20 \quad \Rightarrow \lambda = 2, -4/5$

Putting these values of λ in (1) the required equations of the spheres are $x^2 + y^2 + z^2 + 4x + 2y + 6z - 11 = 0$ and $5\,(x^2 + y^2 + z^2) - 8x - 4y - 12z - 13 = 0$.

3. (*i*) Given spheres are $x^2 + y^2 + z^2 + 3x - 3y + 6 = 0$, $x^2 + y^2 + z^2 - 6y - 6z + 6 = 0$.

$\therefore$ Radical plane of the two spheres is $x + y + 2z = 0$ ... (1)

$\therefore$ Equation to the spheres of the coaxal system with (1) as radical plane is $x^2 + y^2 + z^2 + 3x - 3y + 6 + \lambda\,(x + y + 2z) = 0$.

Now proceed as in Ex. 1.

(*ii*) Given spheres are $S \equiv x^2+y^2+z^2-8x+2y-2z+32=0$ and

$S' \equiv x^2+y^2+z^2-7x+z+23=0$.

Equation of the radical plane π is given by $S-S'=0 \Rightarrow -x+2y-3z+9=0$

Equation of the co-axial system is given by $S+\lambda\pi=0$

$$\Rightarrow x^2+y^2+z^2-8x+2y-2z+32+\lambda\,(-x+2y-3z+9)=0$$

$$\Rightarrow x^2+y^2+z^2+(-8-\lambda)\,x+(2+2\lambda)\,y+(-2-3\lambda)\,z+9\lambda+32=0$$

$$\Rightarrow (x^2+y^2+z^2)+(-8-\lambda)\,x+(2+2\lambda)\,y+(-2-3\lambda)\,z+(9\lambda+32)=0$$

$$\text{centre} = \left(\frac{8+\lambda}{2}, \frac{-(2+2\lambda)}{2}, \frac{2+3\lambda}{2}\right),$$

$$\text{radius} = \sqrt{\left(\frac{8+\lambda}{2}\right)^2+\left(\frac{2+2\lambda}{2}\right)^2+\left(\frac{2+3\lambda}{2}\right)^2-(9\lambda+32)}$$

for limiting points, radius $=0$

$$\Rightarrow \sqrt{\left(\frac{8+\lambda}{2}\right)^2+\left(\frac{2+2\lambda}{2}\right)^2+\left(\frac{2+3\lambda}{2}\right)^2-(9\lambda+32)}=0$$

$$\Rightarrow (8+\lambda)^2+(2+2\lambda)^2+(2+3\lambda)^2-4\,(9\lambda+32)=0$$

$$\Rightarrow \lambda^2+16\lambda+64+4\lambda^2+4+8\lambda+9\lambda^2+4+12\lambda-36\lambda-128=0$$

$$\Rightarrow 14\lambda^2+72-128=0 \Rightarrow 14\lambda^2-56=0 \Rightarrow \lambda^2-4=0 \Rightarrow \lambda=\pm 2$$

Case - (*i*) : $\lambda=2$, limiting point is $(5,-3,4)$

Case - (*ii*) : $\lambda=-2$, limiting point is $(3,1,-2)$

5. Let $A\,(a,b,c)$ and $B\,(p,q,r)$ be two fixed points.

Let $P\,(x_1,y_1,z_1)$ be a point such that $PA=nPB$.

$$PA=nPB \Leftrightarrow PA^2=n^2PB^2$$

$$\Leftrightarrow (x_1-a)^2+(y_1-b)^2+(z_1-c)^2=n^2[(x_1-p)^2+(y_1-q)^2+(z_1-r)^2]$$

$$\Leftrightarrow (x_1^2+y_1^2+z_1^2-2ax_1-2by_1-2cz_1+a^2+b^2+c^2)$$
$$-n^2\,(x_1^2+y_1^2+z_1^2-2px_1-2qy_1-2rz_1+p^2+q^2+r^2)=0$$

$\therefore$ Locus of P is $(x^2+y^2+z^2-2ax-2by-2cz+a^2+b^2+c^2)$
$$-n^2\,(x^2+y^2+z^2-2px-2qy-2rz+p^2+q^2+r^2)=0.$$

It is the form $\lambda_1S+\lambda_2S'=0 \quad (\lambda_1=1, \lambda_2=-n^2)$

where $S=0, S'=0$ are two spheres.

Clearly the locus represents a coaxal system of spheres with radical plane $S-S'=0$.

6. For a coaxal system of spheres limiting points are $(-1, 2, 1)$ and $(-2, 1, -1)$.

$\therefore$ Equation to the limiting point $(-1, 2, 1)$ is $(x+1)^2+(y-2)^2+(z-1)^2=0$

and equation to the limiting point $(-2,1,-1)$ is

$(x+2)^2+(y-1)^2+(z+1)^2=0 \Rightarrow x^2+y^2+z^2+4x-2y+2z+6=0$.

$\therefore$ Radical plane of the coaxal system is $x+y+2z=0$.

7. Let a sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ ... (1)

If it cuts orthogonally the spheres

$x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0$ ... (2)

and $x^2+y^2+z^2+2u''x+2v''y+2w''z+d''=0$ ... (3)

then $2uu'+2vv'+2ww'=d+d'$ ... (4)

$2uu''+2vv''+2ww''=d+d''$ ... (5)

Equation to the coaxal system of spheres with (2) and (3) as its members is

$$x^2+y^2+z^2+2u'x+2v'y+2w'z+d'+\lambda\,[2\,(u'-u'')\,x$$
$$+2\,(v'-v'')\,y+2\,(w'-w'')\,z+(d'-d'')]=0.$$

If (1) cuts the system orthogonally, then

$2\,[u'+\lambda\,(u'-u'')]u+2\,[v'+\lambda\,(v'-v'')]\,v+2\,[w'+\lambda\,(w'-w'')]=d+d'+\lambda\,(d'-d'')$

must be true.

L.H.S. $=2uu'+2vv'+2ww'+\lambda\,[(2uu'+2vv'+2ww')-(2uu'+2vv'+2ww')]$

$=d+d'+\lambda\,[d+d'-d+d'']\;=d+d'+\lambda\,[d'-d'']\;=$ R. H. S.

Hence the sphere which intersects two spheres orthogonally will intersect every member of the coaxal system determined by them orthogonally.

8. Given radical plane is $x+y+2z=0$ ... (1) One limiting point is $(-2, 1, -1)$.

Take $(-2, 1, -1)$ as centre and radius as zero.

Then the equation of the sphere is $(x+2)^2+(y-1)^2+(z+1)^2=0$

$\Rightarrow x^2+y^2+z^2+4x-2y+2z+6=0$ ... (2)

Equation of the co-axial system of spheres is $S+\lambda\pi=0$

$\Rightarrow x^2+y^2+z^2+4x-2y+2z+6+\lambda\,(x+y+2z)=0$

$\Rightarrow x^2+y^2+z^2+(4+\lambda)\,x+(\lambda-2)y+(2\lambda+2)\,z+6=0$

centre $=\left(\dfrac{-(4+\lambda)}{2}, \dfrac{-(\lambda-2)}{2}, \dfrac{-(2\lambda+2)}{2}\right)$

radius $=\sqrt{\left(\dfrac{-(4+\lambda)}{2}\right)^2+\left(\dfrac{(\lambda-2)}{2}\right)^2+\left(\dfrac{(2\lambda+2)}{2}\right)^2-6}$

for limiting points radius = 0

$$\left(\frac{4+\lambda}{2}\right)^2+\left(\frac{\lambda-2}{2}\right)^2+\left(\frac{2\lambda+2}{2}\right)^2-6=0$$

$$(4+\lambda)^2+(\lambda-2)^2+(2\lambda+2)^2-24=0$$

$$\Rightarrow\ \lambda^2+16+8\lambda+\lambda^2-4\lambda+4+4\lambda^2+4+8\lambda-24=0$$

$$\Rightarrow\ 6\lambda^2+12\lambda=0 \qquad 6\lambda(\lambda+2)=0$$

$$\Rightarrow \lambda=0, \lambda=-2$$

Case (*i*) : When $\lambda=0$. Limiting point is $(-2,1,-1)$

Case (*ii*) : When $\lambda=-2$. Limiting point is $(-1,2,1)$

9. Let the four spheres be $\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4$ with equations

$$(x-a)^2+(y-0)^2+(z-0)^2=r_1^2$$

i.e., $x^2+y^2+z^2-2ax+a^2-r_1^2=0$ .... (1)

$x^2+y^2+z^2-2by+b^2-r_2^2=0$ .... (2)

$x^2+y^2+z^2-2cz+c^2-r_3^2=0$ .... (3) and $x^2+y^2+z^2-ax-by-cz=0$ .... (4)

$\therefore$ Radical planes of $\Sigma_1,\Sigma_2;\Sigma_1,\Sigma_3;\Sigma_1\,\Sigma_4$ are

respectively : $2ax-2by+b^2-a^2+r_1^2-r_2^2=0$ .... (5)

$2ax-2cy+c^2-a^2+r_1^2-r_3^2=0$ .... (6)

$ax-by-cz+r_1^2-a^2=0$ .... (7)

Eliminating x,y,z from (5), (6), and (7),

$$\begin{vmatrix} 2a & -2b & 0 \\ 2a & 0 & -2c \\ a & -b & -c \end{vmatrix} = 4abc\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -4abc \neq 0$$

$\therefore$ Radical planes (5), (6), (7) have a unique common point (x,y,z) and the spheres represented by (1), (2), (3) and (4) is the radical centre.

Now $(5)+(6)-3(7)$

$$\Rightarrow 4ax-3ax-2by+3by-2cz+3cz\ \ +b^2-2a^2+3a^2+c^2+2r_1^2-3r_1^2-r_2^2-r_3^2=0$$

$$\Rightarrow ax+by+cz+a^2+b^2+c^2-(r_1^2+r_2^2+r_3^2)=0\ \Rightarrow ax+by+cz=0$$

$\therefore$ (x,y,z) should satisfy the equation.

$\therefore$ The radical centre of the four spheres lies on $ax+by+z=0$

UNIT - IV

THE CONE

Exercise 7 (*a*)

1. If l,m,n are the d.r.'s of the generators lying on the cone.

Then the equation to the generator passing through the origin $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$

Eliminating l,m,n from $3l^2-4m^2+5n^2=0$

The equation of the cone is $3x^2-4y^2+5z^2=0$

2. Equation to the generator through (α,β,γ) is $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$

$\therefore\ l,m,n$ satisfy the equation $al^2+bm^2+cn^2=0$

$\Rightarrow a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2=0$ which is the required cone.

Exercise 7 (*b*)

1. The equation tot he cone passing through the three axes can be taken in form

$fyz+gzx+hxy=0$ .... (1) The line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ lies on (1).

$\Leftrightarrow$ d.r.'s $(1,-2,3)$ satisfy the equation (1).

$\Rightarrow f(-2)(3)+g(3)(1)+h(1)(-2)=0 \Rightarrow 6f-3g+2h=0$ .... (2)

Similarly the line $\frac{x}{2}=\frac{y}{1}=\frac{z}{1}$ lies on (1)

$\Leftrightarrow f(1)(1)+g(1)(2)+h(2)(1)=0 \Rightarrow f+2g+2h=0$ .... (3)

Solving (2) and (3) $\frac{f}{-6-4}=\frac{g}{2-12}=\frac{h}{12+3}\Rightarrow\frac{f}{-10}=\frac{g}{-10}=\frac{h}{15}\Rightarrow\frac{f}{2}=\frac{g}{2}=\frac{h}{-3}$

$\therefore$ Required cone is $2yz+2zx-3xy=0$

2. Equation to the cone containing the coordinate axis be $ayz+bzx+cxy=0$.

The lines $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$, $\frac{x}{-1}=\frac{y}{1}=\frac{z}{1}$ and $\frac{x}{5}=\frac{y}{4}=\frac{z}{1}$ are the generators

$\Leftrightarrow a(-2)(3)+b(3)(1)+c(1)(-2)=0\Rightarrow 6a-3b+2c=0$ I

$a(1)(1)+b(1)(-1)+c(-1)(1)=0\Rightarrow a-b-c=0$ II

$a(4)(1)+b(1)(5)+c(5)(4)=0\Rightarrow 4a+5b+20c=0$ III

Solving I and II : $\frac{a}{3+2}=\frac{b}{2+6}=\frac{c}{-6+3}\Rightarrow\frac{a}{5}=\frac{b}{8}=\frac{c}{-3}$

$\therefore$ Required cone is $5yz+8zx-3xy=0$.

3. Equation to the cone containing the three axis as generators is

$ayz + bzx + cxy = 0$ ..... (1) Also the lines $\frac{x}{3} = \frac{y}{5} = \frac{z}{1}$ ..... (2)

$\frac{x}{1} = \frac{y}{-1} = \frac{z}{2}$ ..... (3) and $\frac{x}{-11} = \frac{y}{5} = \frac{z}{8}$ ..... (4)

are generators of the cone $\Leftrightarrow$

$a(5)(1) + b(1)(3) + c(3)(5) = 0 \Rightarrow 5a + 3b + 15c = 0$ I

$a(-1)(2) + b(2)(1) + c(1)(-1) = 0 \Rightarrow 2a - 2b - c = 0$ II

and $a(5)(8) + b(8)(-11) + c(-11)(5) = 0 \Rightarrow 40a - 88b - 55c = 0$ III

Solving I and II we get $\frac{a}{3+30} = \frac{b}{30-5} = \frac{c}{-10-6} \Rightarrow \frac{a}{33} = \frac{b}{25} = \frac{c}{-16}$

$\therefore$ Equation to the cone is $33yz + 25zx - 16xy = 0$

Exercise 7 (*c*)

1. (*a*) Let a line of intersection be $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

It lies on the cone and on the plane $\Rightarrow l^2 + 9m^2 - 4n^2 = 0$ and $l + 3m - 2n = 0$

eliminating l, $(2n-3m)^2 + 9m^2 - 4n^2 = 0 \Rightarrow m(3m-2n) = 0 \Rightarrow m = 0, 3m - 2n = 0$

(*i*) Let $m = 0$, then $l - 2n = 0 \Rightarrow \frac{l}{2} = \frac{m}{0} = \frac{n}{1}$. $\therefore$ Equation to the line is $\frac{x}{2} = \frac{y}{0} = \frac{z}{1}$

(*ii*) Let $3m - 2n = 0$, then $l = 0 \Rightarrow \frac{l}{0} = \frac{m}{2} = \frac{n}{3}$. $\therefore$ The line is $\frac{x}{0} = \frac{y}{2} = \frac{z}{3}$

(*b*) Let the line be $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$. It lies on the cone and on the plane

$\Leftrightarrow 3mn + 14nl - 30lm = 0$, $l + 7m - 5n = 0$.

Eliminating l, $3mn + 14n(5n - 7m) - 30m(5n - 7m)$

$\Rightarrow 6m^2 - 7mn + 2n^2 = 0 \Rightarrow (2m - n)(3m - 2n) = 0 \Rightarrow 2m = n$ or $3m = 2n$

(*i*) Let $2m = n$ then $l + 7m - 5(2m) = 0 \Rightarrow l - 3m = 0 \Rightarrow \frac{l}{3} = \frac{m}{1}$

$\therefore \frac{l}{3} = \frac{m}{1} = \frac{n}{2}$. $\therefore$ Equation to the line is $\frac{x}{3} = \frac{y}{1} = \frac{n}{2}$

(*ii*) Solving $l + 7m - 5n = 0$.

$0.l + 3m - 2n = 0 = \frac{l}{1} = \frac{m}{2} = \frac{n}{3}$ $\therefore$ Equation to the line is $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

2 (*a*).Let the cone intersect the plane along the generator $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

$\therefore (l, m, n)$ satisfy the cone and the plane

$\therefore\ 3l + m + 5n = 0$ ... (1) $\Rightarrow m = -(3l + 5n)$ ... (2)

Also $6mn - 2nl + 5lm = 0$ ... (3)

Substituting from (2): $-6(3l + 5n)n - 2nl - 5l(3l + 5n) = 0$

$\Rightarrow l^2 + 3ln + 2n^2 = 0$ $\Rightarrow (l + n)(l + 2n) = 0$

$\Rightarrow l + n = 0$ ... (4) and $l + 2n = 0$ ... (5)

Solving (1) and (4): $l + 0.m + n = 0$, $3l + m + 5n = 0$

$$\therefore \frac{l}{0-1} = \frac{m}{3-5} = \frac{n}{1-0} \Rightarrow \frac{l}{-1} = \frac{m}{-2} = \frac{n}{1} \quad \text{... I}$$

Solving (1) and (5) $l + 0.m + 2n = 0$, $3l + m + 5n = 0$

$$\frac{l}{0-2} = \frac{m}{6-5} = \frac{n}{1-0} \Rightarrow \frac{l}{-2} = \frac{m}{1} = \frac{n}{1} \quad \text{... II}$$

Hence the d.r's of the two lines are $(-1, -2, 1)$ and $(-2, 1, 1)$. If θ is If θ is the angle between the lines then

$$\cos\theta = \frac{(-1)(-2) - 2(1) + 1(1)}{\sqrt{1+4+1}\ \sqrt{4+1+1}} = \frac{1}{6} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{6}\right)$$

(*b*) Let the line of intersection be $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

$\therefore\ l + m + n = 0$ ... (1) and $6lm + 3mn - 2nl = 0$ ... (2)

Substituting from (1) $l = -(m + n)$ in (2)

$-6m(m + n) + 3mn + 2n(m + n) = 0$

$\Rightarrow 6m^2 + mn - 2n^2 = 0$ and $\Rightarrow (2m - n)(3m + 2n) = 0$

$\Rightarrow 2m - n = 0$ ... (3) and $3m + 2n = 0$ ... (4)

Solving (1) and (3) *i.e.,* $l + m + n = 0$, $0.l + 2m - n = 0$

$$\frac{l}{-1-2} = \frac{m}{0+1} = \frac{n}{2-0} \Rightarrow \frac{l}{-3} = \frac{m}{1} = \frac{n}{2}$$

Solving (1) and (4): *i.e.,* $l + m + n = 0$, $0.l + 3m + 2n = 0$

$$\frac{l}{2-3} = \frac{m}{0-2} = \frac{n}{3-0} \Rightarrow \frac{l}{-1} = \frac{m}{-2} = \frac{n}{3}$$

$\therefore$ The d.r's of lines of intersection are $(-3, 1, 2)$ and $(-1, -2, 3)$.

Lines of intersection are $\frac{x}{-3} = \frac{y}{1} = \frac{z}{2}$ and $\frac{x}{-1} = \frac{y}{-2} = \frac{z}{3}$

If θ is the angle between the lines

$$\cos\theta = \frac{(-3)(-1) + 1(-2) + 2(3)}{\sqrt{9+1+4}\ \sqrt{1+4+9}} = \frac{7}{14} = \frac{1}{2} \Rightarrow \theta = 60^\circ$$

(*c*) Let the line be $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

It lies on the cone and on the plane $\Leftrightarrow 20l^2 + 7m^2 - 108n^2 = 0$ and $10l + 7m - 6n = 0$

Eliminating n, $20l^2+7m^2-108\left(\frac{10l+7m}{6}\right)^2=0$

$\Rightarrow 20\,l^2+7m^2-3(100\,l^2+49m^2+140\,lm)=0$

$\Rightarrow 20\,l^2+3lm+m^2=0 \Rightarrow (2l+m)\,(l+m)=0$

$\Rightarrow 2l+m=0$ ... (3) and $l+m=0$ .... (4)

(i) Solving $10l+7m-6n=0$ .... (2) $2l+m+0.\,n=0$ ....

$$\frac{l}{0+6}=\frac{m}{-12+0}=\frac{n}{10-14}\Rightarrow\frac{l}{3}=\frac{m}{-6}=\frac{n}{-2}$$

(ii) Solving $10l+7m-6n=0$ .... (2) $l+m+0.n=0$ .... (4)

$$\frac{l}{0+6}=\frac{m}{-6-0}=\frac{n}{10-7}\Rightarrow\frac{l}{2}=\frac{m}{-2}=\frac{n}{1}$$

$\therefore$ Angle between the lines of intersection is θ where

$$\cos\theta=\frac{3\,(2)-6\,(-2)-2\,(1)}{\sqrt{9+36+4}\sqrt{4+4+1}}=\frac{16}{21}\Rightarrow\theta=\text{Cos}^{-1}\left(\frac{16}{21}\right)$$

(e) Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be a line of intersection

then $8mn+3nl-5lm=0$ and $4l-m-5n=0$.

Eliminating m, $8n\,(4l-5n)+3nl-5l\,(4l-5n)=0$

$\Rightarrow l^2-3\,ln+2n^2=0\Rightarrow(l-n)\,(l-2n)=0 \;\Rightarrow l-n=0$ or $l-2n=0$

(i) Solving $l-n=0$ and $4l-m-5n=0\Rightarrow\frac{l}{1}=\frac{m}{-1}=\frac{n}{1}$

(ii) Solving $l-2n=0$ and $4l-m-5n=0$. $\frac{l}{2}=\frac{m}{3}=\frac{n}{1}$.

$\therefore$ Angle between the lines $\cos\theta=\frac{1\,(2)-1\,(3)+1\,(1)}{\sqrt{1+1+1}\sqrt{4+9+1}}=0\Rightarrow\theta=\frac{\pi}{2}$

3. Let the line of intersection be $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$

$\therefore$ (l, m, n) must satisfy the cone and the plane

$u\,l+v\,m+w\,n=0 \;\Rightarrow n=\frac{u\,l+v\,m}{-w}$ ..... (1) and $al^2+bm^2+cn^2=0$ ..... (2)

Substituting (1) in (2) $al^2+bm^2+c\frac{(u\,l+v\,m)^2}{w^2}=0$

$\Rightarrow aw^2l^2+bw^2m^2+c\,(u^2l^2+v^2m^2+2uvlm)=0$

$\Rightarrow l^2\,(aw^2+cu^2)+2lm(uv)+m^2(bw^2+cv^2)=0$

$\Rightarrow\left(\frac{l}{m}\right)^2(aw^2+cu^2)+2uv\frac{l}{m}+(bw^2+cv^2)=0$

which is a quadratic in $\left(\frac{l}{m}\right)$. Let be roots be $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$

$\therefore \left(\frac{l_1}{m_1}\right)\left(\frac{l_2}{m_2}\right) = \frac{bw^2 + cv^2}{aw^2 + cu^2} \Rightarrow \frac{l_1 l_2}{bw^2 + cv^2} = \frac{m_1 m_2}{aw^2 + cu^2}$

by symmetry each $= \frac{n_1 n_2}{av^2 + bu^2} = k$

The lines of intersection are at right angles $\Leftrightarrow l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$\Leftrightarrow k\left[(bw^2 + cv^2) + (cu^2 + aw^2) + (av^2 + bu^2)\right] = 0$

$\Rightarrow (b+c)\,u^2 + (c+a)\,v^2 + (a+b)\,w^2 = 0$

4. Let one of the lines be $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$. Line lies on the cone and the plane

$\Leftrightarrow mn + nl + lm = 0$ and $al + bm + cn = 0$

Eliminating n, $(m+l)\left(-\frac{al+bm}{c}\right) + lm = 0$

$\Rightarrow al^2 + (a+b-c)\,lm + bm^2 = 0 \Rightarrow a\left(\frac{l}{m}\right)^2 + (a+b-c)\left(\frac{l}{m}\right) + b = 0$

Let the roots be l_1/m_1 and l_2/m_2.

$\therefore \frac{l_1}{m_1}.\frac{l_2}{m_2} = \frac{b}{a} \Rightarrow \frac{l_1 l_2}{1/a} = \frac{m_1 m_2}{(1/b)} = \frac{n_1 n_2}{1/c}$ (Symm.)

The lines will be at right angles $\Leftrightarrow l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \Rightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$

5. In the given cone Co.eft. of x^2 + coeff. of y^2 + co eft. of $z^2 = 0$

$= b - c + c - a + a - b = 0$

Hence the cone contains. sets of three mutually perpendicular generators.

Now the plane $lx + my + nz = 0$ cuts the cone in perpendicular generators

$\Leftrightarrow$ the normal of the plane with d.r.'s (l, m, n) lies on the cone

$\Rightarrow (b-c)\,l^2 + (c-a)\,m^2 + (a-b)\,n^2 + 2fmn + 2gnl + 2hlm = 0$

6. By worked example 5. The angle between the lines of intersection of the plane $x + y + z = 0$ and the cone $ayz + bzx + cxy = 0$ is $\frac{\pi}{3} \Leftrightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$

In this problem, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \Rightarrow (b-c) + (c-a) + (a-b) = 0$

Hence the angle between the lines of intersection is $\pi/3$.

EXERCISE 7 (*d*)

1. **(*a*)** Given base curve is $z = 2, x^2 + y^2 = 4$.

Homogenising the conic $x^2 + y^2 = 4$ with the equation of the plane $z = 2$ we get

$x^2 + y^2 = 4\left(\frac{z}{2}\right)^2 \Rightarrow x^2 + y^2 = z^2$ which is the required cone.

(*b*) Given spheres are $S = x^2 + y^2 + z^2 + x - 2y + 3z = 4$,

$$S' = x^2 + y^2 + z^2 + 2x - 3y + 4z = 5$$

$\therefore$ Equation to the common plane of intersection is $S - S' = 0 \Rightarrow x - y + z = 1$.

Homogenising the sphere S = 0 with the plane equation we get the required cone as

$x^2 + y^2 + z^2 + (x - 2y + 3z)(x - y + z) - 4(x - y + z)^2 = 0$

(*c*) Homogenising the conicoid equation $x^2 + y^2 - z^2 - 2x + 1 = 0$ with the plane equation $z = 3$ we get the cone $x^2 + y^2 - z^2 - 2x\left(\frac{z}{3}\right) + 1\left(\frac{z}{3}\right)^2 = 0 \Rightarrow 9x^2 + 9y^2 - 8z^2 - 6zx = 0$

(*d*) Given plane $lx + my + nz = p \Rightarrow \frac{lx + my + nz}{p} = 1$

Homogenising the equation of the conicoid we get the required cone

$\Rightarrow ax^2 + by^2 + cz^2 = \left(\frac{lx + my + nz}{p}\right)^2$

$\Rightarrow p^2(ax^2 + by^2 + cz^2) = l^2x^2 + m^2y^2 + n^2z^2 + 2lmxy + 2mnyz + 2nlzx$

$\Rightarrow (ap^2 - l^2)x^2 + (bp^2 - m^2)y^2 + (cp^2 - n^2)y^2 - 2lmxy - 2mnyz - 2nlzx = 0$

2. Equation to the sphere through the point (0, 0, 0) be

$x^2 + y^2 + z^2 - 2ux - 2vy - 2wz = 0$ ..... (1)

The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ cuts the axes at A $(a, 0, 0)$, B $(0, b, 0)$ and C $(0, 0, c)$

A lies an the sphere (1) $\Leftrightarrow a^2 - 2ua = 0 \Rightarrow 2u = a$, similarly $2v = b$, $2w = c$.

Hence the sphere OABC is $x^2 + y^2 + z^2 - ax - by - cz = 0$

Homogenising the sphere with the given place we get

$x^2 + y^2 + z^2 - (ax + by + cz)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0$

$\Rightarrow yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + (xy)\left(\frac{a}{b} + \frac{b}{a}\right) = 0$, which is the required cone.

3. Let $f(x, y) = ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$

Homogenising the equation with the help of $z = k$ we get

$ax^2+by^2+2hxy+2gx\left(\frac{z}{k}\right)+2fy\left(\frac{z}{k}\right)+c\left(\frac{z}{k}\right)^2=0$. Multiplying (k^2/z^2) we get

$a\left(\frac{xk}{z}\right)^2+b\left(\frac{yk}{z}\right)^2+2h\left(\frac{xk}{z}\right)\left(\frac{yk}{z}\right)+2g\left(\frac{xk}{z}\right)+2f\left(\frac{yk}{z}\right)+c=0$

4. (*a*) Let the equation of the generator through the vertex $(-1,1,2)$ be

$$\frac{x+1}{l}=\frac{y-1}{m}=\frac{z-2}{n}=r \qquad \text{..... (1)}$$

A point on the generator is $(l\,r-1, m\,r+1, n\,r+2)$.

This point lies on the curve $3x^2-y^2=1, z=0$

$\Leftrightarrow 3\,(l\,r-1)^2-(m\,r+1)^2=1,\ \ n\,r+2=0 \Rightarrow r=-\frac{2}{n}$

Eliminating r we get $3\left[-\frac{2l}{n}-1\right]^2-\left[-\frac{2m}{n}+1\right]^2=1$

$\Rightarrow 3\,(2\,l+n)^2-(2\,m-n)^2=n^2$ I

Eliminating l, m, n from I by using (1)

$\Rightarrow 3\left[2\,(x+1)+(z-2)\right]^2-\left[2\,(y-1)-(z-2)\right]^2=(z-2)^2$

$\Rightarrow 3\,(2x+z)^2-(2y-z)^2=(z-2)^2 \Rightarrow 3\,(4x^2+4xz+z^2)-(4y^2-4yz+z^2)=z^2-4z+4$

$\Rightarrow 12x^2-4y^2+z^2+4yz+12zx+4z-4=0$.

(*b*) Let the equation to the generator through the point (1, 1, 1) be

$$\frac{x-1}{l}=\frac{y-1}{m}=\frac{z-1}{n}=r \qquad \text{.... (1)}$$

Any point on the generator is $\{\,l\,r+1, m\,r+1, n\,r+1\,\}$

This point lies on the curve $x^2+y^2=4, z=2$

$\Leftrightarrow (l\,r+1)^2+(m\,r+1)^2=4,\ \ n\,r+1=2 \Rightarrow r=\frac{1}{n}$

Substituting the value of r, $\left(\frac{l}{n}+1\right)^2+\left(\frac{m}{n}+1\right)^2=4 \Rightarrow (l+n)^2+(m+n)^2=4n^2$

Eliminating l, m, n using (1) we get $\left[(x-1)+(z-1)\right]^2+\left[(y-1)+(z-1)\right]^2=4\,(z-1)^2$

$\Rightarrow (x+z-2)^2+(y+z-2)^2=4\,(z^2-2z+1)$

$\Rightarrow x^2+y^2-2z^2+2yz+2zx-4\,x-4y+4=0$.

(*c*) Let the equation to the generator passing through the vertex (1, 2, 3) be $\frac{x-1}{l}=\frac{y-2}{m}=\frac{z-3}{n}=r$ ..... (1)

Any point on the generator is $\{\,l\,r+1, m\,r+2, n\,r+3\,\}$

This point lies on the curve $x^2+y^2+z^2=4$, $x+y+z=1$

$\Leftrightarrow (lr+1)^2+(mr+2)^2+(nr+3)^2=4$ ..... (2)

and $lr+1+mr+2+nr+3=1 \Rightarrow r(l+m+n)=-5 \Rightarrow r=\dfrac{-5}{l+m+n}$

$\therefore$ Substituting in (2) $\left(\dfrac{-5l}{l+m+n}+1\right)^2+\left(\dfrac{-5m}{l+m+n}+2\right)^2+\left(\dfrac{-5n}{l+m+n}+3\right)^2=4$

$\Rightarrow (m+n-4l)^2+(2n+2l-3m)^2+(3l+3m-2n)^2=4(l+m+n)^2$

$\Rightarrow 5l^2+3m^2+n^2-6mn-4nl-2lm=0$

$\Rightarrow 5(x-1)^2+3(y-2)^2+(z-3)^2-6(y-2)(z-3)-4(z-3)(x-1)-2(x-1)(y-2)=0$

$\Rightarrow 5x^2+3y^2+z^2-2xy-6yz-4zx+6x+8y+10z=26$

(*d*) Similar to above example.

5. Let the equation to the generator through (α,β,γ) be $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}=k$ .... (1)

The point $(\alpha+lk,\beta+mk,\gamma+nk)$ lies on the curve $ax^2+by^2=1, z=0$

$\Leftrightarrow a(\alpha+lk)^2+b(\beta+mk)^2=1, \gamma+nk=0 \Rightarrow k=-\dfrac{\gamma}{n}$

Eliminating k, $a\left(\alpha-\dfrac{l\gamma}{n}\right)^2+b\left(\beta-\dfrac{m\gamma}{n}\right)^2=1$

Using (1) we have the required equation , $a(n\alpha-l\gamma)^2+b(n\beta-m\gamma)^2=n^2$

$=a[\alpha(z-\gamma)-\gamma(x-\alpha)]^2+b[\beta(z-\gamma)-\gamma(y-\beta)]^2=(z-\gamma)^2$

$=a[\alpha(z-\gamma)-\gamma(x-\alpha)]^2+b[\beta(z-\gamma)-\gamma(y-\beta)]^2=(z-\gamma)^2$

6. Any line through (a,b,c) is $\dfrac{x-a}{l}=\dfrac{y-b}{m}=\dfrac{z-c}{n}$

It meets $x=0$ in the point $\left[0,b-\dfrac{am}{l},c-\dfrac{an}{l}\right]$

This point lies on given curve $\mathrm{F}(y,z)=0$

$\Leftrightarrow \mathrm{F}\left[b-a\dfrac{y-\beta}{(x-a)},c-a\dfrac{(z-c)}{(x-a)}\right]=0 \Rightarrow \mathrm{F}\left[\dfrac{bx-ay}{x-a},\dfrac{cx-az}{\gamma-a}\right]=0$

This meets the ZX - plane *i.e.* $y=0$ in the curve $\mathrm{F}\left[\dfrac{bx}{x-a},\dfrac{cx-az}{x-a}\right]=0, y=0$

EXERCISE 7 (*e*)

1. Let $S \equiv x^2+y^2+z^2-2x+4z-1=0$ and $P=(x_1, y_1, z_1)=(1,1,1)$

$\therefore$ $S_1 \equiv x(1)+y(1)+z(1)-(x+1)+2(z+1)-1 = y+3z$, $S_{11}=1+1+1-2+4-1=4$

Equation to the enveloping cone is $S_1^2 = SS_{11}$

$\Rightarrow (y+3z)^2 = 4(x^2+y^2+z^2-2x+4z-1) \Rightarrow 4x^2+3y^2-5z^2-6yz-8x+16z-4=0$

2. Let $S \equiv x^2+y^2+z^2-2x-4y=0$. Given vertex = (1, 1, 1)

$\therefore$ $S_1 \equiv x(1)+y(1)+z(1)+(x+1)-2(y+1) = 2x-y+z-1$, $S_{11} \equiv 1+1+1+2-4=1$

Equation to the enveloping cone is $S_1^2 = S \,.\, S_{11}$

$\Rightarrow$ $(2x-y+z-1)^2 = (x^2+y^2+z^2+2x-4y)(1)$

$\Rightarrow$ $3x^2+2yz-4xy-6x+6y-2z+1=0$

3. The axes of the cone makes equal angles θ with the coordinate axes.

$\therefore$ d.r's of the axis are $(\cos\theta, \cos\theta, \cos\theta) \propto (1,1,1)$

Semivertical angle of the cone is $\alpha = 30^\circ$. Vertex $=(2,-3,5)$

$\therefore$ Equation to the cone is $\left[(x-2)^2+(y+3)^2+(z-5)^2\right](1+1+1)\cos^2 30^\circ$

$$=\left[1.(x-2)+1.(y+3)+1.(z-5)\right]^2$$

$\Rightarrow (x^2+y^2+z^2-4x+6y-10z+38)(3)\left(\dfrac{3}{4}\right) = (x+y+z-4)^2$

$\Rightarrow 5(x^2+y^2+z^2)-8(yz+zx+xy)-4x+86y-58z+278=0$

4. Given Vertex = (0, 0, 0). Equation to the axis is $\dfrac{x}{2}=\dfrac{y}{-4}=\dfrac{z}{3}$

$\therefore$ D.r's of the axes an $(2,-4,3)$. Let the semivertical angle be α.

Then the equation to the cone is $(x^2+y^2+z^2)(4+16+9)\cos^2\alpha = (2x-4y+3z)^2$

The cone passes through the point (1, 1, 2)

$\Leftrightarrow$ $(1+1+4)\,29\cos^2\alpha = (2-4+6)^2 \Rightarrow \cos\alpha = \dfrac{8}{87}$

$\therefore$ Required cone is $(x^2+y^2+z^2)\,29\left(\dfrac{8}{87}\right) = (2x-4y+3z)^2$

$\Rightarrow$ $4x^2+40y^2+19z^2-72yz+36zx-48xy=0$

5. Let $P(x,y,z)$ be any point on the surface of the cone.

Hence the d.r.'s of OP are (x,y,z). The d.r.'s of the x axis is on $(1,0,0)$

$\therefore\ \cos\alpha = \dfrac{x.1+y.0+z.0}{\sqrt{x^2+y^2+z^2}\sqrt{1+0+0}} \Rightarrow (x^2+y^2+z^2)\cos^2\alpha = x^2$

$\Rightarrow y^2+z^2 = x^2\tan^2\alpha$

6. Equation to the axis $\dfrac{x-3}{4}=\dfrac{y-2}{1}=\dfrac{z-1}{3}$. Semi vertical angle is $30°$.

$\therefore$ Equation to the cone is $[(x-3)^2+(y-2)^2+(z-1)^2](4^2+1^2+3^2)\cos^2 30$

$=[4(x-3)+1(y-2)+3(z-1)]^2$

$\Rightarrow (x^2+y^2+z^2-6x-4y-2z+14)(26)(3/4) = (4x+y+3z-17)^2$ etc.

7. Vertex $V=(1,0,1)$. D.r.'s of the axis are $(1,1,1)$ passes through $P(1,1,1)$

Equation to the axis is $\dfrac{x-1}{1}=\dfrac{y-0}{1}=\dfrac{z-1}{1}$

$\therefore$ D.r.s of V P $=(1-1,1-0,1-1)=(0,1,0)$

Semi vertical angle α is the angle between VP and the axis

$\Rightarrow \cos\alpha = \dfrac{1(0)+1.1+1.0}{\sqrt{1+1+1}\sqrt{0+1+0}} = \dfrac{1}{\sqrt{3}}$. $\therefore$ Equation to the cone is

$[(x-1)^2+(y)^2+(z-1)^2](1+1+1).(1/\sqrt{3})^2 = [1(x-1)+1.y+1(z-1)]^2$

$\Rightarrow x^2+y^2+z^2-2x-2z+2 = (x+y+z-2)^2 \Rightarrow xy+yz+zx-x-2y-z+1=0$

8. Equation to the axis of the cone is $\dfrac{x-1}{3}=\dfrac{y+2}{3}=\dfrac{z+1}{5}$ and $\alpha=60°$

$\therefore$ Equation to the required cone is $[(x-1)^2+(y+2)^2+(z+1)^2](3^2+4^2+5^2).\cos^2 60$

$=[3(x-1)+4(y+2)+5(z+1)]^2$

$\Rightarrow (x^2+y^2+z^2-2x+4y+2z+6)\,50 = 4[3x+4y+5z+10]^2$ etc.

9. Let $A=(1,2,2), B=(2,1,-2), C=(2,-2,1)$. Clearly $OA=OB=OC=\sqrt{1+4+4}=3$

$\Rightarrow A,B,C$ lie on the sphere centre O and of radius 3.

Equation to the sphere is $x^2+y^2+z^2=3^2$

Let the plane through A be $a(x-1)+b(y-2)+c(z-2)=0$

It passes through $B(2,1,-2) \Leftrightarrow a-b-4c=0$ I

It passes through $C(2,-2,1) \Leftrightarrow a-4b-c=0$ II

Solving I and II : $\dfrac{a}{1-16}=\dfrac{b}{-4+1}=\dfrac{c}{-4+1} \Rightarrow \dfrac{a}{5}=\dfrac{b}{1}=\dfrac{c}{1}$

Equation to the plane is $5(x-1)+1(y-2)+1(z-2)=0 \Rightarrow 5x+y+z=9$

Now Homogenising the equation of the sphere with the help of the plane

$$x^2+y^2+z^2=9\left(\frac{5x+y+z}{9}\right)^2 \Rightarrow 9x^2+9y^2+9z^2=25x^2+y^2+z^2+10xy+2yz+10zx=0$$

$$\Rightarrow 8x^2-4y^2-4z^2+5xy+yz+5zx=0$$

Exercise 7 (*f*)

1. **(*a*)** Let $S \equiv 2x^2+2y^2+7z^2-10yz-10zx+2x+2y+26z-17=0$

$$\frac{\partial S}{\partial x}=4x-10z+2,\ \frac{\partial S}{\partial y}=4y-10z+2,\ \frac{\partial S}{\partial z}=14z-10y-10x+26$$

Required equations are $2x-5z+1=0$ .... (1)

$2y-5z+1=0$ .... (2) $7z-5y-5x+13=0$ .... (3)

Solving the equations we get the vertex (2, 2, 1)

(*b*) Let $S \equiv 4x^2-y^2+2z^2+2xy-3yz+12x-11y+6z+4=0$

$$\therefore\ \frac{\partial S}{\partial x}=8x+2y+12,\ \frac{\partial S}{\partial y}=-2y+2x-3z-11,\ \frac{\partial S}{\partial z}=4z-3y+6$$

Now to obtain the vertex we have to solve the equations

$4x+y+6=0$ .... (1), $2x-2y-3z-11=0$ .... (2) and $3y-4z-6=0$ .... (3)

Solving the equations we get the vertex $=(-1,-2,-3)$

(*c*) Let $S \equiv x^2-2y^2+3z^2-4xy+5yz-6zx+8x-19y-2z-20=0$

$$\therefore\ \frac{\partial S}{\partial x}=2x-4y-6z+8,\ \frac{\partial S}{\partial y}=-4y-4x+5z-19,\ \frac{\partial S}{\partial z}=6z+5y-6x-2$$

Now to obtain the vertex solve the equations.

$x-2y-3z+4=0$ ... (1), $4x+4y-5z+19=0$ ... (2) $6x-5y-6z+2=0$ ... (3)

Solving the equations we get the vertex point as $(1,-2,3)$

2. Equations to the plane $\perp r$ to the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and passing through the vertex (0, 0, 0)

is $x+2y+3z=0$ ... (1)

Let l, m, n be the d.r's of the line of intersection of (1) with the cone

$5yz-8zx-3xy=0$. Then $5mn-8nl-3lm=0$ ... (1) and

$l+2m+3n=0$ .. (2) $\Rightarrow l=-(2m+3n)$

Substituting in (1) $5mn+8n(2m+3n)+3m(2m+3n)=0$

$\Rightarrow m^2+5mn+4n^2=0 \quad \Rightarrow (m+n)(m+4n)=0 \quad \Rightarrow m=-n$ or $m=-4n$

(*i*) Let $m=-n$ then from (2) $l=-n \Rightarrow l=m=-n \quad \Rightarrow \frac{l}{1}=\frac{m}{1}=\frac{n}{-1}$

(*ii*) Let $m = -4n$ then $l = 5n \Rightarrow \frac{m}{-4} = \frac{n}{1}$ and $\frac{l}{5} = \frac{n}{1} \Rightarrow \frac{l}{5} = \frac{m}{-4} = \frac{n}{1}$

Hence the other $\perp r$ generators are $\frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$ and $\frac{x}{5} = \frac{y}{-4} = \frac{z}{1}$

3. Let a line perpendicular to given line be $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ .... (1)

Also $l + m + 2n = 0$ ... (2). (1) lies on the cone $\Leftrightarrow 3mn - 2nl - 2lm = 0$ .... (3)

Eliminating l from (2) and (3) : $3mn + 2n(m+2n) + 2m(m+2n) = 0$

$\Rightarrow 2m^2 + 9mn + 4n^2 = 0 \Rightarrow (2m+n)(m+4n) = 0$

$\Rightarrow 2m + n = 0$ .... (4) $m + 4n = 0$ .... (5)

(*i*) Solving $l + m + 2n = 0$ .... (3) $0.l + 2m + n = 0$ ..... (4)

$\frac{l}{1-4} = \frac{m}{0-1} = \frac{n}{2-0} = \frac{l}{3} = \frac{m}{1} = \frac{n}{-2}$. The line is $\frac{x}{3} = \frac{y}{1} = \frac{z}{-2}$

(*ii*) Solving $l + m + 2n = 0$ ... (3) $0.l + m + 4n = 0$ .... (5)

$\frac{l}{4-2} = \frac{m}{0-4} = \frac{n}{1-0} = \frac{l}{2} = \frac{m}{-4} = \frac{n}{1}$. $\therefore$ Equation to the generator is $\frac{x}{2} = \frac{y}{-4} = \frac{z}{1}$

4. Let (x_1, y_1, z_1) be a point on the sphere $2(x^2+y^2+z^2) = 3r^2$ then $2(x_1^2+y_1^2+z_1^2) = 3r^2$.
Equation to the enveloping cone of $x^2+y^2+z^2 = r^2$ with vertex at (x_1, y_1, z_1) is $S_1^2 = S\,S_{11}$

$\Rightarrow (x x_1 + y y_1 + z z_1 - r^2)^2 = (x^2+y^2+z^2-r^2)(x_1^2+y_1^2+z_1^2-r^2)$

$= (x^2+y^2+z^2-r^2)\left(\frac{3}{2}r^2 - r^2\right)$

$2(x x_1 + y y_1 + z z_1 - r^2)^2 = r^2(x^2+y^2+z^2-r^2)$

Consider: Co.eft. of x^2 + Co.eft. y^2 + Co.eft. z^2

$= (2x_1^2 - r^2) + (2y_1^2 - r^2) + (2z_1^2 - r^2)$

$= 2(x_1^2+y_1^2+z_1^2) - 3r^2 = 2(x_1^2+y_1^2+z_1^2) - 3r^2 = 0$

Hence the enveloping cone has three mutually $\perp r$ generators.

$\Rightarrow$ Given sphere has three mutually $\perp r$ tangent lines.

5. From the given cone Co.eft. of x^2 + Co.eft. y^2 + Co.eft. $z^2 = b - c + c - a + a - b = 0$

Hence the cone contains sets of three mutually $\perp r$ generators.

Now the plane $lx + my + nz = 0$ cuts the cone in $\perp r$ generators

$\Leftrightarrow$ the normal of the plane with d.r's (l, m, n) lies on the cone

$\Leftrightarrow (b-c)l^2 + (c-a)m^2 + (a-b)n^2 + 2fmn + 2gml + 2hlm = 0$

6. Let $P(x_1, y_1, z_1)$ be the point from three mutually $\perp r$ tangent lies are drawn to the cone $S \equiv ax^2 + by^2 + cz^2 - 1 = 0$ ... (1)

Equation to the enveloping cone of (1) with vertex at P is $S_1^2 = S\, S_{11}$

$\Rightarrow (axx_1 + byy_1 + czz_1 - 1)^2 = (ax^2 + by^2 + cz^2 - 1)\ (ax_1^2 + by_1^2 + cz_1^2 - 1)$

This cone has three mutually $\perp r$ generators

$\Leftrightarrow$ Co.eft. of $x^2 +$ Co.eft. $y^2 +$ Co.eft. $z^2 = 0$

$$\Leftrightarrow a^2x_1^2 - a\,(ax_1^2 + by_1^2 + cz_1^2 - 1) + b^2y_1^2 - b\,(ax_1^2 + by_1^2 + cz_1^2 - 1) + c^2z_1^2 - c\,(ax_1^2 + by_1^2 + cz_1^2 - 1) = 0$$

$\Leftrightarrow -a\,(by_1^2 + cz_1^2 - 1) - b(ax_1^2 + cz_1^2 - 1) - c(ax_1^2 + by_1^2 - 1) = 0$

$\Rightarrow ax_1^2(b+c) + by_1^2(c+a) + cz_1^2(a+b) = a+b+c$

$\therefore$ Locus of P is $a(b+c)x^2 + b(c+a)y^2 + c(a+b)z^2 = a+b+c$

7. Let $P(x_1, y_1, z_1)$ be a point and a line through P be $\dfrac{x-x_1}{l} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n} = r$ ... (1)

A point on the line is $(lr + x_1, mr + y_1, nr + z_1)$

The point lies on the curve $\Leftrightarrow (lr + x_1)^2 + (mr + y_1)^2 = 1$, $nr + z_1 = 0 \Rightarrow r = -\dfrac{z_1}{n}$

Eliminating r we get, $\left(-\dfrac{lz_1}{n} + x_1\right)^2 + \left(-\dfrac{mz_1}{n} + y_1\right)^2 = 1$

$(lz_1 - nx_1)^2 + (mz_1 - ny_1)^2 = n^2$. Eliminating l, m, n by using (1),

$$[(x-x_1)z_1 - x_1(z-z_1)]^2 + [(y-y_1)z_1 - (z-z_1)y_1]^2 = (z-z_1)^2$$

Cone has three mutually $\perp$ generators

$\Leftrightarrow$ Co.eft. of $x^2 +$ Co.eft. $y^2 +$ Co.eft. $z^2 \Leftrightarrow z_1^2 + z_1^2 + x_1^2 + y_1^2 - 1 = 0$

Locus of P is $x^2 + y^2 + 2z^2 = 1$

8. Let the equation to the plane at a distance of a from the origin be $lx + my + nz = a$ where l, m, n are direction cosines. Homogenising the equating of the sphere with that of the plane $x^2 + y^2 + z^2 = 3a^2\left(\dfrac{lx + my + nz}{a}\right)^2$. Co.eft of $x^2 +$ Co.eft of $y^2 +$ Co.eft of z^2

$= (1 - 3l^2) + (1 - 3m^2) + (1 - 3n^2) = 3 - 3\,(l^2 + m^2 + n^2) = 3 - 3 = 0$. Hence the problem.

EXERCISE 7 (g)

1. Given cone is $3x^2 + 4y^2 + 5z^2 + 2yz + 4zx + 6xy = 0$.

$$\Rightarrow a = 3, b = 4, c = 5, 2f = 2, 2g = 4, 2h = 6$$

It is required to find the reciprocal cone.

$A = bc - f^2 = 4(5) - (1)^2 = 19$ $\qquad$ $F = gh - af = 2(3) - 3(1) = 3$

$B = ca - g^2 = 5(3) - (2)^2 = 11$ $\qquad$ $G = hf - bg = 3(1) - 4(2) = -5$

$C = ab - h^2 = 3(4) - (3)^2 = 3$ $\qquad$ $H = fg - ch = 1(2) - 5(3) = -13$

$\therefore$ Equation to the reciprocal cone is $19x^2 + 11y^2 + 3z^2 + 6yz - 10zx - 26xy = 0$

2. Let the equation to the cone passing through the three coordinate axes is $ayz + bzx + cxy = 0$ ... (1)

This cone contain the normals of the given planes $x + y + z = 0$ and $2x - y + z = 0$.

$\Leftrightarrow$ The normals $\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$ and $\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ lie on (1)

$\Leftrightarrow$ $a(1)(1) + b(1)(1) + c(1)(1) = 0 \Rightarrow a + b + c = 0$ ...(3)

and $\Leftrightarrow$ $a(-3)(1) + b(1)(2) + c(2)(-3) = 0 \Rightarrow 3a - 2b + 6c = 0$ ... (4)

Solving (3) and (4) we have $\frac{a}{8} = \frac{b}{-3} = \frac{c}{-5}$

Hence the cone (1) is $8yz - 3zx - 5xy = 0$ ... (2)

The reciprocal cone of $ayz + bzx + cxy = 0$ is

$$a^2x^2 + b^2y^2 + c^2z^2 - 2bcyz - 2cazx - 2abxy = 0$$

Similarly the reciprocal cone of (2) is

$64x^2 + 9y^2 + 25z^2 - 2(-3)(-5)yz - 2(-5)(8)zx - 2(8)(-3)xy = 0$

$\Rightarrow 64x^2 + 9y^2 + 25z^2 - 30yz + 80zx + 48xy = 0$

3. Given cone is $2x^2 + 3y^2 + 4z^2 + 2yz + 4zx + 6xy = 0$ ... (1)

$\therefore$ $a = 2, b = 3, c = 4, f = 1, g = 2, h = 3$.

$A = bc - f^2 = (3)(4) - 1 = 11$; $\qquad$ $F = gh - af = 2(3) - 2(1) = 4$

$B = ca - g^2 = 4(2) - 4 = 4$; $\qquad$ $G = hf - bg = 3(1) - 3(2) = -3$

$C = ab - h^2 = 2(3) - 9 = -3$; $\qquad$ $H = fg - ch = 1(2) - 4(3) = -10$

$\therefore$ Eqution to the reciprocal cone of (1) is $11x^2 + 4y^2 - 3z^2 + 8yz - 6zx - 20xy = 0$

4. Equation to reciprocal cone of $ax^2 + by^2 + cz^2 = 0$ is $\frac{x^2}{a} = \frac{y^2}{b} = \frac{z^2}{c} = 0$

Similarly the reciprocal cone of $2x^2 - 3y^2 + z^2 = 0$ is

$\frac{x^2}{2} = \frac{y^2}{-3} = \frac{z^2}{1} = 0 \Rightarrow 3x^2 - 2y^2 + 6z^2 = 0$

The plane $l\,x + m\,y + n\,z = 0$ touches the cone $2x^2 - 3y^2 + z^2 = 0$

$\Leftrightarrow$ The normal of the plane lies on the reciprocal cone

$3x^2 - 2y^2 + 6z^2 = 0 \Leftrightarrow 3l^2 - 2m^2 + 6n^2 = 0$.

5. Let the equation to the cone passing through the three axes be $ayz + bzx + cxy = 0$ ... (1)

Now the normals of the planes.

$x - y + z = 0$ ... (2), $\quad 2x + 3y + z = 0$ ... (3) and $4x - y - 5z = 0$ ... (4)

lie an the cone (1) $\Leftrightarrow a-b+c=0$, $3a+2b+6c=0$ and $5a-20b-4c=0$.

Solving these equations we get $\frac{a}{8}=\frac{b}{3}=\frac{c}{-5}$

$\therefore$ Equation to the cone is $8yz+3zx-5xy=0$ ... (5)

Now we write be reciprocal cone of (5) as $64x^2+9y^2+25z^2+30yz+80zx-48xy=0$

6. Clearly the point $(1,1,1)$ lies on the cone.

$\therefore$ Equation to the generator is $\frac{x-1}{1}=\frac{y-1}{1}=\frac{z-1}{1}$

Equation to the cone is $S \equiv x^2+2y^2-3z^2+2yz-5zx+3xy=0$

Equation to the tangent plane at $P(1,1,1)$ is $S_1=0$.

$\Rightarrow x(1)+2y(1)-3z(1)+[y(1)+1(z)]-\frac{5}{2}[z(1)+1(x)]+\frac{3}{2}[x(1)+1(y)]=0$

$\Rightarrow 2x+4y-6z+2y+2z-5z-5x+3x+3y=0$

$\Rightarrow 9y-9z=0 \qquad \Rightarrow y-z=0$.

Exercise 7 (*h*)

1. Any cone through the intersection of the line cones is $S+\lambda S'=0$

$\Rightarrow x^2-2y^2+3z^2-4yz+5zx-6xy+\lambda(2x^2-3y^2+4z^2-5yz+6zx-10xy)=0$

The line with d.r.'s $(1,1,1)$ lies on it

$\Leftrightarrow 1-2+3-4+5-6+\lambda(2-3+4-5+6-10)=0 \Rightarrow \lambda=-\frac{1}{2}$

$\therefore$ Equation to the cone is

$$x^2-2y^2+3z^2-4yz+5zx-6xy-\frac{1}{2}(2x^2-3y^2+4z^2-5yz+6zx-10xy)=0$$

2. Any cone through the intersection of the given two cones is

$27x^2+20y^2-32z^2+\lambda(2yz+zx-4xy)=0$

Let us take the other plane passing through the lines of intersection be $lx+my+nz=0$

(1) must represent the pair of planes

$\Rightarrow 27x^2+20y^2-32z^2+\lambda(2yz+zx-4xy) = (3x+2y-4z)(lx+my+nz)$

Comparing the coeffs. $3l=27, 2m=20, -4n=-32 \Rightarrow l=9, m=10, n=8$

$\therefore$ The other plane is $9x+10y+8z=0$

We can verify by comparing with the coeffs of yz, zx and xy . Hence the problem.

UNIT -V

THE CYLINDER

Exercise 8 (*a*)

1. Let $P(x_1, y_1, z_1)$ be a point on the cylinder

$\therefore$ Equation to the line parallel to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$

and passing through P is $\frac{x-x_1}{1}=\frac{y-y_1}{-2}=\frac{z-z_1}{3}=r$.

Any point on the line is $P(r+x_1, -2r+y_1, 3r+z_1)$.

P lies on the base curve $x^2+2y^2=1$, $z=3$

$\Leftrightarrow (x_1+r)^2+2(y_1-2r)^2=1$ and $3r+z_1=3 \Rightarrow r=\frac{3-z_1}{3}$

Eliminating r we have $\left(x_1+\frac{3-z_1}{3}\right)^2+2\left[y_1-2\frac{(3-z_1)}{3}\right]^2=1$

$\Rightarrow (3x_1-z_1+3)^2+2(3y_1+2z_1-6)^2=9$

$\therefore$ Locus of P is $3x^2+6y^2+3z^2+8yz-2zx+6x+24y-18z+24=0$

2. Let $P(x_1, y_1, z_1)$ be any point on the cylinder, then the equations to the generator through P are $\frac{x-x_1}{1}=\frac{y-y_1}{-2}=\frac{z-z_1}{3}$. The line meets the plane $z=0$ at the point

given by $\frac{x-x_1}{1}=\frac{y-y_1}{-2}=\frac{z-z_1}{3} \Rightarrow \left(x,-\frac{z_1}{3}, y_1+\frac{2z_1}{3}, 0\right)$.

The generator intersects the conic $\Leftrightarrow \left(x_1-\frac{z_1}{3}\right)^2+2\left(y_1+\frac{2z_1}{3}\right)^2=1$

$\therefore$ Locus of P is $\left(x-\frac{z}{3}\right)^2+2\left(y+\frac{2z}{3}\right)^2=1 \Rightarrow 3x^2+6y^2+3z^2-2zx+8yz-3=0$

3. Since the axis of the cylinder is parallel to z - axis, by eliminating z

$ax^2+by^2=2z$ and $lx+my+nz=p$

We get $ax^2+by^2=2\left(\frac{p-lx-my}{n}\right)$. $\therefore$ Cone is $n(ax^2+by^2+2lx+2my-np)=0$

4. Equation to the generator parallel to $z-$axis and passing through $P(x_1, y_1, z_1)$ is

$\frac{x-x_1}{0}=\frac{y-y_1}{0}=\frac{z-z_1}{1}=r$. Any point on the genertor is $P(x_1, y_1, r+z_1)$

P lies on the guiding curve $x^2+y^2=z, x+y+z=1$

$\Leftrightarrow x_1^2+y_1^2=(r+z_1)^2$ and $x_1+y_1+z_1+r=1 \Rightarrow r=1-x_1-y_1-z_1$

Eliminating $r \Rightarrow x_1^2+y_1^2=(1-x_1-y_1)^2$. $\therefore$ Locus of P is $2xy-2x-2y+1=0$

EXERCISE 8 (*b*)

1. Equation to the axis of the cylinder is $\frac{x-1}{2}=\frac{y}{1}=\frac{z-3}{2}$

$\therefore$ Equation to the rt. circular cylinder of radius 2 is

$\left[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-r^2\right](l^2+m^2+n^2)=[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2$

$\Rightarrow\left[(x-1)^2+(y-2)^2+(z-3)^2-4\right](4+1+4)=[2.(x-1)+1.(y-2)+2.(z-3)]^2$

$\Rightarrow 9(x^2+y^2+z^2-2x-4y-6z+10)=(2x+y+2z-10)^2$

$\Rightarrow 5x^2+8y^2+5z^2-4xy-4yz-8zx+22x-16y-14z-10=0$

2. Equation to the axis is $\frac{x}{2}=\frac{y}{1}=\frac{z}{-2}$, Radius of the cylinder = 4

$\therefore$ Equation to the cylinder is $(x^2+y^2+z^2-16)(4+1+4)=[2x+y-2z]^2$

$\Rightarrow 9(x^2+y^2+z^2-16)=4x^2+y^2+4z^2+4xy-4yz-8zx$

$\Rightarrow 5x^2+8y^2+5z^2+4yz+8zx-4xy-144=0$.

3. Let $A=(1,0,0)$, $B=(0,1,0)$ and $C=(0,0,1)$

ABC is an equilateral triangle $\Rightarrow$ Centroid of Δ ABC = circumcentre of Δ ABC

$\therefore$ Centre of the circle $=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$

Radius of the circle $=\sqrt{\left(1-\frac{1}{3}\right)^2+\left(0-\frac{1}{3}\right)^2+\left(0-\frac{1}{3}\right)^2}=\sqrt{\frac{2}{3}}=r$

Equation to the plane ABC is $x+y+z=1$

Equation to the normal which also passes through (0, 0, 0)

$\therefore$ D.r's of the normal are (1, 1, 1). $\therefore$ Equation to the cylinder is

$\left[(x-0)^2+(y-0)^2+(z-0)^2-r^2\right](l^2+m^2+n^2)=(lx+my+nz)^2$

$\Rightarrow\left(x^2+y^2+z^2-\frac{2}{3}\right)(1+1+1)=(x+y+z)^2$

$\Rightarrow (3x^2+3y^2+3z^2-2)=x^2+y^2+z^2+2xy+2yz+2zx$

$\Rightarrow x^2+y^2+z^2-yz-zx-xy-1=0$

4. Equation to the axis of the cylinder is $\frac{x-1}{2}=\frac{y-2}{2}=\frac{z-2}{2}=$

radius of the cylinder $r=2$. $\therefore$ Equation of the cylinder is

$[(x-1)^2+(y-2)^2+(z-2)^2-(2)^2](2^2+2^2+2^2)=[2(x-1)+2(y-2)+2(z-2)]^2$

$\Rightarrow 12[x^2+y^2+z^2-2x-4y-4z+5]=(2x+2y+2z-10)^2$

$\Rightarrow x^2+y^2+z^2-yz-zx-xy+2x-y-z-5=0$

5. Let $P(x_1, y_1, z_1)$ be any point on the cylinder

$\therefore$ Equation of the generator is $\frac{x-0}{2} = \frac{y-0}{3} = \frac{z-0}{6}$

Radius of the cylinder $r = 5$

Equation to the cylinder is $[x^2 + y^2 + z^2 - 5^2](2^2 + 3^2 + 6^2) = [2x + 3y + 6z]^2$

$= 49(x^2 + y^2 + z^2 - 25) = 4x^2 + 9y^2 + 36z^2 + 12xy + 36yz + 24zx$

$\Rightarrow 45x^2 + 40y^2 + 13z^2 - 12xy - 36yz - 24zx - 1225 = 0$

6. Given circle $x^2 + y^2 + z^2 - x - y - z = 0$ ... (1) $x + y + z = 1$ ... (2)

The axis of the cylinder will be perpendicular to the plane (1) and passes through the centre of the sphere.

The plane $x + y + z = 1$ meets the axes at $A(1, 0, 0)$, $B(0, 1, 0)$ and $C(0, 0, 1)$

$\Rightarrow$ ABC is an equilateral trangle.

$\Rightarrow$ Circum centre = incentre = centroid = $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = G$

$\therefore$ Circum radius $= AG = \sqrt{\left(1 - \frac{1}{3}\right)^2 + \left(0 - \frac{1}{3}\right)^2 + \left(0 - \frac{1}{3}\right)^2} = \sqrt{\frac{2}{3}} \Rightarrow r^2 = \frac{2}{3}$

By symmetry the normal to the plane through G clearly passes through the origin.

$\therefore$ Equation to the axis of the cylinder $\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$

$\therefore$ Equation to the curve right circular cylinder is

$$[x^2 + y^2 + z^2 - (2/3)](1 + 1 + 1) = [1.x + 1.y + 1.z]^2$$

$\Rightarrow (3x^2 + 3y^2 + 3z^2 - 2) = x^2 + y^2 + z^2 + 2yz + 2zx + 2xy$

$\Rightarrow x^2 + y^2 + z^2 - yz - zx - xy = 1$

7. Equation to the axis of the cylinder is $\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{-1}$

radius of the cylinder $= 3$. Equation to the right circular cylinder is

$[(x-1)^2 + (y-3)^2 + (z-5)^2 - 3^2][2^2 + 2^2 + (-1)^2] = [2(x-1) + 2(y-3) - 1(z-5)]^2$

$\Rightarrow 9[x^2 + y^2 + z^2 - 2x - 6y - 10z + 26] = (2x + 2y - z - 3)^2$

$\Rightarrow 9x^2 + 9y^2 + 9z^2 - 18x - 54y - 90z + 234$

$= 4x^2 + 4y^2 + z^2 + 8xy + 6z + 9 - 4xz - 12x - 4yz$

$\Rightarrow 5x^2 + 5y^2 + 8z^2 + 4yz + 4zx - 8xy + 6x - 42y - 96z + 225 = 0$

EXERCISE 8 (*c*)

1. Let $P(x_1, y_1, z_1)$ be any point on the cylinder

$\therefore$ Equation of the generator is $\dfrac{x-x_1}{1} = \dfrac{y-y_1}{2} = \dfrac{z-z_1}{3} = k$ (say)

The point $(k+x_1, 2k+y_1, 3k+z_1)$ lies on the sphere

$\Leftrightarrow (k+x_1)^2 + (2k+y_1)^2 + (3k+z_1)^2 = 25$

$\Rightarrow 14k^2 + 2k\,(x_1 + 2y_1 + 3z_1) - (x_1^2 + y_1^2 + z_1^2 - 25) = 0$

The line touches the sphere $\Leftrightarrow 4\,(x_1 + 2y_1 + 3z_1)^2 = 4\,(14)\,(x_1^2 + y_1^2 + z_1^2 - 25) = 0$

$\therefore$ Locus of P is $(x+2y+3z)^2 = 14\,(x^2 + y^2 + z^2 - 25)$

$\Rightarrow 13x^2 + 10y^2 + 5z^2 - 4xy - 6zx - 12yz - 350 = 0$

2. Try yourself as in Worked Ex. 1.

3. Let $P(x_1, y_1, z_1)$ be a point on the cylinder.

The equation to the generator is $\dfrac{x-x_1}{1} = \dfrac{y-y_1}{1} = \dfrac{z-z_1}{1} = r$

The $(r+x_1, r+y_1, r+z_1)$ lies on $ax^2 + by^2 + cz^2 = 1$ ... (2)

$\Leftrightarrow a\,(r+x_1)^2 + b\,(r+y_1)^2 + c\,(r+z_1)^2$

$\Rightarrow r^2(a+b+c) + 2r(ax_1 + by_1 + cz_1) + (ax_1^2 + by_1^2 + cz_1^2 - 1)$

(1) touches the conicoid (2)

$\Leftrightarrow 4\,(ax_1 + by_1 + cz_1)^2 = 4\,(a+b+c)\,(ax_1^2 + by_1^2 + cz_1^2 - 1)$

Locus of P is $(ax+by+cz)^2 = (a+b+c)\,(ax^2 + by^2 + cz^2 - 1)$

$\Rightarrow (b+c)\,x^2 + (c+a)y^2 + (a+b)z^2 - 2abxy - 2bcyz - 2cazx - (a+b+c) = 0$

THE CENTRAL CONICOID

Exercise 9 (*a*)

1. Equation to the conicoid $3x^2-6y^2+9z^2+17=0$

$$\frac{3}{(-17)}x^2-\frac{6}{(-17)}y^2+\frac{9}{(-17)}z^2=1 \quad \text{... (1)}$$

Any plane parallel to $x+4y-2z=\lambda$ can be taken as $x+4y-2z=\lambda$ ...(2)

Now (2) is a tangent plane to (1) $\Leftrightarrow$ $\lambda=\frac{(1)^2}{-3/17}+\frac{(4)^2}{(+6/17)}+\frac{(-2)^2}{(-9/17)}$

$$\Rightarrow \lambda^2=17\left[-\frac{1}{3}+\frac{8}{3}-\frac{4}{9}\right]=17\left(+\frac{17}{9}\right)=+\frac{289}{9}\Rightarrow \lambda=\pm\frac{17}{3}$$

$\therefore$ Required tangent plane is $x+4y-2z=\pm\frac{17}{3} \Rightarrow 3x+12y-6z=\pm 17$.

2. (*a*) Equation to any plane passing through the given line is

$7x-6y+9+\lambda(z-3)=0 \Rightarrow 7x-6y+\lambda z+(9-3\lambda)=0 \Rightarrow 7x-6y+\lambda z=3\lambda-9$

This plane touches the conicoid $7x^2-3y^2+z^2+21=0$

i.e., $\frac{x^2}{-3}+\frac{y^2}{7}+\frac{z^2}{21}=1 \Rightarrow (3\lambda-9)^2=-(7)^2+7(-6)^2+21(\lambda^2)$

$$\Rightarrow 2\lambda^2+9\lambda+4=0 \Rightarrow (2\lambda+1)(\lambda+4)=0 \Rightarrow =-\frac{1}{2},-4$$

Required tangent planes are (*i*) $7x-6y+9-\frac{1}{2}(z-3)=0 \Rightarrow 14x-12y-z+21=0$

(*ii*) $7x-6y+9-4(z-3)=0 \Rightarrow 7x-6y-4z+21=0$

(*b*) Equation to the plane passing through the given line is

$7x+10y-30+\lambda(5y-3z)=0 \Rightarrow 7x+(5\lambda+10)y-3\lambda z=30$.

This plane touches the conicoid $\frac{7x^2}{60}+\frac{5y^2}{60}+\frac{3z^2}{21}=1$

$$\Leftrightarrow (7)^2\frac{60}{7}+(5\lambda+10)^2\frac{60}{5}+(-3\lambda)^2\frac{(60)}{3}=30^2$$

$$\Rightarrow 7+5(\lambda+2)^2+3\lambda^2=15 \Rightarrow 2\lambda^2+5\lambda+3=0 \Rightarrow (2\lambda+3)(\lambda+1)=0 \Rightarrow \lambda=-\frac{3}{2},-1$$

$\therefore$ Required planes are (*i*) $7x+10y-30-\frac{3}{2}(5y-3z)=0 \Rightarrow 14x+5y+9z=60$

(*ii*) $7x+10y-30-1(5y-3z)=0 \Rightarrow 7x+5y+3z=30$

(*c*) Given line is $\frac{x}{3}=\frac{y-3}{-3}=\frac{z}{1} \Rightarrow x=-y+3,\ x=3z \Rightarrow x+y-3=0,\ x-3z=0$ ... (1)

Any plane passing through the line (1) is $x+y-3+\lambda(x-3z)=0 \Rightarrow x(1+\lambda)+y-3\lambda z=3$

This plane touches the conicoid $\frac{x^2}{6}+\frac{y^2}{3}+\frac{z^2}{2}=1$

$\Leftrightarrow 6\,(1+\lambda)^2+3\,(1)+2\,(-3\lambda)^2=3^2 \Rightarrow 2\,(\lambda^2+2\lambda+1)+1+6\lambda^2=3$

$\Rightarrow 2\,\lambda^2+\lambda=0 \;\Rightarrow \lambda\,(2\lambda+1)=0 \Rightarrow \lambda=0,-\frac{1}{2}$

Required tangent planes are (i) $x+y-3=0$ and

(ii) $x+y-3-\frac{1}{2}(x-3z)=0 \;\Rightarrow x+y+3z=6$

3. The plane $3x+12y-6z=17$ touches the conicoid $-\frac{3}{17}x^2+\frac{6}{17}y^2-\frac{9}{17}z^2=1$

$\Leftrightarrow (3)^2\left(\frac{17}{-3}\right)+(12)^2\left(\frac{17}{6}\right)+(-6)^2\left(\frac{-17}{9}\right)=(17)^2 \Leftrightarrow 17(-3+24-4=(17)^2 \Rightarrow (17)^2=(17)^2$

Hence the plane touches the conicoid.

Point of contact is $=\left(\frac{l^2}{ap},\frac{m^2}{bp},\frac{n^2}{cp}\right)$

$\Rightarrow \left\{(3)^2\left(\frac{17}{-3}\right)\frac{1}{17},(12)\left(\frac{17}{6}\right)\left(\frac{1}{17}\right),(-6)\left(\frac{-17}{9}\right)\left(\frac{1}{17}\right)\right\}=\left(-1,2,\frac{2}{3}\right)$

4. ***(i)*** Let $G\,(x_1,y_1,z_1)$ be the centraid of Δ ABC .

Then $A=(3x_1,0,0), B=(0,3y_1,0), C=(0,0,3z_1)$

$\therefore$ Equation to the plane ABC is $\frac{x}{3x_1}+\frac{y}{3y_1}+\frac{z}{3z_1}=1$ .... (1)

(1) touches the conicoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$\Leftrightarrow a^2\left(\frac{1}{3x_1}\right)^2+b^2\left(\frac{1}{3y_1}\right)^2+c^2\left(\frac{1}{3z_1}\right)^2=1^2$ $\therefore$ Locus of G is $\frac{a^2}{x^2}+\frac{b^2}{y^2}+\frac{c^2}{z^2}=9$

(ii) Let $G\,(x_1,y_1,z_1)$ be the centroid of the tetrahedran OABC .

Then $A=(4x_1,0,0), B=(0,4y_1,0), C=(0,0,4z_1)$

Equation to the plane ABC is $\frac{x}{4x_1}+\frac{y}{4y_1}+\frac{z}{4z_1}=1$ .... (1)

(1) touches the conicoid $\Leftrightarrow a^2\left(\frac{1}{4x_1}\right)^2+b^2\left(\frac{1}{4y_1}\right)^2+c^2\left(\frac{1}{4z_1}\right)^2=1$

$\therefore$ Locus of G is $\frac{a^2}{x^2}+\frac{b^2}{y^2}+\frac{c^2}{z^2}=16$

5. Let the tangent plane be $lx+my+nz=p$.

This touches the conicoid $\Leftrightarrow a^2l^2+b^2m^2+c^2n^2=p^2$

$\therefore$ Equation to the tangent plane is $lx + my + nz = \pm\sqrt{a^2l^2 + b^2m^2 + c^2n^2}$

Distance from the origin to the plane $= r \Rightarrow \dfrac{\sqrt{a^2l^2 + b^2m^2 + c^2n^2}}{\sqrt{l^2 + m^2 + n^2}} = r$

$\Rightarrow a^2l^2 + b^2m^2 + c^2n^2 = r^2(l^2 + m^2 + n^2)$

$\Rightarrow l^2(a^2 - r^2) + m^2(b^2 - r^2) + n^2(c^2 - r^2) = 0$ ... (1)

Equation to the normal from (0, 0, 0) to the tangent plane is $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$ ... (2)

Locus of (1) is obtained by eliminating l, m, n between (1) and (2)

$\therefore$ Locus is $x^2(a^2 - r^2) + y^2(b^2 - r^2) + z^2(c^2 - r^2) = 0$

6. Equation to the tangent plane be $lx + my + nz = \sqrt{\dfrac{l^2}{a} + \dfrac{m^2}{b} + \dfrac{n^2}{c}}$

It passes through $(\alpha, \beta, \gamma) \Leftrightarrow l\alpha + m\beta + n\gamma = \sqrt{\dfrac{l^2}{a} + \dfrac{m^2}{b} + \dfrac{n^2}{c}}$ ... (1)

Equation to the normal to the tangent plane from $(0,0,0)$ is $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$ ... (2)

Eliminating l, m, n from (1) and (2) the locus of the normal is

$(\alpha x + \beta y + \gamma z)^2 = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}$

7. Tangent plane makes equal angles with the axes

$\Rightarrow$ normal makes equal angles with the axes $\Rightarrow$ d.r's of the normal are (1, 1, 1)

Equation to the tangent plane is $x + y + z = p$ where $p = \sqrt{a^2 + b^2 + c^2}$

Tangent planes cut the axes at A $= (p, 0, 0)$, B $= (0, p, 0)$, and C $= (0, 0, p)$

$\Rightarrow$ OA $= p$, OB $= p$, OC $= p$

Area of the tetrahedran $= \dfrac{1}{6}$. OA . OB . OC $= \dfrac{1}{6}p^3 = \dfrac{1}{6}(a^2 + b^2 + c^2)^{3/2}$

EXERCISE 9 (*b*)

1. Given conicoid $S \equiv ax^2 + by^2 + cz^2 - 1 = 0$. Let P (x_1, y_1, z_1) be the point.

Equation to the enveloping cone with vertex at P is $S_1^2 = S\,(S_{11})$

$\Rightarrow (axx_1 + byy_1 + czz_1 - 1)^2 = (ax^2 + by^2 + cz^2 - 1)(ax_1^2 + by_1^2 + cz_1^2 - 1)$

Cone contains three mutually $\perp r$ generaters

$\Leftrightarrow$ co-eft. of x^2 + co.eft. of y^2 + co-eft. of $z^2 = 0$

$\Rightarrow a^2x_1^2 - a(ax_1^2 + by_1^2 + cz_1^2 - 1) + b^2y_1^2 - b(ax_1^2 + by_1^2 + cz_1^2 - 1)$

$+ c^2z_1^2 - c(ax_1^2 + by_1^2 + cz_1^2 - 1) = 0$

$\Rightarrow a\,(by_1^2 + cz_1^2 - 1) + b\,(cz_1^2 + ax_1^2 - 1) + c\,(ax_1^2 + by_1^2 - 1) = 0$

$\therefore$ Locus of P is $a\,(b+c)\,x^2 + b\,(c+a)\,y^2 + c\,(a+b)\,z^2 = 0$.

2. Let $P\,(x_1, y_1, z_1)$ be a point on the generator $\|$ to the given line.

$\therefore$ Equation to the generator is $\dfrac{x-x_1}{0} = \dfrac{y-y_1}{\sqrt{a^2-b^2}} = \dfrac{z-z_1}{c} = r$ ... (1)

Any point on the line is $Q\,\{x_1, y_1 + r\sqrt{a^2-b^2}, z_1 + cr\}$.

Q lies on the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1 \Leftrightarrow \dfrac{x_1^2}{a^2} + \dfrac{\left(y_1 + r\sqrt{a^2-b^2}\right)^2}{b^2} + \dfrac{(z_1+cr)^2}{c^2} = 1$

$$\Rightarrow r^2\left[\frac{a^2-b^2}{b^2} + 1\right] + 2r\left[\frac{y_1\sqrt{a^2-b^2}}{b^2} + \frac{cz_1}{c^2}\right] + \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1 = 0$$

Line (1) is tangent line to the cylinder $\Leftrightarrow$ Disc = 0.

$$\Leftrightarrow 4\left(\frac{y_1\sqrt{a^2-b^2}}{b^2} + z_1\right)^2 - 4\left(\frac{a^2}{b^2}\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right) = 0$$

Locus of P is $\left(\dfrac{y\sqrt{a^2-b^2}}{b^2} + z\right)^2 - \dfrac{a^2}{b^2}\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1\right) = 0$

This conicoid meets the plane $z = 0$ along the cone

$$\left(\frac{y\sqrt{a^2-b^2}}{b^2}\right)^2 - \frac{a^2}{b^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = 0 \Rightarrow \frac{y^2(a^2-b^2)}{b^4} - \frac{a^2}{b^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = 0$$

$\Rightarrow (a^2-b^2)y^2 - (b^2x^2 + a^2y^2 - a^2b^2) = 0$

$\Rightarrow b^2(x^2+y^2) = a^2b^2 \Rightarrow x^2 + y^2 = a^2$ which is a circle.

SRI KRISHNA DEVARAYA UNIVERSITY

B.A /B.Sc Degree Examination Apri/May 2018

SECOND SEMESTER

Part II: Mathematics

Paper II: Solid Geometry (New Regulation 2016-17)(Common for B.A/B.Sc)

Time : 3 hours **Maximum : 75 marks**

SECTION - A

I. Answer any Five Questions. **$(5 \times 5 = 25)$**

1. Find the image of the point (1, 3, 4) in the plane $2x - y + z = 3$.

Ans. Refer Problem 20(1), Page No. 73.

2. Find the equation of the plane through (1, 0, –2) and perpendicular to the planes $2x + y - z = 2$, $x - y - z = 3$.

Ans. Refer Example 5, Page No. 47.

3. Find the equation of the line through (1, 2, 3) and parallel to the line $x - y + 2z = 5$, $3x + y + z = 6$.

Ans. Does is Problem 21, Page No. 73.

4. Prove that $\dfrac{x+4}{3} = \dfrac{y+6}{5} = \dfrac{z-1}{-2}$ and $3x - 2y + z + 5 = 0 = 2x + 3y + 4z - 4$ lines are coplanar and also find their intersecting point.

Ans. Refer Example 4, Page No. 87.

5. Find the equation of the shperes passing through the circle $x^2 + y^2 + z^2 = 4$, $z = 0$ and itneresected by the plane $x + 2y + 2z = 0$ in a circl eof radius '3'.

Ans. Refer Example 3, Page No. 42.

6. Find the radius and centre of the circle of intersectio of the sphere $x^2 + y^2 + z^2 = 25$, $2x + 3y + 2z = 9$.

Ans. Refer Problem 3(i), Page No. 144.

7. Find the limiting points of the coaxial system of the sphere determine dby the spheres $x^2 + y^2 + z^2 - 8x + 2y - 2z + 32 = 0$, $x^2 + y^2 + z^2 - x + z + 23 = 0$.

Ans. Refer Problem 4, Page No. 173.

8. Show that $\dfrac{x}{1} = \dfrac{y}{-1} = \dfrac{z}{-1}$ is a generator of the cone $5yz + 8zx - 2xy = 0$.

Ans. Refer Example 2 Page No. 179.

9. Find the equation of the right circular cone whose vertex is (0, 0, 0) axis the line $x = t$; $y = 2t$; $z = 3t$ and whose semi-vertical angle is 60°.

Ans. Refer Example 4, Page No. 195.

10. Find the equation of the cylinder whose generators are paralle to $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ and passing through the guiding curve $x^2 + 2y^2 = 1, z = 3$.

Ans. Refer Example 1, Page No. 220.

SECTION - B

Answer ALL questions $5 \times 10 = 50$

NIT I

11. (a) A variable plane is at a constant distance 3P from the origin and meets the coordinate axes in A, B, C. Show that the locus of the centroid of ΔABC is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.

Ans. Refer Problem - 23, Page No. 49.

(OR)

(b) The equation $6x^2 + 4y^2 - 11xy + 3yz + 4zx = 0$ represent a pair of planes and find its distance between them.

Ans. Refer Problem 1(iv), Page No. 61.

UNIT II

12. (a) Find the equation of the plane which contains the line $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z-3}{4}$ and is perpendicular to the plane $x + 2y + z = 12$.

Ans. Refer

(b) Find the shortest distance between and the equaitons of shortest distance between the lines $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-1}{2}, \frac{x-4}{4}=\frac{y-5}{5}=\frac{z-2}{3}$.

Ans. Refer Example 1(a), Page No. 103.

UNIT III

13. (a) A plane passes through a fixed point (a, b, c) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere of OABC is $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$.

Ans. Refer Example 6, Page No. 54.

(b) Show that the plane $4x + 9y + 14z - 64 = 0$ touches the sphere $3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0$. Find also the point of contact.

Ans. Refer Problem 4, Page No. 155.

UNIT IV

14. (a) Find the limiting points of the coaxial system of spheres $x^2 + y^2 + z^2 - 20x + 30y - 40z + 29 + \lambda(2x - 3y + 4z) = 0$.

Ans. Refer Problem - 1, Page No. 173.

(b) Find the equation of the cone with vertex (5, 4, 3) and $3x^2 + 2y^2 = 6$, $y + z = 0$ as base.

Ans. Refer Example 3, Page No. 189.

UNIT - V

15. (a) Find the equation of right circular cone whose vertex is P(2, –3, 5) axis PQ which makes equal angles with the axes and which passes through A(1, –2, 3).

Ans. Refer Example 2, Page No. 194.

OR

(b) Find the equation of the enveloping cylinder of sphere $x^2 + y^2 + z^2 = 25$, whose generators are parallel to $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$.

Ans. Refer Example 1, Page No. 225.

VIKRAM SIMHAPURI UNIVERSITY

THREE YEAR B.Sc/B.A (CBCS) Degree Examination Apri/May 2018

SECOND SEMESTER

Part II: Mathematics

Paper II: Solid Geometry

Time : 3 hours **Maximum : 75 marks**

SECTION - A

I. Answer any Five Questions. **(5 × 5 = 25)**

1. Find the angle between the planes $2x - y + z = 0$ and $x + y + 2z = 7$.

Ans. Refer 17(ii), Page No. 49.

2. Find the equation of the plane passing through (1, 0, –2) and perpendicular to the plane $2x + y - z = 2$ and $x - y - z = 3$.

Ans. Refer Example 5, Page No. 47.

3. Find the points of intersection of the lines $\frac{x-1}{-3} = \frac{y-2}{2} = \frac{z-3}{2}$ and $\frac{x-1}{3} = \frac{y-5}{1} = \frac{z}{-5}$.

Ans. Refer Problem - 9, Page No. 72.

4. Show that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ are coplanar.

Ans. Refer Example 5, Page No. 185.

5. Find the centre and radius of the sphere $2x^2 + 2y^2 + 2z^2 - 2x + 4y + 2z + 1 = 0$.

Ans. Refer Example 1(ii), Page No. 135.

6. Find the equation of the sphere through the circle $x^2 + y^2 + z^2 + 2x + 3y + 6 = 0$, $x - 2y + 4z = 9$ and the point (1, 2, 3).

Ans. Refer Problem (iii) 5, Page No. 145.

7. Find the equation of the tangent plane to the sphere $x^2 + y^2 + z^2 - 2x + 4y + 2z - 3 = 0$, to the point (–1, 4, –2).

Ans. Refer Problem 3, Page No. 155.

8. Show that the spheres are orhtozonal

$x^2 + y^2 + z^2 + 6y + 2z + 8 = 0, \quad x^2 + y^2 + z^2 + 6x + 8y + 4z + 20 = 0,$

Ans. Refer Problem - 1, Page No. 169.

9. Find the equation of the cone with vector (1, 1, 1) and guiding curve $x^2 + y^2 = 4$, $z = 2$.

Ans. Refer Problem 4(a), Page No. 182.

10. Find the equation of the cylinder whose generators are parallel to the lines $\frac{x}{-1}=\frac{y}{-2}=\frac{z}{3}$ and whose guiding curve int he ellipse $x^2 + 2y^2 = 1$, $z = 0$.

Ans. Refer Problem - 1 Page No. 220.

PART - B

Answer any FIVE of the following questions, choosing at least ONE question from each Unit. **5 × 10 = 50**

SECTION - A

UNIT - I

11. Find the equations of the planes bisecting the angle between the planes. $2x - y - 2z + 3 = 0$, $3x - 2y + 6z + 8 = 0$.

Ans. Refer Problem 9(iii), Page No. 55.

12. Prove that the equations $2x^2 - 3y^2 + 4z^2 + xy + 6zx - yz = 0$ represents a pair of planes and find the angle between them.

Ans. Refer Problem 1(i), Page No. 61.

UNIT - II

13. Find the image of the point (1, 3, 4) in the plane $2x - y + z + 3 = 0$.

Ans. Refer Problem 20(1), Page No. 73.

14. Find the length and equaions to the lines of shortest distance between the lines $\frac{x-3}{-1}=\frac{y-3}{2}=\frac{z+2}{1}, \frac{x-1}{1}=\frac{y+7}{3}=\frac{z+2}{2}$.

Ans. Refer Problem 2(i) , Page No. 103.

UNIT - III

15. Find the centre and radius of the circle $x - 2y + 2z = 15$, $x^2 + y^2 + z^2 - 2y - 4z - 11 = 0$.

Ans. Refer Exerices 3(ii), Page No. 144.

16. Find whether the following circle is a great circle or small circle $x^2 + y^2 + z^2 + 7y + 2z + 2 = 0$, $2x - 3y + 4z = 8$.

Ans. Refer Exerices 15(i), Page No. 145.

SECTION - B

UNIT - IV

17. Find the equation of the sphere which touches the planes $2x + 2y - z + 2 = 0$ at $(1, -2, 1)$ and cuts orthogonally the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$.

Ans. Refer Example 2, Page No. 177.

18. Find the limiting points of the coaxial system defined by the spheres $x^2 + y^2 + z^2 + 4x - 2y + 2z + 4 = 0$ and $x^2 + y^2 + z^2 + 2x - 4y - 2z + 6= 0$.

Ans. Refer Example 1, Page No. 172.

ANDRA UNIVERSITY

Three Year B.Sc./B.A. (CBCS) Degree Examination Apri/May 2018

SECOND SEMESTER

Part II: Mathematics

Paper II: Solid Geometry

Time : 3 hours **Maximum : 75 marks**

SECTION - A

I. Answer any Five Questions. **(5 × 5 = 25)**

1. Find the equation of the plane through (4, 4, 0) and perpendicular to the planes $x + 2y + 2z = 5$ and $3x + 3y + 2z - 8 = 0$.

Ans. Refer Example 4, Page No. 47.

2. Show that the equation $x^2 + 4y^2 + 9z^2 - 12yz - 6zx + 4xy + 5x + 10y - 15z + 6 = 0$ represents a pair of parallel planes and find the distance between them.

Ans. Refer Example 3, Page No. 60.

3. Find the image of the point (2, –1, 3) in the planes $3x - 2y + z = 9$.

Ans. Refer Example 2, Page No. 68.

4. Find the equations of the line through the point (1, 1, 1) and interesecting lines.

$2x - y - z - 2 = 0 = x + y + z = 1$

$x - y - z + 3 = 0 = 2x + 4y - z = -4$

Ans. Refer Example 8, Page No. 89.

5. Find the equation of the sphere having its centre on the line $5y + 2z = 0 = 2x - 3y$ and passing through the points (2, – 1, –1) (0, – 2, – 1).

Ans. Refer Problem 5, Page No. 136.

6. Find the vertex of the cone $2x^2 + 2y^2 + 7z^2 - 10yz - 10zx + 2x + 2y + 26 - 17 = 0$.

Ans. Refer Problem 1(a), Page No. 205.

7. Find the equation to the right circular cylinder whose guiding circle is $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$.

Ans. Refer Example 1, Page No. 222.

8. Find the enveloping cylidner of the sphere $x^2 + y^2 + z^2 + 2x - 4y + 1 = 0$ having its generators parallel to $x = y = z$.

Ans. Refer Example 1, Page No. 225.

SECTION - B

Answer ALL of the following questions. **(5 × 10 50)**

11. (a) If A = (1, 3, 2), B = (–5, 0, 2), C = (1, 1, – 4) find the distance of (2, 3, 4) from the plane $\overline{ABC}$ without find the equation to $\overline{ABC}$.

Ans. Refer Problem 26, Page No. 50.

(b) A variable plane passes through in a fixed point (a, b, c). It meets the axes of reference in A, B and C. Show that the l....... of the point of intersection of the planes through A, B, C and parallel of the planes through A, B, C and parallel to the cordinate plane as $ax^{-1} + by^{-1} + cz^{-1} = 1$.

Ans. Refer Example 6, Page No. 54.

10. (a) Find the image of the lines $\frac{x-1}{9} = \frac{y-2}{1} = \frac{z+3}{-3}$ in the plane $3x - 3y + 10z - 26 = 0$.

Ans. Refer Example 6, Page No. 70.

Or

(b) Find S.D and the requestions of the line of S.D between the lines $3x - 9y + 5z = 0 = x + y - z$ and $6x + 8y + 3z - 10 = 0 = x + 2y + z = 3$.

Ans. Refer Example 4, Page No. 99.

11. (a) The plane of equation $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the axes in A, B, C. Find the sanction of the ciecumcircle of ΔABC and inface find its centre.

Ans. Refer

Or

(b) Find the limiting points of the coaxal system defined by the spheres $x^2 + y^2 + z^2 + 4x + 2y + 2z + 6 = 0$ and $x^2 + y^2 + z^2 + 2x - 4y - 2z + 6 = 0$.

Ans. Refer Example 1, Page No. 172.

12. (a) Prove that the angle between the lines of intersection of the plane $x + y + z = 0$ with the cone $ayz + bzx + cxy = 0$ us $\frac{\pi}{3}$ if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$.

Ans. Refer Example 5, Page No. 185.

Or

(b) Prove that the cone $E(x, y, z) = 0$ will have three mutually perpendicular generators as $a + b + c = 0$.

Ans. Refer Theorem 7.14, Page No. 201.

13. (a) Find the equation of the cylinder whose generators have direction raties $(1, -2, 3)$ and whose guiding curve is $x^2 + 2y^2 = 1, z = 0$.

Ans. Refer Problem - 1, Page No. 220.

Or

(b) Find the equation of the enveloping cyclinder of the conicoid $ax^2 + by^2 + cz^2 = 1$. Whose generators are parallelo to $x = y = z$.

Ans. Refer Problem - 3 Page No. 225.

SRI VENKATESWARA UNIVERSITY

Three Year B.Sc./B.A. (CBCS) Degree Examination April 2018
Choice Based Credit System
SECOND SEMESTER
Part II: Mathematics
Paper I: Solid Geometry (w.e.f 2015-2016)

Time : 3 hours **Maximum : 75 marks**

SECTION - A

Answer any FIVE of the following **(5 × 5 = 25)**

1. Find the equation to the plane through the points (1, –2, 4), (3, –4, 5) and perpendicular to XY plane. (5)

Ans. Refer Problem 8(iii), Page No. 48.

2. Prove that the equation $2x^2 - 6y^2 - 12z^2 + 18yz + 2zx + xy = 0$ represents a pair of planes, and find the angle between them. (5)

Ans. Refer Example 1, Page No. 59.

3. Find the image of the point (1, 3, 4) in the plane $2x - y + z + 3 = 0$ (5)

Ans. Refer Problem - 20(i), Page No. 73.

4. Find the equation to the plane containing the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ if perpendicular to the plane $x + 2y + z - 12 = 0$.

Ans. Refer Problem 6, Page No. 77.

5. Find the centre and radius of the circle $x^2 + y^2 + z^2 - 2y - 4z - 11 = 0, x + 2y + 2z - 15 = 0$. (5)

Ans. Refer Problem 3(ii), Page No. 144.

6. Show that the plane $2x - 2y + z + 12 = 0$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ and find the point of contact. (5)

Ans. Refer Exerices 4, Page No. 151.

7. Find the equation to the cone which passes through the three co-ordiante axes and the lines $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and $\frac{x}{2} = \frac{y}{1} = \frac{z}{1}$.

Ans. Refer Problem 1, Page No. 181.

8. Find the equation of the lines of intersection of the plane $2x + y - z = 0$ and the cone $4x^2 - y^2 + 3z^2 = 0$. (5)

Ans. Refer Example 1, Page No. 183.

9. Find the equation of the cylinder whose generators are parallel to $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and which passes through the curve $x^2 + y^2 = 16, z = 0$.

Ans. Refer Example 1, Page No. 183.

10. Find the equation to the right circular cylinder whose axis is $\frac{x-1}{2}=\frac{y}{3}=\frac{z-3}{1}$ and of radius 2.

Ans. Refer Example 1, Page No. 223.

SECTION - B

Answer ALL questions. Each carries 10 Marks. **5 × 10 = 50**

11. Find the equation of the plane bisecting the obtuse angle between the planes $3x + 4y - 5z + 1 = 0$ and $5x + 12y - 13z = 0$. (10)

Ans. Refer Problem 10, Page No. 55.

12. If $H = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ represents pair of planes and θ is angle between them. Then show that

$$\cos\theta = \left|\frac{a+b+c}{\sqrt{(a+b+c)^2+4\left(f^2+g^2+h^2-ab-bc-ca\right)}}\right| \quad (10)$$

Ans. Refer Theorem 3.32, Page No. 58.

13. Find the image of the line $\frac{x-1}{9}=\frac{y-2}{1}=\frac{z+3}{-3}$ in the plane $3x-3y+10z-26=0$. (10)

Ans. Refer Example 6, Page No. 70.

14. Find the length and equations of the line of S.D between the lines $\frac{x}{1}=\frac{y}{2}=\frac{z}{1}$ and $x + y + 2z - 3 = 0 = 2x + 3y + 3z - 4$. (10)

Ans. Refer Example 3, Page No. 56.

15. A sphere of constant radius 'r' passes through the original 'O' and cuts the axes in A, B, C. Prove that the foot of the perpendicular from 'Q' to the plane $\overleftrightarrow{ABC}$ lies on $(x^2 + y^2 + z^2)^2 (x^2 + y^2 + z^2) = 4r^2$ (10)

Ans. Refer Example 6, Page No. 134.

Or

16. Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at $(1, -2, 1)$ and cuts orthogonally the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$. (10)

Ans. Refer Problem 15, Page No. 156.

17. Prove that the angle between the lines of intersection of the plane $x + y + z = 0$ with the cone $ayz + bzx + cxy = 0$ is $\frac{\pi}{3}$ if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$. (10)

Ans. Refer Example 5, Page No. 185.

Or

18. Show that the equation $2y^2 - 8yz - 4zx - 8xy + 6x - 4y - 2z + 5 = 0$ represents a cone and find its vertex. (10)

Ans. Refer Exerices 3, Page No. 201.

19. Find the equation to the right circular cylinder whose guiding circle is $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$. (10)

Ans. Refer Example 1, Page No. 222.

Or

20. Find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 - 2x + 4y - 1 = 0$, having its generators parallel to the line $x = y = z$. (10)

Ans. Refer Example 1, Page No. 225.

KRISHNA UNIVERSITY

Three Year B.Sc./B.A. (CBCS) Degree Examination April 2018

Choice Based Credit System

SECOND SEMESTER

Part II: Mathematics

Solid Geometry (Regulation 2015-2016)

Time : 3 hours **Maximum : 75 marks**

SECTION - A

Answer any FIVE of the following **(5 × 5 = 25)**

1. Find the equation of the plane through the point (4, 4, 0) and perpendicular to each of the plane $x + 2y + 2z - 5 = 0$ and $3x + 3y + 2z - 8 = 0$.

Ans. Refer Example 4, Page No. 47.

2. Show that the equation $2x^2 - 3y^2 + 4z^2 + xy + 6xz - yz = 0$ represents a pair of planes.

Ans. Refer Problem 1(i), Page No. 61.

3. Find the symmetric form of the equations of the line $x + y + z + 1 = 0 = 4x - y - 2z + 2$.

Ans. Refer Problem 16, Page No. 73.

4. Find the angle between the line $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$ and the plane $3x + y + z = 7$.

Ans. Solve $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$ ....(1)

given trans $3x + y + z - 7 = 0$

Let θ be angle between the plane and the list

$$\text{then } \sin\theta = \frac{\pm 3.2 + 1.3 + 16}{\sqrt{(3^2 + 1^2 + 1^2)}\sqrt{2^2 + 3^2 + 6^2}} = \pm\frac{6+3+6}{\sqrt{11}\sqrt{49}} = \pm\frac{15}{7\sqrt{11}}$$

5. Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at $(1, -2, 1)$ and passes therough the origin.

Ans. Refer Problem - 15, Page No. 156.

6. Find the equation of the sphere through the points $(1, -4, 3)$ $(1, -5, 2)$ $(1, -3, 0)$ and passes through the origin.

Ans. Refer Problem 8, Page No. 136.

7. Find the equation of the cone whose vertex is (1, 2, 3) and guiding curve is $y^2 = 4ax$, $z = 0$.

Ans. Refer Example 5, Page No. 190.

8. Find the equation to the right circular cylinder whose guiding circle is $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$.

Ans. Refer Example 1, Page No. 222.

PART B

Answer All Questions **5 × 10 = 50 Marks**

9. A variable plane is at a constant distance 3p from the origin and meets the axes in A, B, C show that the locus of the centroid of the triangle ABC is $x^2 + y^2 + z^2 = p^2$.

Ans. Refer Problem 22, Page No. 49.

Or

10. Find the equation of the bisector of the angles between the planes $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$ and distinguish them.

Ans. Refer Problem 9(ii), Page No. 55.

UNIT II

11. Show that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}; \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ are coplanar. Find the point of intersection and the plane containing the lines.

Ans. Refer Example 1, Page No. 85.

Or

12. Find the shortest distance and the equations of S.D between the lines $3x - 9y + 5z = 0 = x + y - z$; $6x + 8y + 3z - 10 = 0 = x + 2y + z - 3$.

Ans. Refer Example 4, Page No. 99.

UNIT - III

13. If r_1, r_2 are the radii of two orthogonal spheres, then the radius of the circle of their undersection is $\frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$.

Ans. Refer Theorem, Page No. 163.

14. If $(-2, 1, -1)$ is a limiting point of a coaxal system for which $x + y + 2z = 0$ is the radical plane then show that the other limiting point is $(-1, 2, 1)$.

Ans. Refer Problem - 8, Page No. 173.

UNIT - IV

15. Find the vertex of the cone $x^2 - 2y^2 + 3z^2 - 4xy + 5yz - 6zx + 8x - 19y - 2z - 20 = 0$.

Ans. Refer Exerice Problem 1(b), Page No. 205.

16. Find the equation of the right circular cone whose vertex is $(1, -2, -1)$ axis the line $\frac{x-1}{3} = \frac{y+2}{4} = \frac{z+1}{5}$ and the semivertical angle 60°.

Ans. Refer

UNIT - V

17. Find the equation to the cylinder whose generators are parallel to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ guiding curve is the ellipse $x^2 + 2y^2 = 1$, $z = 3$.

Ans. Refer Example Problem 1, Page No. 220.

Or

18. Find the equation of the enveloping cylidner of the sphere $x^2 + y^2 + z^2 - 2x + 4y - 1 = 0$ having its generators parallel to the line $x = y = z$.

Ans. Refer Example 1, Page No. 225.

ADIKAVI NANANAYA UNIVERSITY

At the end of Second Semester Degree Examination

Choice Based Credit System

SECOND SEMESTER

Part II: Mathematics

Solid Geometry (W.e.f Admitted Batch 2016-17) (CBCS PATTERN)

Time : 3 hours **Maximum : 75 marks**

SECTION - A

Answer any FIVE of the following **($5 \times 5 = 25$)**

1. Find the equations of the planes through the intersection of the planes $x + 3y + 6 = 0$, $3x - y - 4z = 0$ such that the perpendicular distance of each from the origin is unity.

Ans. Refer Example 2, Page No. 53.

2. Find teh equation of the line through the point (1, 2, 4) and parallel to the line $3x + 2y - z = 4$, $x - 2y - 2z = 5$.

Ans. Refer Problem 21, Page No. 73.

3. Show that the line $\frac{x-3}{3} = \frac{2-y}{4} = \frac{z+1}{1}$ intersects the line $2x + 4y + 3z + 3 = 0$, $x + 2y + 3z = 0$ in the point (9, –6, 1)

Ans. Refer Problem 8, Page No. 72.

4. A plane passes through a fixed point (a, b, c) and intersects the axes in A, B, C. Show that the center of the sphere OABC $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$.

Ans. Refer Example 5, Page No. 134.

5. Find the polar line of $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ w.r to the sphere $x^2 + y^2 + z^2 = 16$.

Ans. Refer Example 3, Page No. 161.

6. Show that the spheres $x^2 + y^2 + z^2 + 6y + 2z + 8 = 0$; $x^2 + y^2 + z^2 + 6x + 8y + 4z + 20 = 0$ are orthogonal.

Ans. Refer Problem - 1, Page No. 169.

7. Find the equation of the quarinic cone through the coordinate axes and three lines.

$\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and $\frac{x}{2} = \frac{y}{1} = \frac{z}{1}$

Ans. Refer Problem - 1, Page No. 181.

8. Find the condition that one plane $uz + vy + wz = 0$ way touch the cone $ax^3 + by^2 + cz^2 = 0$.

Ans. Refer Example 6, Page No. 211.

SECTION - B

II. Answer ALL questions. $5 \times 10 = 50$

9. (a) A variable palne is at a constant distance "p" from the origin and meets the axes in A, B, C. Show that the louis of the coentroid of the tetrahedron OABC is $x^{-2} + y^{-2} + z^{-2} = 16p^2$.

Ans. Refer Problem - 23, Page No. 49.

10. Find the sanction of the shortest distance between the lines $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$ cure $x + y + 2z - 3 = 0 = 2x + 3y + 3z - 4$.

Ans. Refer Example 6, Page No. 211.

11. (a) Show that the two circles $x^2 + y^2 + z^2 - y + 2z = 0$, $x - y + z = 2$ and $x^2 + y^2 + z^2 + x - 3y + z - 5 = 0$, $2x - y + 4z - 1 = 0$ line on the same sphere and its equation.

Ans. Refer Example 5, Page No. 143.

(OR)

(b) Find the pole of the plane $x - y + 5z - 3 = 0$ w.r.to the sphere $x^2 + y^2 + z^2 = 9$.

Ans. Refer Probelm 3, Page No. 162.

12. (a) Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at $(1, -2, 1)$ and cuts orthogonally the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$.

Ans. Refer Example 2, Page No. 167.

(b) Find the limiting points of the coaxal system of spheres of which two members are $x^2 + y^2 + z^2 + 3x - 3y + 6 = 0$, $x^2 + y^2 + z^2 - 6y - 6z + 6 = 0$.

Ans. Refer Problem 3, Page No. 173.

13. (a) Show that if a right circular cone has sets of three mutually perpendicular generations, its semivertical angle must be $\tan^{-1}\sqrt{2}$.

Ans. Refer Example 3, Page No. 203.

Or

(b) Find he plane which touches the cone $x^2 + y^2 - 3z^2 + 2yz - 5zx + 3xy = 0$, among the generator whose direction ratios are $(1, 1, 1)$.

Ans. Refer Problem 6, Page No. 212.

ANDHRA UNIVERSITY, VISAKAPATNAM

B.Sc. (Under CBCS) Degree Examination

SECOND SEMESTER

Part II: Mathematics

Part I: Solid Geometry (Common with B.A/B.Sc)

(with Effective from 2016-2017 admitted batch)

Time : 3 hours **Maximum : 75 marks**

SECTION - A

Answer any FIVE of the following **(5 × 5 = 25)**

1. Find the equation of the plane through (4, 4, 0) and perpendicular to the planes $x + 2y + 2z = 5$ and $3x + 3y + 2z = 8$.

Ans. Refer Example 4, Page No. 47.

2. Find the equation of the plane bisecting the acute angle between the planes $x + 2y + 2z - 3 = 0$, $3x + 4y + 12z + 1 = 0$.

Ans. Does is Example 7, Page No. 54.

3. Find the equations of the line through the point (1, 1, 1) and intersection the lines
$2x - y - z - 2 = 0 = x + y + z - 1$
$x - y - z - 3 = 0 = 2x + + 4y - z - 4$.

Ans. Refer Example 8, Page No. 89.

4. Find the equation of the plane contaiing the line $\frac{x}{a} + \frac{z}{c} = 1$, y = 0.

Ans. Does is Example 5, Page No. 100.

5. Show that the spheres $x^2 + y^2 + z^2 = 25$; $x^2 + y^2 + z^2 = 225 = 0$ touch externally, and find the point of contact.

Ans. Refer Problem 17, Page No. 156.

6. Determine the equation of the sphere through the points (4, –1, 2), (0, –2, 3), (1, –5, 1), (2, 0, 1) and find its radius.

Ans. Refer Problem 2(ii), Page No. 135.

7. Find the equation of the cone which passes through the three coordiante axes and the lines $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and $\frac{x}{3} = \frac{y}{-1} = \frac{z}{1}$.

Ans. Refer Problem 1, Page No. 181.

8. Find the equation to the cylinder whose generators are parallel to $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and guiding curve $x^2 + y^2 = 16$; $z = 0$.

Ans. Refer Example 1, Page No. 219.

SECTION - B

Answer the following (ONE question from each Unit)

UNIT - I

9. (a) Obtain the equation of the plane which passes through the point $(-1, 3, 2)$ and is perpendicular to each of the planes $x + 2y + 2z = 5$, $3 + 2y + 2z = 8$.

Ans. Refer Problem -9, Page No. 48.

Or

(b) Find the equation of the plane which is perpendicular to the plane $5x + 3y + 6z + 8 = 0$ and which contains the line of intersection of $x + 2y + 3z - 4 = 0$, $2x + y - z + 5 = 0$.

Ans. Refer Problem - 6, Page No. 55.

UNIT II

10. (a) Find the length and equation to the line of shortest distance between the lines

$$\frac{x}{4} = \frac{y+1}{3} = \frac{z-2}{2},\ 5x - 2y - 3z + 6 = 0 = x - 3y + 2z - 3.$$

Ans. Refer Problem - 4, Page No. 103.

(b) Prove that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$; $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ are coplanar. Find their point of intersection and the plane containing the lines.

Ans. Refer Example 1, Page No. 85.

UNIT - III

11. (a) Obtain the equations of the sphere which passes through the circle $x^2 + y^2 + z^2 - 2x + 2y - 4z + 3 = 0$, $2x + y + z = 4$ and touch the plane $3x + 4y = 14$.

Ans. Refer

Or

(b) Show that the polar line of $\frac{x+1}{2} = \frac{y-2}{3} = z+3$ with respect to the sphere $x^2 + y^2 + z^2 = 1$ is the line $\frac{7x+3}{11} = \frac{2-7y}{5} = \frac{z}{-1}$.

Ans. Refer Problem 10(i), Page No. 162.

UNIT IV

12. (a) State and prove a necessary and sufficient condition for a cone to admit a set of 3 mutually perpendicular generators.

Ans. Refer Theorem 7.14, Page No. 201.

Or

(b) Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at $(1, -2, 1)$ and cuts orthogonally the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$.

Ans. Refer Example 2, Page No. 167.

UNIT - V

13. (a) Find the equation of the right circular cone whose vertex is $(1, -2, -1)$, axis the line $\frac{x-1}{3}=\frac{y+2}{4}=\frac{z+1}{5}$ and semi-vertical angle $60°$.

Ans. Refer Problem 8, Page No. 198.

Or

(b) Find the equation of the right circular cylinder of radius 5 units and having its axis the line $\frac{1}{2}x=\frac{1}{3}y=\frac{1}{6}z$.

Ans. Refer Problem 5, Page No. 224.